Preface to the First Edition

This book introduces the reader to a broad collection of integration theories, focusing on the Riemann, Lebesgue, Henstock-Kurzweil and McShane integrals. By studying classical problems in integration theory (such as convergence theorems and integration of derivatives), we will follow a historical development to show how new theories of integration were developed to solve problems that earlier integration theories could not handle. Several of the integrals receive detailed developments; others are given a less complete discussion in the book, while problems and references directing the reader to future study are included.

The chapters of this book are written so that they may be read independently, except for the sections which compare the various integrals. This means that individual chapters of the book could be used to cover topics in integration theory in introductory real analysis courses. There should be sufficient exercises in each chapter to serve as a text.

We begin the book with the problem of defining and computing the area of a region in the plane including the computation of the area of the region interior to a circle. This leads to a discussion of the approximating sums that will be used throughout the book.

The real content of the book begins with a chapter on the Riemann integral. We give the definition of the Riemann integral and develop its basic properties, including linearity, positivity and the Cauchy criterion. After presenting Darboux’s definition of the integral and proving necessary and sufficient conditions for Darboux integrability, we show the equivalence of the Riemann and Darboux definitions. We then discuss lattice properties and the Fundamental Theorem of Calculus. We present necessary and sufficient conditions for Riemann integrability in terms of sets with Lebesgue measure 0. We conclude the chapter with a discussion of improper integrals.
We motivate the development of the Lebesgue and Henstock-Kurzweil integrals in the next two chapters by pointing out deficiencies in the Riemann integral, which these integrals address. Convergence theorems are used to motivate the Lebesgue integral and the Fundamental Theorem of Calculus to motivate the Henstock-Kurzweil integral.

We begin the discussion of the Lebesgue integral by establishing the standard convergence theorem for the Riemann integral concerning uniformly convergent sequences. We then give an example that points out the failure of the Bounded Convergence Theorem for the Riemann integral, and use this to motivate Lebesgue’s descriptive definition of the Lebesgue integral. We show how Lebesgue’s descriptive definition leads in a natural way to the definitions of Lebesgue measure and the Lebesgue integral. Following a discussion of Lebesgue measurable functions and the Lebesgue integral, we develop the basic properties of the Lebesgue integral, including convergence theorems (Bounded, Monotone, and Dominated). Next, we compare the Riemann and Lebesgue integrals. We extend the Lebesgue integral to $n$-dimensional Euclidean space, give a characterization of the Lebesgue integral due to Mikusinski, and use the characterization to prove Fubini’s Theorem on the equality of multiple and iterated integrals. A discussion of the space of integrable functions concludes with the Riesz-Fischer Theorem.

In the following chapter, we discuss versions of the Fundamental Theorem of Calculus for both the Riemann and Lebesgue integrals and give examples showing that the most general form of the Fundamental Theorem of Calculus does not hold for either integral. We then use the Fundamental Theorem to motivate the definition of the Henstock-Kurzweil integral, also known as the gauge integral and the generalized Riemann integral. We develop basic properties of the Henstock-Kurzweil integral, the Fundamental Theorem of Calculus in full generality, and the Monotone and Dominated Convergence Theorems. We show that there are no improper integrals in the Henstock-Kurzweil theory. After comparing the Henstock-Kurzweil integral with the Lebesgue integral, we conclude the chapter with a discussion of the space of Henstock-Kurzweil integrable functions and Henstock-Kurzweil integrals in $\mathbb{R}^n$.

Finally, we discuss the “gauge-type” integral of McShane, obtained by slightly varying the definition of the Henstock-Kurzweil integral. We establish the basic properties of the McShane integral and discuss absolute integrability. We then show that the McShane integral is equivalent to the Lebesgue integral and that a function is McShane integrable if and only if
it is absolutely Henstock-Kurzweil integrable. Consequently, the McShane integral could be used to give a presentation of the Lebesgue integral which does not require the development of measure theory.
Preface to the Second Edition

The second edition of this text contains several additions and changes in Chapters 3, 4 and 5. In Chapter 3 on the Lebesgue integral, we have added material about the convolution product and spaces of Lebesgue integrable functions. As an application of the Fubini-Tonelli Theorems, the convolution product of two integrable functions is defined in Section 3.8.1. Approximate identities are defined and used with the convolution product to establish several approximation results, including the Weierstrass Approximation Theorem. In Section 3.9 on the space $L^1(E)$ of Lebesgue integrable functions, we have added examples of dense subsets of $L^1(E)$ and given applications including a proof of the Riemann-Lebesgue Lemma.

We have included a change of variables theorem for the Lebesgue integral as a consequence of the Riesz-Fischer Theorem on the completeness of $L^1(E)$.

The notion of uniform integrability is introduced in Chapters 4 and 5 on the Henstock-Kurzweil and McShane integrals. New proofs of the major convergence theorems, the Monotone and Dominated Convergence Theorems, are given for both integrals based on the notion of uniform integrability. A new integration-by-parts result for the Henstock-Kurzweil integral is added and used to establish a version of the Riemann-Lebesgue Lemma for the Henstock-Kurzweil integral.

More exercises have been added.