Introduction

Part 1

This is a book for people who really like probability problems. There are, I think, a lot of people who fall into that category. Indeed, the editors of *Parade*, a magazine insert in millions of Sunday newspapers across America, thought a probabilistic question intriguing enough to put it on the cover of their issue of August 10, 1997. For the real connoisseur of probability, however, it was actually a pretty tame problem: "Your dog has a litter of four. Is it most likely that two are males and two are females?"

That question was posed in the "Ask Marilyn" column by the famously intelligent Marilyn vos Savant, who answered, "Nope! The most likely split is three males and one female, or three females and one male." That is correct, too, for the case of female/male births being equally likely. Two males and two females has a probability of $\frac{3}{8}$, while the second case has probability of $\frac{1}{2}$. Vos Savant, who has carved a successful writing career partly out of posing old math questions (with answers that have been known for centuries) to readers who find them new, doesn't give her fans the math behind the answer, but for this book, the doggy problem is just too elementary to be included as a legitimate "probability puzzler."

Please don't misunderstand me. While I occasionally think Vos Savant is just a bit too unrevealing of her debt to ancient mathematical lore (I suspect that many of her readers think she is the originator of the problems in her column), I do think she does provide a useful service by publishing such problems. Who could deny that it is a refreshing change to see math of any sort in a newspaper column, as compared to the more typical, seemingly endless rehashing of the supposed details of celebrity lives, or other similar sophomoric speculations? Vos Savant's column is written in the spirit of Laplace's famous dictum, "The theory of probabilities is at bottom nothing but common sense reduced to calculus," but, while clever, Laplace did overstate his argument just a bit. Once beyond the doggy type of question, probability theory can quickly become nonintuitive in the extreme, even for experienced analysts. That is, perhaps surprisingly, one of its most seductive and charming features.

It isn't hard to understand vos Savant's reluctance to put real math in her column, of course, as the quote from Agatha Christie at the beginning of this book accurately reflects how most math-innocents view technical analyses of any sort. As W. Somerset Maugham wrote in the first paragraph of his short story "Mr. Harrington's Washing," "Man has always found it easier to sacrifice his life than to learn the multiplication table." But if you have read this far, then you certainly don't fall into that category. The doggy problem is so simple (see Part II of this introduction) that one could literally write down all sixteen possibilities (sixteen, because there are four consecutive births, each with two possible outcomes, and $2 \times 2 \times 2 \times 2 = 16$) and then just count how many times each of the different situations occurs. In the jargon of mathematics, "we have several different events defined on a finite sample space, with each sample point the result of a Bernoulli sequence of four trials, with the probability of a success being one-half."

Now that is a mouthful. And I am not going to define any of those terms; if you know what that last sentence means, then you just passed the test qualifying you to get the most out of this book. If you don't know what that sentence said, then this book may be just a bit too much for you, at least for now. But that doesn't mean you shouldn't buy it. Do buy it and use it as a study supplement as you take an elementary course in probability.

As I declared before, the doggy problem is really just a routine drill problem, the sort of question that textbook authors put a dozen or so of at the end of each section of their books. Such problems are important to do as learning exercises, and every beginning student should do a number of them when first learning any new math topic. There are lots of such problems in all of the mostly excellent probability textbooks available today; so many, in fact, that Vos Savant will never run out of recyclable drill problems with which to dazzle her readers.

Once beyond the drill problem stage, however, most probability students are eager to try their new and powerful skills on more challenging, more interesting problems. That is the sort of problem you'll find in this book. And where did these problems come from, you may wonder. During the past twenty-five years, I have taught (and continue to teach) probability theory to undergraduate electrical engineering students at the University of New Hampshire. (My debt to hundreds of students who have patiently listened to me talk and scribble on the blackboard in EE647 is a very large one, indeed.) At the end of each term there are always pleas to provide some sort of extra credit work with which to bolster grades, and I have responded by offering what I call "Challenge Problems." These are optional problems (students have to accept the challenge before seeing the problem, and after seeing it, they can't change their minds) to be done as "take-homes" during the week before the final exam (independent work only), with unlimited time, and no partial credit. If a student gets the problem right, then I add five points to his or her final exam score. But if he or she gets it wrong, then I subtract five points.

Over the last twenty-five years, I have created perhaps a hundred or so such questions, and the ones I think are the best are included here. None, to my knowledge, has ever appeared in print before, at least not in the way posed here. The level of these problems is elementary, but that simply means that they all can be done with no mathematics beyond freshman calculus (and at least one of them can be solved with just arithmetic). Each problem has a detailed solution and extended discussion (often including computer illustrations using the powerful scientific application software called MATLAB) in the second half of the book. The problems are much like the famous Birthday Problem or the Buffon Needle Problem, neither of which is included here because they have become so easy to find in textbooks. The problems here are actually no more conceptually difficult than are those two classics, however, and I hope you have fun trying your luck on them.

Part II.

Binary Numbers, the Doggy Problem, the Gulf War, and Shooting at Targets

The doggy problem can be solved by simply counting. If we let 1 denote a female birth and 0 denote a male birth, then the sixteen possibilities for a litter of four dogs are represented by the sixteen four-digit binary numbers from 0 to 15:

0 = 0000	8 = 1000
1 = 0001	9 = 1001
2 = 0010	10 = 1010
3 = 0011	11 = 1011
4 = 0100	12 = 1100
5 = 0101	13 = 1101
6 = 0110	14 = 1110
7 = 0111	15 = 1111.

As a mathematician would put it, "Each number is a sample point in the sample space of the experiment of having a litter of four puppies." If a male and female are equally likely to be born, then each sample point is equally likely, with a probability of $\frac{1}{16}$. There are six sample points with two 1's and two 0's (the numbers 3, 5, 6, 9, 10, and 12), and there are eight sample points with either three 1's and one 0 or one 1 and three 0's (the numbers 7, 11, 13, and 14, and the numbers 1, 2, 4, and 8, respectively). Thus, the probability of two males and two females is $\frac{6}{16} = \frac{3}{8}$, and the probability of either three males and one female or one male and three females is $\frac{8}{16} = \frac{1}{2}$.

It was surprising to me that as elementary as this counting technique is, it at first appeared as if Marilyn vos Savant had been unaware of it. I say this because in her *Parade* column of June 14, 1998, she printed a letter asking her a question (the details of which are not important here); the correspondent ended by informing vos Savant that he was also asking "my other two heroes—Stephen Hawking and Kurt Vonnegut—[the same question]. I figure that my chances are 7 out of 8 that I will get at least one response from the three of you." To that, vos Savant replied, "I have a question for you: How in the world did you figure those chances?!"

The answer to Marilyn's question is an even simpler binary counting exercise than is the doggy problem. Just let 1 denote a response and 0 denote no response, and we see immediately that there are eight possibilities; the binary numbers 000 (no responses at all) to 111 (three responses). Seven of those eight binary numbers have at least one 1, so if the correspondent was assuming each of the eight sample points is equally likely (the probability of each response is $\frac{1}{2}$ and that the responses are independent of each other), we then have the stated chances by inspection. If the probability of a response is not $\frac{1}{2}$, then the chances are different but no more difficult to calculate. Suppose p is the probability of a response from a hero; thus, $1-p$ is the probability of no response. Then the probability of no responses at all is $(1-p)^3$ and thus the probability of at least one response is $1-(1-p)^3$. For example, if $p=\frac{1}{6}$, then the chances for at least one response are 91 out of 216.

My original impression that vos Savant was unaware of all this turned out to be wrong, however, because in her September 6, 1998, column in *Parade*, she addressed the problem again by printing a letter from a reader asking just how the original correspondent calculated his odds. Marilyn's answer made it clear that she hadn't really meant to imply that she didn't know how to count in binary, as she correctly listed all eight possible combinations of reply possibilities from herself, Hawking, and Vonnegut. But then she stumbled again, with the following words about the original correspondent:

> Then he incorrectly figured that these eight possibilities were equally likely. This is a lot like saying

> there are two possibilities regarding sunrise tomorrow: (1) The sun will rise in the morning; or (2) the sun will not rise in the morning. And, therefore, the chances are only fifty-fifty that the sun will rise!

Marilyn's first sentence is simply not a valid objection, and the rest of that passage is just irrelevant. The original correspondent certainly did not "figure incorrectly" by assuming each reply had a probability of $\frac{1}{2}$. The probability of each reply could be anything from 0 to 1, and he was well within his rights to make the special (if perhaps overly optimistic) assumption of $\frac{1}{2}$. Vos Savant's remarks concerning the sun are beside the point: The probabilities of the individual replies are due to individual human decisions, while the event of the sun rising tomorrow is the result of the physical laws of gravity and orbital mechanics. Probability has nothing to do with it. But, curiously, her particular imagery is reminiscent of Laplace's famously incorrect use of Bayes's theorem of conditional probability to compute the odds of the sun's rising tomorrow. Could this be the case of yet another classic puzzler that she once read about, and of which she has since forgotten the proper historical setting? (Laplace presented this calculation in his famous 1814 "Essai Philosophique des Probabilités.")

The counting method works well for problems that involve a small number of different possibilities, but in general we need a more powerful approach. The sophisticated math behind the doggy problem is simply the binomial theorem applied to a Bernoulli sequence of trials (which is a sequence with two characteristics: The trials are independent, and each trial has precisely two possible outcomes). If there are n births, with the probability of a female birth being p (and so the probability of a male birth is $1-p$), then the probability of k females and $n-k$ males is given by

$$\binom{n}{k} p^k (1-p)^{n-k}$$

where the binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ denotes the number of different ways of selecting k things from n things. The factorial function is defined for positive integers as $n! = n(n-1)(n-2) \ldots (2)(1)$. Notice, too, that since $\binom{n}{n} = 1$, i.e., there is just one way to select all n things, then the binomial coefficient formula reduces to the special and important result that $0! = 1$, *not* the zero that beginning students so often write.

For the doggy problem we have $n = 4$ and $p = \frac{1}{2}$. So, the probability of two females ($k = 2$) and two males is

$$\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{4!}{2!\,2!} \cdot \frac{1}{2^4} = \frac{6}{16} = \frac{3}{8},$$

and the probability of either one female ($k = 1$) and three males, or three females ($k = 3$) and one male, is

$$\binom{4}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{4!}{1!3!} \cdot \frac{1}{16} + \frac{4!}{3!1!} \cdot \frac{1}{16} = \frac{4}{16} + \frac{4}{16} = \frac{1}{2}.$$

A far more interesting application of this simple math was reported in the *Boston Globe* on January 24, 1992 (p. 3), as part of the retrospective analyses then being conducted on the Gulf War. In particular, the Pentagon had gone on record with a claim that the Patriot antiaircraft missile system had "successfully engaged over 80 percent" of the Scud missiles Iraq had launched at Saudi Arabia. An MIT physicist, Theodore Postol, disputed that claim; it was a remarkable claim, too, as the Patriot was designed to counter relatively slow manned aircraft, not supersonic ballistic missiles.

Postol based his skepticism on what he saw after watching videotapes of fourteen Patriot-Scud engagements. There were thirteen misses and one probable hit. The *Globe* article ended with this quote from Professor Postol: "What are the odds I would see 13 misses and one hit if the Patriot was successfully shooting down 80 percent of the Scuds?"

No answer was given in the newspaper, but we can easily calculate it for ourselves using the doggy-problem math. Simply think of a hit as having the claimed probability of 0.8 (thus, a miss has a probability of 0.2), and if we assume independent engagements, we thus have

$$\binom{14}{1}(0.8)\,(0.2)^{13}$$

as the probability that Professor Postol would see what he saw. The numbers work out to give a probability of less than 10^{-8}, a value so small that most people would reject the Pentagon's claim of $p = 0.8$. Flipping a fair coin and getting twenty-six consecutive heads is more likely. That is, if the Pentagon's claim had merit, then an event of incredibly low probability had been observed to actually occur: an event with odds of more than a hundred million to one against it. Even most megabuck lotteries have better odds than that. It is, in fact, far more likely that the Pentagon claim did not have merit.

(An aside: I have come to call such rare occurrences *Daphne events*, after Daphne Tams, the contracts and copyright manager for Princeton University Press, for whom it once snowed on seventeen (!) consecutive birthdays. This would be most incredible, of course, if her birthday were in the middle of summer, but even for a winter birthday, it seems to be an a priori rare event. For more on Daphne's snowy birthdays, see Problem 5.)

I have used Postol's problem as lecture material on Bernoulli trials simply because it is "interesting" (and I'll leave that word undefined and just say students seem to find the military context alone fascinating). Calculating the answer to Postol's question involves no technical difficulties, and it can be done on the most elementary of scientific hand calculators; I am using the one I keep in my office desk drawer for emergencies, one I bought for fifteen dollars a couple of years ago in a discount drugstore. However, another aspect of Bernoulli problems that I emphasize in class is that they can easily evolve into questions that, while easy to set up, can present calculation challenges. Don't forget that the goal for engineers, scientists, and applied mathematicians is the calculation of an answer. If you can't do the final calculation you've failed, no matter how pretty your preliminary analysis may be.

As my final example, then, consider the following problem. You are told that a markswoman can hit a target with each shot from her rifle with a fixed probability p that is, before she starts shooting, equally likely to be $\frac{1}{2}$ or $\frac{2}{3}$. (Perhaps there are two rifles available to her, one with $p=\frac{1}{2}$ and the other with $p=\frac{2}{3}$, and the actual rifle used is determined by the flip of a fair coin.) To determine the value of p you conduct an experiment; you ask her to shoot at the target 300 times. If $p=\frac{1}{2}$, then you expect about $300\times\frac{1}{2}=150$ hits, and if $p=\frac{2}{3}$, then you expect about $300\times\frac{2}{3}=200$ hits. When she is done shooting, however, you see she has hit the target 175 times, exactly midway between 150 and 200 hits. So what is your decision?

This is a very simple example of what mathematicians call *hypothesis testing*; that is, you have to decide *after* the experiment which hypothesis ($p=\frac{1}{2}$ or $p=\frac{2}{3}$), that is equally likely *before* the experiment, to accept. You can use doggy math to do this by applying what is called *maximum likelihood*, which is a fancy way of saying, "Let's first calculate the probabilities of 175 hits out of 300 shots for $p = \frac{1}{2}$ and then again for $p = \frac{2}{3}$, and then decide in favor of the value that gives the larger probability." So, all you have to do is calculate the two numbers

$$P\left(\frac{1}{2}\right) = \binom{300}{175}\left(\frac{1}{2}\right)^{175}\left(\frac{1}{2}\right)^{125} \text{ and } P\left(\frac{2}{3}\right) = \binom{300}{175}\left(\frac{2}{3}\right)^{175}\left(\frac{1}{3}\right)^{125}$$

and see which is larger. (In fact, the previous example is of the same nature; we calculated the probability of the hypothesis $p = 0.8$ and found it to be too small to be credible. So we accepted what mathematicians call the *null* hypothesis: $p \neq 0.8$.)

I like to assign this as a homework problem to see how students handle the binomial coefficients. The factorials are much too large for direct calculation, and some students are stumped. Others, however, see the trick for simply avoiding the factorials by calculating the *ratio* of $P\left(\frac{2}{3}\right)$ and $P\left(\frac{1}{2}\right)$. Then the binomial coefficients cancel to give

$$\frac{P\left(\frac{2}{3}\right)}{P\left(\frac{1}{2}\right)} = \frac{\left(\frac{2}{3}\right)^{175}\left(\frac{1}{3}\right)^{125}}{\left(\frac{1}{2}\right)^{175}\left(\frac{1}{2}\right)^{125}} = \frac{2^{175}/3^{300}}{1/2^{300}} = 2^{175}\left(\frac{2}{3}\right)^{300},$$

a calculation even my cheap drugstore machine can handle. It gives $P\left(\frac{2}{3}\right) = 0.71264\ P\left(\frac{1}{2}\right)$. This tells us that $P\left(\frac{1}{2}\right)$ is significantly larger than $P\left(\frac{2}{3}\right)$ and, indeed, direct calculations give

$$P\left(\frac{1}{2}\right) = 7.1 \times 10^{-4} \text{ and } P\left(\frac{2}{3}\right) = 5.06 \times 10^{-4}.$$

Thus, $p = \frac{1}{2}$ is the maximum likelihood choice, although with some probability it is the *wrong* choice.

These last two examples are far more interesting applications of doggy problem math than is Vos Savant's but, still, they are really just drill problems too. I think you'll find the problems that follow to be a distinct step up in their challenge. I hope you find them both instructive and fun. At least I did, and that's because I subscribe to words attributed to Beresford Parlett, an applied mathematician at the University of California at Berkeley: "Only wimps do the general case. True teachers tackle examples."

In other words, don't let abstraction, no matter how beautiful, blind you to the hard realities of the practical world. The quantum physicist Werner Heisenberg (1901–1976), winner of the 1934 Nobel prize, said this even more bluntly when, on being told that a famous mathematician had declared that "space is simply the field of linear operators," replied, "Nonsense, space is blue and birds fly through it." Keep that message in mind as you read the problems in this book.