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Yamabe-type Equations on Complete, Noncompact Manifolds

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Introduction

This book originates from a graduate course given at the University of Milan in 2007.

Our goal is twofold: first, to present a self-contained introduction to the geometric and analytic aspects of the Yamabe problem on a complete noncompact Riemannian manifold, treating existence, nonexistence, uniqueness and *a priori* estimates of the solutions. Secondly, we intend to describe in a way accessible to the nonspecialist a range of methods and techniques that can be successfully applied to more general nonlinear equations which arise in applications.

The classical Yamabe problem concerns the possibility of pointwise conformally deforming a metric of scalar curvature $S(x)$ on the manifold M to a new metric with prescribed scalar curvature $K(x)$. In the case where K is constant it is a natural higher dimensional generalization of the Poincaré–Köbe Uniformization Theorem for Riemann surfaces and can be seen as a way to select a privileged metric on the manifold.

If $\langle\ ,\ \rangle$ is the original metric of the Riemannian manifold M and we denote with $\widetilde{\langle\ ,\ \rangle} = \varphi^2 \langle\ ,\ \rangle$, $\varphi > 0$, a conformally deformed metric, then the two scalar curvatures $S(x)$ and $\widetilde{S}(x)$ are related by the equation

$$\varphi^2 \widetilde{S}(x) = S(x) - 2(m-1)\frac{\Delta\varphi}{\varphi} - (m-1)(m-4)\frac{|\nabla\varphi|^2}{\varphi^2}$$

(see equation (2.7) in Chapter 2), where Laplacian, gradient, and norm are those of the metric $\langle\ ,\ \rangle$. In the case where the dimension m of the manifold is greater than or equal to three, it is useful to set

$$\varphi = u^{\frac{2}{m-2}}$$

so that the above equation takes the form

$$\widetilde{S} u^{\frac{m+2}{m-2}} = Su - 4\frac{m-1}{m-2}\Delta u.$$

Thus the *Yamabe problem* amounts to finding a positive solution u of the familiar *Yamabe equation*

$$c_m \Delta u - Su + K u^{\frac{m+2}{m-2}} = 0, \tag{1}$$

where $c_m = 4\frac{m-1}{m-2}$ and $K = \widetilde{S}$, the prescribed scalar curvature of the conformally deformed metric. If M is compact and K is constant, after an initial attempted solution by H. Yamabe [Yam60], the problem was solved thanks to efforts of N. Trudinger [Tru68], T. Aubin [Aub76] and R. Schoen [Sch84] (see the nice survey paper by J.M. Lee and T.H. Parker, [LP87], for a complete and self-contained treatment). The solution was obtained using variational methods, and one of the main analytic difficulties stems from the fact that $\frac{m+2}{m-2}$ is the critical exponent for the Sobolev embedding $W^{1,2} \hookrightarrow L^{\frac{2m}{m-2}}$.

A natural generalization of the classical Yamabe problem is the case where K is nonconstant and/or M is noncompact. In this direction we mention the pioneering work of J. L. Kazdan and F. W. Warner, [KW74a], [KW74b], [KW75a], [KW75b]. It should also be mentioned that even the classical Yamabe problem of deforming the metric to one of constant scalar curvature in the noncompact setting is in general not solvable, as first shown by Z. R. Jin, [Jin88].

The Yamabe problem for noncompact manifolds with variable prescribed curvature is the subject of the present monograph. Indeed, we describe methods which allows us to consider the more general *Yamabe-type equations* (resp. inequalities) of the form

$$\Delta u + a(x)u - b(x)u^{\sigma} = 0 \quad (\text{resp. } \geq 0) \tag{2}$$

where $\sigma > 1$, and we study nonexistence, *a priori* estimates, uniqueness and existence.

Equations of the form (2) and still more general differential inequalities of the form

$$u\Delta u + a(x)u^2 - b(x)u^{\sigma+1} \geq -A|\nabla u|^2 \tag{3}$$

arise in complex analysis (e.g. in the study of the structure of complete Kähler manifolds, [LY90], [Li90] and [LR96], in the Schwarz Lemma for the ratio of volume elements of Kähler manifolds of the same dimension, [Gri76], in the study of pluriharmonic functions on a Kähler manifold, [PRS08]), in the study of harmonic maps with bounded dilation ([EL78] and [PRS08] Chapter 8), in the classification of locally conformally flat manifolds ([PRS07]), in the study of Yang-Mills fields, and in population dynamics, to quote only a few examples.

Existence and nonexistence of positive solutions of (2) clearly depend on the geometry of the underlying manifold, typically encoded by curvature or volume growth of geodesic balls, on properties of the coefficients (typically the relative signs of the coefficients $a(x)$ and $b(x)$) and their asymptotic behavior and on the mutual interplay of the two. This interplay can be taken into account in terms of the relative asymptotic behavior of the coefficients versus the geometry at infinity of the manifold or, at a deeper level, in terms of spectral properties of Schrödinger operators naturally associated to the equation.

From the geometrical interpretation of the equation, it is natural to expect it will be easier to have existence when a and b are "close" enough, for instance they have the same sign, while it will be more difficult to have existence (and therefore it will be easier to prove nonexistence) when a and b are farther apart, for instance when they have opposite sign. This expectation is confirmed by both the existence and the nonexistence results that we will describe.

The geometry of the manifold also plays a natural role in the uniqueness results as well in the *a priori* estimates on the solutions. The latter have a particular geometric interest since they are responsible for the completeness/noncompleteness of the deformed metric.

As mentioned above, we use a variety of techniques adapted to the geometric situation at hand in which the lack of compactness and of symmetry and homogeneity prevents the use of more standard tools typical of compact situations or of the Euclidean setting.

In particular, for existence we will essentially use the method of *sub-super solutions*, [Ama76], [Sat73]. Nonexistence will be obtained using *Liouville-type results* which in turn are obtained using either integral formulas or a method based on the coupling of the supposed solution of the Yamabe-type inequality with that of an appropriate Schrödinger-type inequality associated to it, in a manner reminiscent of the classical generalized maximum principle. Uniqueness will be obtained using variants of the *weak maximum principle* (see, e.g., [PRS05b]) and of clever integration by parts arguments. Finally, *a priori* estimates will be typically obtained using an elaboration of the old idea of the proof of the Schwarz's Lemma by L. H. Ahlfors, [Ahl38], which is at the heart of the maximum principle.

The book is divided into seven chapters.

In the first chapter we give a quick review of Riemannian geometry using the method of *moving frames*. While we assume basic knowledge of Riemannian geometry, several computations will be carried out in full detail in order to acquaint the reader with notation and formalism. We concentrate on derivation of the symmetry properties of the curvature tensors together with a number of other identities that will be repeatedly used in the sequel. In particular, we will describe the commutation rules for covariant derivatives up to fourth order. Then we describe comparison results for the Laplacian of the Riemannian distance function, and for the volume of geodesic balls in terms of lower bounds for the Ricci curvature. We point out that our treatment, which follows that of [PRS05b], does not use Jacobi fields.

In Chapter 2 we first derive equations for the change of curvature tensors under a conformal change of the metric and introduce the Yamabe equation. As a side product of our computations we obtain decomposition of the Riemann curvature tensor in its irreducible components and we exhibit the conformal invariance of the Weyl tensor. Then, we briefly consider the case where M is compact to illustrate the interplay between geometry and analysis, with a few illuminating examples such as the Kazdan-Warner obstruction, a result of Obata on Einstein manifolds and a far-reaching "generalization" due to Véron-Véron, through which we prove further results of Escobar. Along the way we give a detailed proof, which inspires to Petersen's treatise [Pet06a], of a famous rigidity result of Obata. The goal is also to give some geometrical feeling on the subject that will enable us to proceed with the noncompact case: the case of the rest of our investigation.

The core of the monograph begins with Chapter 3, devoted to nonexistence results. As mentioned above, since our methods apply to more general situations which have a wide range of applications, we consider in fact differential inequalities of the form (2) and (3). We describe several nonexistence results; in most of them we assume that u satisfies suitable integrability conditions, that $b(x)$ is nonnegative and that there exists a positive solution φ to the differential inequality

$$\Delta\varphi + Ha(x)\varphi \leq -K\frac{|\nabla\varphi|^2}{\varphi}$$

with H, K parameters satisfying $H > 0$, $K > -1$. Note that in the special case where $K = 0$ the latter condition amounts to the fact that the bottom of the spectrum of the operator $-\Delta - Ha(x)$ is nonnegative. Since $-\Delta$ is a nonnegative operator, the condition is trivially satisfied if $a(x) \leq 0$ on M and may be interpreted as a measure of smallness in a spectral sense of the positive part of $a(x)$. This agrees with the heuristic intuition on the effect of the relative signs of $a(x)$ and $b(x)$ on the existence of solutions. The existence of the positive function φ enters the proof in two different ways. In Theorem 3.2 one uses the functions φ to obtain an integral inequality involving u and its gradient from which one concludes that u is constant, and therefore necessarily identically zero. In a second group of results, the function φ is combined with the solution u to give rise to a diffusion-type differential inequality for which we prove a Liouville theorem. This yields the desired triviality. We also show that when σ is greater than or equal to the critical exponent $(m+2)/(m-2)$, then, by performing an appropriate change of the metric and of the solution, the nonexistence results can be improved to allow even some controlled negativity of the coefficient $b(x)$.

Chapter 4 is devoted to establishing *a priori* upper and lower estimates for the asymptotic behavior of solutions of the differential inequalities

$$\Delta u + a(x)u - b(x)u^\sigma \geq 0, \text{ resp. } \Delta u + a(x)u - b(x)u^\sigma \leq 0,$$

under assumptions on $a(x)$ and $b(x)$ related to an assumed radial lower bound for the Ricci curvature. As briefly mentioned above, the results are obtained by applying Alhfors's old idea, namely, one considers an auxiliary function defined in terms of the solution u which by construction attains an extremum, and applies the usual maximum principle. Clearly, the heart of the method consists in finding the best auxiliary function for the problem at hand. We exhibit examples showing that our estimates are essentially sharp. Some further estimates, which cannot be obtained with the previous method, are provided by direct comparison with the aid of the maximum principle (see section 4.3). The chapter ends with some nonexistence results for the Yamabe problem, which complement those described in Chapter 3.

In Chapter 5 we discuss some uniqueness results for positive solutions of Yamabe-type equations (2). The first, the very general Theorem 5.1, states that if the coefficient $b(x)$ is nonnegative and not identically zero, then two solutions whose difference is L^2-integrable are necessarily the same. Although very general, it is sharp, and, remarkably, the assumption on the L^2-integrability cannot be replaced by an L^p condition with $p > 2$. The result is obtained by means of a clever elementary integral inequality. The second result, Theorem 5.2, follows by a comparison argument which relies on a version of the weak maximum principle (see Theorem 5.3) which is interesting in its own. While in most of the results available in the literature, uniqueness is obtained by requiring that solutions have a rather precisely determined asymptotic behavior, our result applies to solutions whose behavior at infinity is specified in a much less stringent manner, see (5.13); moreover, the conclusion is reached assuming only conditions on the volume growth of the manifold. Counterexamples show the sharpness of each result. The chapter ends with a geometric application to the group of conformal diffeomorphisms of a complete manifold and to the uniqueness of solutions of the geometric Yamabe problem.

Chapter 6 deals with existence results for Yamabe-type equations (2) on the complete, noncompact, Riemannian manifold M. The main tool is the *monotone iteration scheme* in various forms, and we give a rather detailed description of it in the appendix at the end of the chapter. The application of the scheme in this context goes back to W. M. Ni, [Ni82], in the Euclidean setting and to P. Aviles and R. C. McOwen, [AM85] and [AM88], for noncompact manifolds. After having introduced some preliminary material on spectral theory, and a useful comparison result, the main body of the chapter is then devoted to the construction of (global and local) super- and subsolutions for the problem. In general terms, supersolutions are obtained under assumptions on the sign of $b(x)$ and of the first eigenvalue of $L = \Delta + a(x)$ on appropriate domains. Because of the combination of signs of the coefficients, subsolutions are harder to find. We give a number of sufficient conditions which ensure that such subsolutions do exist: among them the spectral condition $\lambda_1^L(M) < 0$, for which we provide a new sufficient condition contained in Theorem 6.11. Furthermore, we mention Theorem 6.15, in which existence is guaranteed under a very week growth condition on $b(x)$, and also Theorem 6.16, where a further weakining on the condition on the sign of $b(x)$ is balanced by the necessity of imposing a constant negative lower bound on the Ricci curvature. We explicitly note that the assumptions of our existence theorems match those of the nonexistence results in the previous chapters.

In the last chapter, Chapter 7, we consider some particular cases where the symmetry of the geometry allows one to use special techniques and to obtain stronger results. Typically this happens in Euclidean and Hyperbolic spaces, and more generally in the case of *models* (in the sense of R. Greene and H. Wu, [GW79]), or manifolds with special symmetry. The specific feature of models which make the analysis more precise is that the Laplacian of the distance function from the origin is given explicitly, as opposed to the case of a general manifold where only upper and lower bounds may be obtained under suitable curvature assumptions, by means of the the Laplacian and Hessian comparison theorems, and where the possible presence of the cut locus raises additional difficulties.

We describe refined techniques adapted to the situation at hand and obtain results that, as a by-product, show the degree of sharpness of the general theory and methods we have developed dealing with generic complete Riemannian manifolds. It seems worth remarking that, in the specific case of Hyperbolic space, we provide a nonexistence result with the aid of a Rellich-Pohozaev type formula (see Theorem 7.7) and, even more, in Proposition 7.9 we introduce an integral obstruction to the existence of a conformal deformation which is of a different nature with respect to the Kazdan-Warner condition.

Many of the results presented in this monograph have been obtained over the years by the authors jointly with many collaborators. To all of them we wish to extend our thanks and appreciation. In particular we are indebted to S. Pigola and M. Rimoldi who provided us with the proof of Theorem 2.10 in Chapter 2.