

Introduction

Mumford-Tate groups are the fundamental symmetry groups in Hodge theory. They were introduced in the papers [M1] and [M2] by Mumford. As stated there the purpose was to interpret and extend results of Shimura and Kuga ([Sh1], [Sh2], and [Ku]). Since then they have played an important role in Hodge theory, both in the formal development of the subject and in the use of Hodge theory to address algebro-geometric questions, especially those that are arithmetically motivated. The informative sets of notes by Moonen [Mo1] and [Mo2] and the recent treatment in [PS] are two general accounts of the subject.

We think it is probably fair to say that much, if not most, of the use of Mumford-Tate groups has been in the study of abelian varieties or, what is essentially the same, polarized Hodge structures of level one¹ and those constructed from this case. The papers [De1], [De2], and [De3] formulated the definitions and basic properties of Mumford-Tate groups in what is now the standard way, a formulation that provides a setting in which Mumford-Tate groups were particularly suited for the study of Shimura varieties, which play a central role in arithmetic geometry. Noteworthy is the use of Mumford-Tate groups and Shimura varieties in Deligne's proof [DMOS] that Hodge classes are absolute in the case of abelian varieties, and their role in formulating conjectures concerning motivic Galois groups (cf. [Se]). See [Mi2] for a useful and comprehensive account and [R] for a recent treatment of Shimura varieties, and [Ke] for a Hodge-theoretic approach.

As will be explained, the perspective in this monograph is in several ways complementary to that in the literature. Before discussing these, we begin by noting that Chapter I is an introductory one in which we give the basic definitions and properties of Mumford-Tate groups in both the case of Hodge structures and of mixed Hodge structures. Section II.A is also introductory where we review the definitions of period domains and their compact duals as well as the canonical exterior differential system on them.

¹Level one means a Tate twist of a polarized Hodge structure of *effective weight one*. In general for $n \geq 1$, a Hodge structure has *effective weight n* if the non-zero Hodge (p, q) components with $p + q = n$ have $p \geq 0, q \geq 0$. When no confusion seems likely, we shall omit the term *effective*.

As will be shown, Mumford-Tate groups M will be reductive algebraic groups over \mathbb{Q} such that the derived or adjoint subgroup of the associated real Lie group $M_{\mathbb{R}}$ contains a compact maximal torus. In order to keep the statements of the results as simple as possible, we will emphasize the case when $M_{\mathbb{R}}$ itself is semi-simple. The extension to the reductive case will be usually left to the reader.

Before turning to a discussion of the remaining contents in this monograph, we first note that throughout we shall use the notation V for a \mathbb{Q} -vector space and $Q : V \otimes V \rightarrow \mathbb{Q}$ for a non-degenerate form satisfying $Q(u, v) = (-1)^n Q(v, u)$ where n is the weight of the Hodge structure under consideration. In many cases there will be given a lattice $V_{\mathbb{Z}}$ with $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$.

One way in which our treatment is complementary is that we have used throughout the interpretation of Mumford-Tate groups in the setting of period domains D and their compact duals \check{D} . The latter are rational, homogeneous varieties defined over \mathbb{Q} . Variations of Hodge structure are integral manifolds of a canonical exterior differential system (EDS), defined on all of \check{D} , and also on quotients of D by discrete subgroups. This leads to a natural extension of the definition of the Mumford-Tate group M_{φ} of a Hodge structure $\varphi \in D$ to the Mumford-Tate group $M_{F^{\bullet}}$ associated the Hodge filtration given by a point $F^{\bullet} \in \check{D}$, and to the Mumford-Tate group $M_{(F^{\bullet}, E)}$ of an integral element $E \subset T_{F^{\bullet}} \check{D}$ of the EDS. Both of these extensions will be seen to have important geometric and arithmetic implications.

A second complementary perspective involves the emphasis throughout on *Mumford-Tate domains* $D_{M_{\varphi}}$ (cf. Section (II.B)), defined as the orbit of the point $\varphi \in D$ by the group $M_{\varphi}(\mathbb{R})$ of real points of the Mumford-Tate group M_{φ} .² One subtlety, discussed in Section IV.G, is that the Mumford-Tate domain depends on its particular representation as a homogeneous complex manifold. The same underlying complex manifold may appear in multiple, and quite different, ways as a Mumford-Tate domain.

For later reference we note that Mumford-Tate domains will have compact duals, which are rational, homogeneous varieties that as homogeneous varieties are defined over a number field.

We shall denote by $M_{\varphi}(\mathbb{R})^0$ the identity component of $M_{\varphi}(\mathbb{R})$ in the classical topology and by $D_{M_{\varphi}}^0$ the component of $D_{M_{\varphi}}$ through φ . To a point $\varphi \in D$, i.e., a polarized Hodge structure V_{φ} on V , is associated the algebra of *Hodge tensors* $\text{Hg}_{\varphi}^{\bullet, \bullet} \subset T^{\bullet, \bullet} := \bigoplus_{k, l} V^{\otimes k} \otimes \check{V}^{\otimes l}$.

²Later in this monograph when discussing the geometry of Mumford-Tate domains, when confusion seems unlikely, we shall not distinguish between the \mathbb{R} -algebraic group $M_{\mathbb{R}}$ and its real points $M(\mathbb{R})$ and the corresponding real Lie group $M_{\mathbb{R}}$.

For reasons discussed below, it is our opinion that the classical Noether-Lefschetz loci (cf. Section (II.C)), defined traditionally by the condition on $\varphi \in D$ that a vector $\zeta \in V$ be a Hodge class, should be replaced by the *Noether-Lefschetz locus* NL_φ associated to $\varphi \in D$, where by definition

$$NL_\varphi = \left\{ \psi \in D : Hg_{\varphi}^{\bullet, \bullet} \subseteq Hg_{\psi}^{\bullet, \bullet} \right\}^3$$

We will then prove the

(II.C.1) THEOREM: *The component $D_{M_\varphi}^0$ of the Mumford-Tate domain D_{M_φ} is the component of NL_φ through $\varphi \in D$.*

An application of this result is the estimate given in theorem (III.C.5) for the codimension of the Noether-Lefschetz locus, in the extended form suggested above, in the parameter space of a variation of Hodge structure. This estimate seems to be unlike anything appearing classically; it illustrates both the role of Mumford-Tate groups and, especially, the integrability condition in the EDS in “dimension counts.”

For a simple first illustration of this, since \check{D} is a projective variety defined over \mathbb{Q} we may speak of a \mathbb{Q} -generic point $F^\bullet \in \check{D}$, meaning that the \mathbb{Q} -Zariski closure of F^\bullet is \check{D} . In the literature there are various criteria, some of them involving *genericity* of one kind or another, that imply that M_φ is equal to the \mathbb{Q} -algebraic group $G = \text{Aut}(V, Q)$. We show that, except when the weight $n = 2p$ is even and the only non-zero Hodge number is $h^{p,p}$, if F^\bullet is a \mathbb{Q} -generic point of \check{D} , then the Mumford-Tate group M_{F^\bullet} is equal to G . A converse will also be discussed. These issues will also be addressed in a more general context in Section VI.A (cf. (VI.A.5)).

A remark on terminology: For Hodge structures of weight one, what we are here calling Mumford-Tate domains have been introduced in [M2] and used in [De2], [De3]. For reasons to be explained in Section II.B, we shall define *Shimura domains* to be the special case of Mumford-Tate domains where M_φ can be described as the group fixing a set of Hodge tensors in degrees one and two.⁴ There are then strict inclusions of sets

$$\left(\begin{array}{c} \text{period} \\ \text{domains} \end{array} \right) \subset \left(\begin{array}{c} \text{Shimura} \\ \text{domains} \end{array} \right) \subset \left(\begin{array}{c} \text{Mumford-Tate} \\ \text{domains} \end{array} \right).$$

³In the classical weight $n = 1$ this point of view is taken in the original papers [M1] and [M2] on the subject, as well as in [De2].

⁴The degree of $t \in V^{\otimes k} \otimes \check{V}^{\otimes l}$ is $k + l$.

We remark that a Shimura domain and a Mumford-Tate domain may be considered as period domains with additional structures. When that additional data is trivial we have the traditional notion of a period domain.⁵

Another result relating Mumford-Tate groups and period domains, the structure theorem stated below, largely follows from results in the literature (cf. [Schm1], [A1]) and the use of Mumford-Tate domains. To state it, we consider a global variation of Hodge structure (cf. Section (III.A))

$$\Phi : S \rightarrow \Gamma \backslash D$$

where S is smooth and quasi-projective. We assume that the \mathbb{Q} -vector space V has an integral structure $V_{\mathbb{Z}}$ and for $G_{\mathbb{Z}} = G \cap \text{Aut}(V_{\mathbb{Z}})$ we denote by $\Gamma \subset G_{\mathbb{Z}}$ the monodromy group. As explained below, we consider Φ up to finite data, which in effect means that we consider Φ up to isogeny, meaning that we can replace S by a finite covering and take the induced variation of Hodge structure. We also denote by M_{Φ} the Mumford-Tate group associated to the variation of Hodge structure. It is also a reductive \mathbb{Q} -algebraic group, and we denote by

$$M_{\Phi} = M_1 \times \cdots \times M_{\ell} \times A$$

the almost product decomposition of M_{Φ} into its \mathbb{Q} -simple factors M_i and abelian part A . We also denote by $D_i \subset D$ the $M_i(\mathbb{R})$ -orbit of a lift to D of the image $\Phi(\eta)$ of a very general point $\eta \in S$. Thus D_i is a Mumford-Tate domain for M_i . Then we have the

(III.A.1) THEOREM: (i) *The D_i are homogeneous complex submanifolds of D .* (ii) *Up to finite data, the monodromy group splits as an almost direct product $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$, $k \leq l$, where for $1 \leq i \leq k$ the \mathbb{Q} -Zariski closure $\overline{\Gamma_i}^{\mathbb{Q}} = M_i$.* (iii) *Up to finite data, the global variation of Hodge structure is given by*

$$\Phi : S \rightarrow \Gamma_1 \backslash D_1 \times \cdots \times \Gamma_k \backslash D_k \times D'$$

where $D' = D_{k+1} \times \cdots \times D_{\ell}$ is the part where the monodromy is trivial.

A consequence of the proof will be that

The tensor invariants of Γ coincide with those of the arithmetic group $M_{1,\mathbb{Z}} \times \cdots \times M_{k,\mathbb{Z}}$ where $M_{i,\mathbb{Z}} = M_i \cap G_{\mathbb{Z}}$.

⁵The motivation for the terminology is that for us this case needs to be distinguished from the general case, when the algebra of Hodge tensors is not generated in degrees one and two. In weight one it is the case originally introduced by Shimura in the 1960's. The somewhat subtle distinctions in terminology will be explained when we discuss what is meant in this work by the "classical and non-classical" cases.

It is known (cf. [De-M1], [De5]) that Γ need not be an arithmetic group,⁶ i.e., a group commensurable with $M_{1,\mathbb{Z}} \times \cdots \times M_{k,\mathbb{Z}}$. However, from the point of view of its tensorial invariants it is indistinguishable from that group.

Because of this result and the arithmetic discussion in Chapters V–VIII it is our feeling that *Mumford-Tate domains are natural objects for the study of global variations of Hodge structure*. In particular, the Cattani-Kaplan-Schmid study of limiting mixed Hodge structures in several variables [CKS] and the recent Kato-Usui construction [KU] of extensions, or partial compactifications, of the moduli space of equivalence classes of polarized Hodge structures might be carried out in the context of Mumford-Tate domains. A previously noted subtlety here is that as a complex homogeneous manifold, the same complex manifold D may have several representations $D = G_{\mathbb{R},i}/H_i$ as a homogeneous space. It is reasonable that the extension of the Kato-Usui theory will depend on the particular \mathbb{Q} -algebraic group G_i . For this reason, as well as for material needed later in this monograph, in Section I.C we give a brief introduction to the Mumford-Tate groups associated to mixed Hodge structures.

Classically there is considerable literature (cf. [Mo1] and [Mo2]) on the question: What are the possible Mumford-Tate groups of polarized Hodge structures whose corresponding period domain is Hermitian symmetric?⁷ In those works the question, “What are the possible Mumford-Tate groups?” is also posed.

In Chapter IV for general polarized Hodge structures we discuss and provide some answers to the questions:

- (i) *Which semi-simple \mathbb{Q} -algebraic groups M can be Mumford-Tate groups of polarized Hodge structures?*⁸

and, more importantly,

- (ii) *What can one say about the different realizations of M as a Mumford-Tate group?*
- (iii) *What is the relationship among the corresponding Mumford-Tate domains?*

To address these questions, we use a third aspect in which this study differs from previous ones in that we invert the first question. For this we use the notion

⁶So far as we know, all non-arithmetic monodromy groups are subgroups of $SU(n, 1)$. A question of which we are not aware if there is an answer to is whether Γ is arithmetic in case the real Lie group associated to $\bar{\Gamma}^{\mathbb{Q}}$ has no simple factors of real rank one. In this regard we note the paper [K14]

⁷In this regard we call attention to the papers [Z1] and [Z2] where this question is addressed.

⁸A variant of this question over \mathbb{R} is treated in [Simp1] — see footnote 12.

of a *Hodge representation* (M, ρ, φ) , which is given by a reductive \mathbb{Q} -algebraic group M , a representation

$$\rho : M \rightarrow \text{Aut}(V, Q),$$

and a circle

$$\varphi : \mathbb{U}(\mathbb{R}) \rightarrow M(\mathbb{R}),$$

where $\mathbb{U}(\mathbb{R})$ is a maximal compact subgroup of the real algebraic group $\mathbb{S} =: \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$,⁹ such that $(V, Q, \rho \circ \varphi)$ gives a polarized Hodge structure.¹⁰ For any Hodge representation, the circle $\varphi(\mathbb{U}(\mathbb{R}))$ lies in a maximal *compact* torus $T \subset M(\mathbb{R})$ whose compact centralizer H_φ is the subgroup of $M(\mathbb{R})$ preserving the polarized Hodge structure $(V, Q, \rho \circ \varphi)$.

We shall say that a representation $\rho : M \rightarrow \text{Aut}(V)$ leads to a *Hodge representation* if there is a Q and φ such that $(V, Q, \rho \circ \varphi)$ is a Hodge representation. We define a *Hodge group* to be a reductive \mathbb{Q} -algebraic group M that has a Hodge representation. In Chapter IV our primary interest will be in the case where M is semi-simple. The other extreme case when M is an algebraic torus will be discussed in Chapter V.

Given a Hodge representation (M, ρ, φ) when M is semi-simple, we observe that there is an associated Hodge representation $(M_a, \text{Ad}, \varphi)$ where the polarized Hodge structure on $(\mathfrak{m}, B, \text{Ad } \varphi)$ is induced from the inclusion $\mathfrak{m} \subset \text{End}_Q(V)$, noting that the Cartan-Killing form B is induced by Q . Here, M_a is the adjoint group, which is a finite quotient of M by its center. For the conjugate $\varphi_m = m^{-1}\varphi m$ by a generic $m \in M_a(\mathbb{R})$, it is shown that M_a is the Mumford-Tate group of $(\mathfrak{m}, B, \text{Ad } \varphi_m)$. Thus, at least up to finite coverings,¹¹ the issue is to use the standard theory of roots and weights to give criteria to have a Hodge representation, and then to apply these criteria in examples.

Because the real points $M(\mathbb{R})$ map to

$$M(\mathbb{R}) \xrightarrow{\rho} \text{Aut}(V_{\mathbb{R}}, Q),$$

⁹See Section I.A for a discussion of the Deligne torus.

¹⁰Without essential loss of generality, we assume as part of the definition that the induced representation $\rho_* : \mathfrak{m} \rightarrow \text{End}(V, Q)$ is injective.

¹¹The issue of finite coverings and the related faithfulness of Hodge representations is interesting in its own right; it will be analyzed in the text. The point is that irreducible Hodge representations of M will be parameterized by pairs (λ, φ) where λ is the highest weight of an irreducible summand of the $\mathfrak{m}_{\mathbb{C}}$ -module $V_{\mathbb{C}}$. Given λ there are conditions on the lattice of groups with Lie algebra \mathfrak{m} that λ be the highest weight of a representation of a particular M in the lattice. To be a Hodge representation will impose conditions on the pair (λ, φ) .

the issue arises early on of what the *real form* $M(\mathbb{R})$ can be.¹² The first result is that:

- (1) If (M, ρ, φ) is a Hodge representation, then as noted above $M(\mathbb{R})$ contains a compact real maximal torus T with $\dim T = \text{rank}_{\mathbb{C}} M_{\mathbb{C}}$. We will abbreviate this by saying that $M(\mathbb{R})$ contains a compact maximal torus. Furthermore,
- (2) If M is semi-simple, then the condition (1) is sufficient; indeed, the adjoint representation of M leads to a Hodge representation.

The natural starting point for an analysis of Hodge representations is the well-developed theory of complex representations of complex semi-simple Lie algebras. To pass from the theory of irreducible complex representations of a complex semi-simple Lie algebra to the theory of irreducible real representations of real semi-simple Lie groups,¹³ there are three elements that come in:

(i) The theory of real forms of complex simple Lie algebras. These were classified by Cartan, and there is now the beautiful tool of Vogan diagrams with an excellent exposition in [K].

(ii) By Schur's lemma, irreducible real representations $V_{\mathbb{R}}$ of a real Lie algebra $\mathfrak{m}_{\mathbb{R}}$ break into three cases depending on whether

¹²In [Simp1] a variant of this question is studied and solved. Namely Simpson defines the notion of a *Hodge type*, which by a result in his paper is given by the pair consisting of a real, semi-simple Lie group $M_{\mathbb{R}}$ together with a circle S^1 in $M_{\mathbb{R}}$ whose centralizer $Z_{M_{\mathbb{R}}}(S^1) = H$ is a compact subgroup of $M_{\mathbb{R}}$. For a Hodge group (M, φ) as defined in this monograph, the associated real Lie group $M(\mathbb{R})$ and image of $\varphi : S^1 \rightarrow M(\mathbb{R})$ give a Hodge type. He shows that for a Hodge group the adjoint representation gives a real polarized Hodge structure. As discussed in Section IV.F, a Hodge type gives a homogeneous complex manifold and these are exactly those that are discussed in [GS].

Simpson also proves an existence theorem ([Simp1], Theorem 3 and Corollary 46), which roughly stated implies that if X is a compact Kähler manifold and $\rho : \pi_1(X) \rightarrow M_{\mathbb{R}}$ is a homomorphism that is \mathbb{R} -Zariski dense in $M_{\mathbb{R}}$, then $M_{\mathbb{R}}$ is the real Mumford-Tate group of a complex variation of Hodge structure and therefore $M_{\mathbb{R}}$ is of Hodge type. This is a wonderful existence result, saying informally that certain Hodge-theoretic data is "motivic over \mathbb{R} ." We note also the interesting papers [KI2] and [KI3].

¹³If M is a \mathbb{Q} -simple algebraic group, then the group $M(\mathbb{R})$ will be a semi-simple — but not necessarily simple — real Lie group. An example of this is $\text{Res}_{k/\mathbb{Q}} \text{SL}_2(k)$ where $k = \mathbb{Q}(\sqrt{-d})$ with d a positive rational number. Then $\text{SL}_2(k)(\mathbb{R}) \cong \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. For ease of exposition, we shall assume that $M(\mathbb{R})$ is also simple. The analysis can, in a straightforward fashion, be extended to the general case.

Of course we ultimately need to consider representations of M that are defined over \mathbb{Q} . This will be discussed in Section IV.A.

$$\text{End}_{\mathfrak{m}_{\mathbb{R}}}(V_{\mathbb{R}}) = \begin{cases} \mathbb{R} & \text{(Real case)} \\ \mathbb{C} & \text{(Complex case)} \\ \mathbb{H} & \text{(Quaternionic case)}. \end{cases}$$

Which case we end up in is determined by the weight λ associated to the representation and by the real form, as encoded in its Vogan diagram.¹⁴ The possible invariant bilinear forms Q on V depend on which case we are in. In the real case, Q is unique up to a scalar and its parity — symmetric or alternating — is determined, but in the complex and quaternionic cases, invariant Q 's of both parities exist. Perhaps the most delicate point in our analysis in Chapter IV is (IV.E.4), a theorem in pure representation theory, which we need in order to deal with the parity of Q in the real case; this result allows us to distinguish between real and quaternionic representations.

(iii) Given a real form $\mathfrak{m}_{\mathbb{R}}$, there is a one-to-one correspondence

$$\begin{array}{ccc} \text{Connected real Lie} & & \text{Subgroups } M_{P'}, \\ \text{groups having Lie} & \iff & P \supseteq P' \supseteq R. \\ \text{algebras } \mathfrak{m}_{\mathbb{R}} & & \end{array}$$

Here P and R are respectively the weight and root lattice. In the case of interest to us when $\mathfrak{t} \subseteq \mathfrak{m}_{\mathbb{R}}$, the associated maximal torus in $M_{P'}(\mathbb{R})$ is

$$T \cong \mathfrak{t}/\Lambda \quad \text{where } \Lambda \cong \text{Hom}(P', \mathbb{Z}).$$

The representation of $\mathfrak{m}_{\mathbb{C}}$ having highest weight λ lives on $M_{P'}(\mathbb{R})$ if, and only if, $\lambda \in P'$. Note that the center

$$Z(M_{P'}(\mathbb{R})) \cong P'/R, \quad \pi_1(M_{P'}(\mathbb{R})) \cong P/P'.$$

The disparity between P and R requires some analysis when $\lambda \in P$ but $\lambda \notin R$.

Because Mumford-Tate groups are \mathbb{Q} -algebraic groups, the issue of describing in terms of dominant weights the irreducible representations over \mathbb{Q} of the simple \mathbb{Q} -algebraic groups whose maximal torus is anisotropic arises. Here again there is a highly developed theory. For those simple \mathbb{Q} -algebraic groups whose maximal torus is anisotropic the theory simplifies significantly, and those aspects needed for this work are summarized in part III of Section IV.A. The upshot is that for the purposes of Mumford-Tate groups as discussed in this monograph, one may focus the detailed root-weight analysis on the real case.

The outline of the steps to be followed in our analysis of Hodge representations is given in (IV.A.3). To state the result, we need to introduce some notation

¹⁴The weight λ associated to the representation will be explained (footnote 15).

that will be explained in the text. Given a real, simple Lie group $M(\mathbb{R})$ with Lie algebra $\mathfrak{m}_{\mathbb{R}}$ having Cartan decomposition

$$\mathfrak{m}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p},$$

where \mathfrak{k} is the Lie algebra of a maximal compact subgroup $K \subset M(\mathbb{R})$ containing a compact maximal torus T , there is the root lattice $R \subset i\check{\mathfrak{t}}$ and weight lattice $P \subset i\check{\mathfrak{t}}$ with $R \subset P$. We then define a map $\psi : R \rightarrow \mathbb{Z}/2\mathbb{Z}$ by

$$\psi(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is a compact root} \\ 1 & \text{if } \alpha \text{ is a non-compact root.} \end{cases}$$

Since the Cartan involution is a well-defined Lie algebra homomorphism, it follows that ψ is a homomorphism. We next define a homomorphism

$$\Psi : R \rightarrow \mathbb{Z}/4\mathbb{Z}$$

by $\Psi = "2\psi"$ — i.e., $\Psi(\alpha) = 0$ for compact roots and $\Psi(\alpha) = 2$ for non-compact roots. The reason for working both “mod 2” and “mod 4” will be explained in the remark.

Given a choice of positive, simple roots there is defined a Weyl chamber \mathbf{C} and weights $\lambda \in P \cap \mathbf{C}$. To each $\lambda \in \mathbf{C}$ there is associated an irreducible $\mathfrak{m}_{\mathbb{C}}$ -module W^λ , and one may define the irreducible $\mathfrak{m}_{\mathbb{R}}$ -module $V_{\mathbb{R}}$ associated to λ .¹⁵ This $\mathfrak{m}_{\mathbb{R}}$ -module will be seen to have an invariant form Q . As noted, in some cases there is more than one invariant form Q and we must choose it based on the result. One of our main results is then:

(IV.E.2) THEOREM: *Assume that M is a simple \mathbb{Q} -algebraic group that contains an anisotropic maximal torus. Assume that we have an irreducible representation*

$$\rho : M \rightarrow \text{Aut}(V)$$

defined over \mathbb{Q} . We let δ be the minimal positive integer such that $\delta\lambda \in R$. Then ρ leads to a Hodge representation if, and only if, there exists an integer m such that

$$\Psi(\delta\lambda) \equiv \delta m \pmod{4}.$$

Implicit in this result is that the invariant form Q may be chosen to be defined over \mathbb{Q} . This and the related results and computation of examples are expressed in terms of congruences mod 2 and mod 4. The reason for the “mod

¹⁵If $\text{Res}_{\mathbb{C}/\mathbb{R}} W^\lambda$ denotes the irreducible $\mathfrak{m}_{\mathbb{R}}$ -module obtained by restriction of scalars, then $\text{End}_{\mathfrak{m}_{\mathbb{R}}}(\text{Res}_{\mathbb{C}/\mathbb{R}} W^\lambda)$ is a division algebra and the definition of $V_{\mathbb{R}}$ depends on whether this algebra is \mathbb{R} , \mathbb{C} or \mathbb{H} .

2” is the sign in $Q(u, v) = \pm Q(v, u)$, and the reason for the “mod 4” is that the 2nd Hodge-Riemann bilinear relations

$$i^{p-q}Q(v, \bar{v}) > 0, \quad v \in V^{p,q}$$

depend on $p - q \pmod 4$.

As an illustration of the application of this analysis, we have the following (the notations are given in the appendix to Chapter IV): The only real forms of simple Lie algebras that give rise to Hodge representations of odd weight are

- $\mathrm{su}(2p, 2q)$, $p + q$ even, compact forms included
- $\mathrm{su}(2k + 1, 2l + 1)$
- $\mathrm{so}(4p + 2, 2q + 1)$, $\mathrm{so}^*(4k)$
- $\mathrm{sp}(2n, \mathbb{R})$ ¹⁶
- EV and EVII (real forms of E_7).

A complete list of the real forms having Hodge representations is given in the table after Corollary (IV.E.3).

At the end of Section IV.E, in the subject titled **Reprise**, we have summarized the analysis of which pairs (M, λ) give faithful Hodge representations and which pairs give odd weight Hodge representations. It is interesting that there are \mathbb{Q} -forms of some of the real, simple Lie groups that admit Hodge representations but do not have faithful Hodge representations.

In Section IV.B we turn to the adjoint representation. Here the Cartan-Killing form B gives an invariant form Q , and as a special case of theorem (IV.E.2) the criteria (IV.B.3) to have a Hodge representation may be easily and explicitly formulated in terms of the compact and non-compact roots relative to a Cartan decomposition of the Lie algebra. A number of illustrations of this process are given. The short and direct proof of this result is also given.

An interesting question of Serre (see 8.8 in [Se]) is whether G_2 is a motivic Galois group. This has recently been settled by Dettweiler and Reiter [DR] using a slight modification of the definition that replaces motives by motives for motivated cycles.¹⁷ One has also the related (or equivalent, assuming the Hodge conjecture) question of whether G_2 is the Mumford-Tate group of a motivic Hodge structure. This too follows from [DR], as will be explained in a forthcoming work of the third author with G. Pearlstein [KP2].

¹⁶Our convention for the symplectic groups is that $2n$ is the number of variables.

¹⁷See in addition the interesting works [GS], [HNY], [Yun], and [Ka].

In this section we also give another one of the main results in this work, which provides a converse to the observation above about the adjoint representation:

THEOREM: *Given a representation $\rho : M \rightarrow \text{Aut}(V, Q)$ defined over \mathbb{Q} and a circle $\varphi : \mathbb{U}(\mathbb{R}) \rightarrow M(\mathbb{R})$, $(V, \pm Q, \rho \circ \varphi)$ gives a polarized Hodge structure if and only if, $(\mathfrak{m}, B, \text{Ad } \varphi)$ gives a polarized Hodge structure.*

Given a representation $\rho : M \rightarrow \text{Aut}(V, Q)$, we identify $\rho_*(\mathfrak{m}) \subset \text{End}_{\mathbb{Q}}(V)$ with \mathfrak{m} . Then, as noted above, if $\rho \circ \varphi$ gives a polarized Hodge structure, there is induced on $(\mathfrak{m}, B) \subset \text{End}_{\mathbb{Q}}(V)$ a polarized sub-Hodge structure. The interesting step is to show that conversely a polarized Hodge structure $(\mathfrak{m}, B, \text{Ad } \varphi)$ induces one for $(V, \pm Q, \rho \circ \varphi)$. This requires aspects of the structure theory of semi-simple Lie algebras. The proof makes use of the explicit criteria (IV.B.3) that $\text{Ad } \varphi$ for a co-character $\varphi : \mathbb{U}(\mathbb{R}) \rightarrow T \subset M(\mathbb{R})$ give a polarized Hodge structure on (\mathfrak{m}, B) .

From our analysis we have the following conclusion: *Mumford-Tate groups are exactly the \mathbb{Q} -algebraic groups M whose associated real Lie groups $M(\mathbb{R})$ have discrete series representation in $L^2(M(\mathbb{R}))$ (cf. [HC1], [HC2], [Schm2], [Schm3]).* The discrete series, and the limits of discrete series, are of arithmetic interest as the infinite components of cuspidal automorphic representations in $L^2(M(\mathbb{Q}) \backslash M(\mathbb{A}))$. This potential connection between arithmetic issues of current interest and Hodge theory seems to us unlikely to be accidental.

In Section IV.F we establish another fundamental result:

(IV.F.1) **THEOREM:** (i) *The subgroup $H_{\varphi} \subset M(\mathbb{R})$ that stabilizes the polarized Hodge structure associated to a Hodge representation (M, ρ, φ) is compact and is equal to the subgroup that stabilizes the polarized Hodge structure associated to the polarized Hodge structure $(\mathfrak{m}, \text{Ad}, \varphi)$.*

(ii) *Under the resulting identification of the two Mumford-Tate domains with the homogeneous complex manifold $M(\mathbb{R})/H_{\varphi}$, the infinitesimal period relations coincide.*

This suggests introducing the concept of a *Hodge domain* $D_{\mathfrak{m}, \varphi}$, which is a homogeneous complex manifold $M(\mathbb{R})_a/H_{\varphi}$ where H_{φ} is the compact centralizer of a circle $\varphi : \mathbb{U}(\mathbb{R}) \rightarrow M(\mathbb{R})_a$. Thus a Hodge domain is equivalent to the data (M, φ) where φ satisfies the conditions in (IV.B.3). We observe that a Hodge domain carries an invariant exterior differential system corresponding to the infinitesimal period relation associated to the polarized Hodge structure $(\mathfrak{m}, B, \text{Ad } \circ \varphi)$.¹⁸ *A given Hodge domain may appear, as a complex manifold,*

¹⁸ In the literature there is a reference to Griffiths-Schmid domains, defined to be homogeneous complex manifolds of the form $M_{\mathbb{R}}/H$ where $M_{\mathbb{R}}$ is a real, generally non-compact

in many different ways as a Mumford-Tate domain.¹⁹ Moreover, there will in general be many relations among various Hodge domains.²⁰ A particularly striking illustration is the realization of each of the two G_2 -invariant exterior differential systems on a 5-manifold found by E. Cartan and the Lie-Klein correspondence between them (cf. [Ca] and [Br]). Other interesting low dimensional examples will also be analyzed in detail at the end of Section IV.F.

A fourth aspect of this work is our emphasis throughout on the properties of Mumford-Tate groups in what we call the *non-classical case*. Here we want to make an important and somewhat subtle point of terminology. By the *classical case* we shall mean the case where (i) the Hodge domain $D_{M_\varphi} = M(\mathbb{R})_a/K$ is Hermitian symmetric; thus, K is a maximal compact subgroup of the adjoint group $M(\mathbb{R})_a$ whose center contains a circle $S_0^1 = \{z \in \mathbb{C}^* : |z| = 1\}$; (ii) only $z, 1, z^{-1}$ appear as the characters of $\text{Ad } S_0^1$ acting on $\mathfrak{m}_\mathbb{C}$, and (iii) $\text{Ad}(i)$ is the Cartan involution.²¹ These are the Hermitian symmetric domains that may be equivariantly embedded in Siegel's upper-half space. The inverse limit of quotients $\Gamma \backslash D_{M_\varphi}$ by arithmetic groups are essentially the complex points of components of Shimura varieties, about which there is a vast and rich theory (cf. [Mi2] and [R] for general references).

The non-classical case itself separates into two classes. The first is when D_{M_φ} is Hermitian symmetric,²² but where condition (ii) is not satisfied. In this case the infinitesimal period relation (IPR) may or may not be trivial. When it is trivial, which we shall refer to as the *unconstrained case*, it is possible, but does not in general seem to be known, whether (speaking informally) D_{M_φ} is

Lie group and H is a compact subgroup that contains a compact maximal torus T ; therefore, $\dim T = \text{rank } \mathfrak{m}_\mathbb{C}$. Hodge domains, as discussed in this monograph, differ in three respects: First, $M_\mathbb{R}$ are the real points $M(\mathbb{R})$ of a \mathbb{Q} -algebraic group M . Secondly, the circle $\varphi : \mathbb{U}(\mathbb{R}) \rightarrow M(\mathbb{R})$ is included as part of the definition, the point being that several different φ 's may give equivalent complex structures but inequivalent polarized Hodge structures on (\mathfrak{m}, B) . Third, the exterior differential system giving the infinitesimal period relation is also implied by the definition.

¹⁹Again we mention the subtlety that arises in the several representations of a fixed complex manifold D as *homogeneous* complex manifolds. For example, $D = \mathcal{U}(2, 1)/T_\mathcal{U}$ is a Mumford-Tate domain for polarized Hodge structures of weight $n = 3$ and with Hodge numbers $h^{3,0} = 1, h^{2,1} = 2$. But $D = \text{SU}(2, 1)/T_{\text{SU}}$ is not a Mumford-Tate domain for polarized Hodge structures of any odd weight. Here, $T_\mathcal{U}$ and T_{SU} are the respective maximal tori in $\mathcal{U}(2, 1)$ and $\text{SU}(2, 1)$. This issue is discussed and illustrated in Section IV.G.

²⁰These relations may or may not be complex analytic. Both cases are of interest, the non-holomorphic ones from the perspective of representation theory.

²¹There is a subtlety here in that the adjoint action of the circle $S_0^1 = Z(K)$ is the "square-root" of a character that gives a weight two polarized Hodge structure on (\mathfrak{m}, B) . This is explained in the remark at the end of Section IV.F, where the φ in writing $D_{M_\varphi} = M(\mathbb{R})/K$ will be identified.

²²We will see that all Hermitian symmetric domains are Hodge domains.

“motivic” in the sense that it parametrizes Hodge structures that arise algebro-geometrically.²³ In any case, the quotient $\Gamma \backslash D_{M_\varphi}$ by an arithmetic group is a quasi-projective, complex algebraic variety. The story of its field of definition seems to be worked out in the most detail in the classical case when the Mumford-Tate domain, factored by an arithmetic group, represents the solution to a moduli problem.

There are also interesting cases where D_{M_φ} is Hermitian symmetric but the IPR is non-trivial. In this case it is automatically integrable; D_{M_φ} is therefore foliated and variations of Hodge structure lie in the leaves of the foliation. Thus we have the situation where on the one hand D_{M_φ} cannot be motivic, while on the other hand for an arithmetic subgroup $\Gamma \subset M$, $\Gamma \backslash D_{M_\varphi}$ is a quasi-projective algebraic variety.

The second possibility in the non-classical case is when the Hodge domain D_{M_φ} is not Hermitian symmetric, which implies that the IPR is non-trivial with again the resulting implication that D_{M_φ} cannot be motivic in the above sense. In general, when the IPR is non-trivial from the viewpoint of algebraic geometry, Hodge theory is a *relative* subject.²⁴ In writing this monograph we have consistently sought to emphasize and illustrate what is different in the non-classical case.

For clarity, we summarize this discussion: The *classical case* is when D_{M_φ} may be equivariantly embedded in the Siegel-upper-half space; equivalently, D_{M_φ} parametrizes abelian varieties whose algebra of Hodge tensors contains a given algebra.²⁵ The *non-classical cases* are the remaining D_{M_φ} 's.

A fifth way in which our treatment is complementary to much of the literature is the discussion of the arithmetic aspects of Mumford-Tate domains and Noether-Lefschetz loci in the non-classical case. Before summarizing some of what is in the more arithmetically oriented Chapters V–VIII, we wish to make a few general observations. For these we assume given a global variation of Hodge structure

$$\Phi : S(\mathbb{C}) \rightarrow \Gamma \backslash D$$

where $S(\mathbb{C})$ are the complex points of a smooth, quasi-projective variety S defined over a field k , D is a period domain, and Γ is the monodromy group. If

²³In this regard we call attention to the very interesting recent paper [FL], where it is shown that in the unconstrained case D_{M_φ} is a Mumford-Tate domain parametrizing a variation of Hodge structure of Calabi-Yau type.

²⁴By this we mean that the basic algebro-geometric objects are variations of Hodge structure $\Phi : S \rightarrow \Gamma \backslash D_{M_\varphi}$, rather than the quotients $\Gamma \backslash D_{M_\varphi}$ by themselves.

²⁵This is the case when the arithmetic aspects, especially Galois representations, of automorphic representation, are most highly developed. The principal reason is that in this case algebraic geometry, specifically the action of Galois groups on l -adic cohomology, may be employed.

one assumes a positive answer to the well-known question, “Is the *spread* of a global variation of Hodge structure again a variation of Hodge structure?”, then we can assume that k is a number field. Associated to a point $p \in S(\mathbb{C})$ there are then two fields:

- (i) the field of definition $k(p)$ of $p \in S(\mathbb{C})$;
- (ii) the field of definition of the Plücker coordinate of any lift of $\Phi(p)$ to \check{D} .

Under special circumstances there are three additional fields associated to p .

- (iii) the field $\text{End}(\Phi(p))$ of endomorphisms associated to any lift of $\Phi(p)$ to a point of D ;
- (iv) the field of definition of the period point $\Phi(p)$ associated to a point $p \in S(\mathbb{C})$;
- (v) the field of definition of the variety X_p .

In (iii), if we assume that the polarized Hodge structure $\Phi(p)$ is simple, then $\text{End}(\Phi(p))$ is a division algebra, and when it is commutative we obtain a field. In (iv), we assume that $\Gamma \backslash D$ is a quasi-projective variety defined over a number field. In (v) we assume that there is a family of smooth projective varieties $\mathcal{X} \rightarrow S$ defined over k , and X_p is the fibre over $p \in S(\mathbb{C})$, whose period map is Φ .

There is a vast and rich literature about the relationships among these fields in the classical case where D is a bounded symmetric domain equivariantly embedded in Siegel’s upper half space. This is the theory of Shimura varieties of Hodge type (cf. [Mi1] and [R]). In the non-classical case very little seems to be known, and one of our objectives is to begin to clarify what some interesting issues and questions are, especially as they may relate to Mumford-Tate groups.²⁶

If we have a global variation of Hodge structure whose monodromy group Γ is an arithmetic group, then there are arithmetic aspects associated to particular irreducible representations of $G(\mathbb{R})$ in $L^2(\Gamma \backslash G(\mathbb{R}))$, as well as their adelic extensions. Here we have little to say other than to recall, as noted, that in the non-classical case the discrete series representations of $G(\mathbb{R})$ correspond to *automorphic cohomology* on $\Gamma \backslash D$, rather than to automorphic forms as in the

²⁶We may think of (ii) and (iii) as reflecting arithmetic properties “upstairs”; i.e., on D or \check{D} , and (i), (iv), (v) as reflecting arithmetic properties “downstairs.” They are related via the transcendental period mapping Φ , and the upstairs arithmetic properties of $\Phi(p)$ when p is defined over a number field is a very rich subject. Aspects of this will be discussed in Section VI.D.

classical case (cf. [GS] and, especially, [WW]).²⁷ So far as we know, there is not yet even speculation about a connection between the arithmetic aspects of $L^2(\Gamma \backslash G(\mathbb{R}))$ and the various arithmetic aspects, as mentioned previously, of a global variation of Hodge structure.²⁸

In the largely expository Chapter V we will discuss Hodge structures with a high degree of symmetry; specifically, Hodge structures with complex multiplication or *CM Hodge structures*. For any Hodge structure $(V, \tilde{\varphi})$, defined by $\tilde{\varphi} : \mathbb{S}(\mathbb{R}) \rightarrow \text{Aut}(V_{\mathbb{R}})$ and not necessarily pure, the endomorphism algebra $\text{End}(V, \tilde{\varphi})$ reflects its internal symmetries. It is reasonable to expect, and is indeed the case as will be discussed in the text, that there is a relationship between $\text{End}(V, \tilde{\varphi})$ and the Mumford-Tate group of $(V, \tilde{\varphi})$. For CM-Hodge structure $(V, \tilde{\varphi})$ with Mumford-Tate group $M_{\tilde{\varphi}}$, there is an equivalence between

- $V_{\tilde{\varphi}}$ is a CM-Hodge structure;
- $M_{\tilde{\varphi}}(\mathbb{R})$ is contained in the isotropy group $H_{\tilde{\varphi}}$;
- $M_{\tilde{\varphi}}$ is an algebraic torus; and
- $M_{\tilde{\varphi}}(\mathbb{Q})$ is contained in the endomorphism algebra $E_{\tilde{\varphi}} = \text{End}(V_{\tilde{\varphi}})$.

In case $V_{\tilde{\varphi}}$ is a simple Hodge structure, $\text{End}(V, \tilde{\varphi})$ is a totally imaginary number field of degree equal to $\dim V$. There is a converse result in (V.3).

We broaden the notion of CM type by defining an *n-orientation of a totally imaginary number field* (OIF(n)) and construct a precise correspondence between these and certain important kinds of CM Hodge structures. In the classical case of weight $n = 1$, we recover the abelian variety associated to a CM type.

Next, we generalize the notion of the *Kubota rank* and *reflex field* associated to a CM Hodge structure $V_{\tilde{\varphi}}$ to the OIF(n) setting. This may then be used to compute the dimension, rational points, and Lie algebra of the Mumford-Tate group of $V_{\tilde{\varphi}}$. When the Kubota rank is maximal, the CM Hodge structure is non-degenerate. In the classical case the corresponding CM abelian variety A is non-degenerate, and by a result of Hazama and Murty [Mo1] all powers of A satisfy the Hodge conjecture.

Chapter VI is devoted to the arithmetic study of Mumford-Tate domains and Noether-Lefschetz loci, both in D and in \check{D} . We use the notation $\mathcal{Z} \subset \check{D}$ for the

²⁷This issue is discussed further near the end of this introduction.

²⁸Here we note the paper [GS] and that there are some promising special results [C1], [C2], [C3]. The authors would like to thank Wushi Goldring for bringing this work to our attention.

set of flags F^\bullet that fail to give Hodge structures; i.e., for which at least one of the maps

$$F^p \oplus \overline{F}^{n-p+1} \rightarrow V_{\mathbb{C}}$$

fails to be an isomorphism. Then points in $\check{D} \setminus \mathcal{Z}$ give Hodge structures that may not be polarized. More precisely, $\check{D} \setminus \mathcal{Z}$ is a union of open $G(\mathbb{R})$ -orbits, each connected component of which corresponds to *Hodge structures having possibly indefinite polarizations* in the sense that the Hermitian forms in the second Hodge representation bilinear relations are non-singular but may not be positive or negative definite.²⁹

For $\varphi \in D$ a polarized Hodge structure with Mumford-Tate group $M_\varphi := M$, we are interested in the loci

$$\begin{aligned} \text{NL}_M &= \left\{ \begin{array}{l} Q\text{-polarized Hodge structures with} \\ \text{Mumford-Tate group contained in } M \end{array} \right\} \\ \check{\text{NL}}_M &= \left\{ \begin{array}{l} \text{flags } F^\bullet \in \check{D} \text{ with Mumford-} \\ \text{Tate group contained in } M \end{array} \right\}. \end{aligned}$$

We show that no component of $\check{\text{NL}}_M$ is contained in \mathcal{Z} , and the components of $\check{\text{NL}}_M$ are indexed in terms of “*Mumford-Tate Hodge orientations*,” extending to non-abelian Mumford-Tate groups the CM types/orientations discussed in Chapter V. Using this we give a computationally effective procedure to determine the components in terms of Lie algebra representations and Weyl groups.

A first consequence of this is that $M(\mathbb{C})$ acts transitively on each component of the locus $\check{\text{NL}}_M$. Another result, which contrasts the classical and non-classical period domains, is that in the classical case NL_M is a single $M(\mathbb{R})$ -orbit in D , whereas in the non-classical case this is definitely *not* true; there is more than one Q -polarized Hodge type.

Next, we observe that the components of NL-loci are all defined over $\overline{\mathbb{Q}}$, the reason being that each component contains CM-Hodge structures, which as points in \check{D} are defined over $\overline{\mathbb{Q}}$, and each component is an orbit in \check{D} of $M(\mathbb{C})$ and M is a \mathbb{Q} -algebraic subgroup of G . Thus the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (or equivalently $\text{Gal}(\mathbb{C}/\mathbb{Q})$) acts on $\check{\text{NL}}_M$ permuting the components. We then show that the normalizers of M in G are the groups stabilizing the NL-loci. In case M is nondegenerate and $\check{\text{NL}}_M$ contains a simple nondegenerate CM Hodge structure, we show that:

²⁹The orbit structure of the action of $M(\mathbb{R})$ on \check{D} is extensively discussed in [FHW]; cf. the remarks at the end of this introduction.

- the action of the normalizer $N_G(M, \mathbb{C})$ on the components of \widetilde{NL}_M turns out to give a “continuous envelope” for the very discontinuous action of $\text{Gal}(\mathbb{C}/\mathbb{Q})$; and
- the endomorphism algebra of a generic $\varphi \in \widetilde{NL}_M$ is a field, and this field contains the field of definition of each component.

In Chapter VII we develop an algorithm for determining all Mumford-Tate subdomains of a given period domain. This result is then applied to the classification of all CM Hodge structures of rank 4 and when the weight $n = 1$ and $n = 3$, to an analysis of their Hodge tensors and endomorphism algebras, and the number of components of the Noether-Lefschetz loci. The result is that one has a complex but very rich arithmetic story; e.g., the results for the $n = 3$ case are summarized in the list given in Theorem (VII.F.1). We note in particular the intricate structure of the components of the Noether-Lefschetz loci in D and in its compact dual \check{D} , and the two interesting cases where the Hodge tensors are generated in degrees 2 and 4, and not just in degree 2 by $\text{End}(V_\varphi)$. One application (cf. VII.H, type (i)) is that a particular class of period maps appearing in mirror symmetry never has image in a proper subdomain of D .

As an observation, it is generally understood that in Hodge theory or algebraic geometry — and especially at their interface — “special points” are of particular interest. These are points with symmetries that have an internal structure that relates arithmetic and geometry. We believe that the tables in Chapter VII give a good picture of the intricacies of the Hodge-theoretic special points. Of course, these relate directly to algebraic geometry when the weight $n = 1$, but when the weight $n = 3$ the story is in its earliest stages.³⁰

In Chapter VIII we discuss some arithmetic aspects of the situation

$$\Phi : S(\mathbb{C}) \rightarrow \Gamma \backslash D$$

where S parametrizes a family $\mathcal{X} \rightarrow S$ of smooth, projective varieties defined over a number field k . We recall the notion of absolute Hodge classes (AH) and strongly absolute classes (SAH). There are well-known conjectures (here H denotes Hodge classes):

(i) $H \Rightarrow \text{AH}$,

(ii) $H \Rightarrow \text{SAH}$.

³⁰The case considered here is when the Hodge structure is of “mirror-quintic-type.” On the algebro-geometric side there is a vast literature, much of it motivated by the connection in physics.

We denote by NL a Noether-Lefschetz locus in $\Gamma \backslash D$, and show that (cf. [Vo])

(ii) *implies that $\Phi^{-1}(NL)$ is defined over a number field.*

Recalling that $NL \subset \check{D}$ is defined over \mathbb{Q} , these observations may be thought of as relating “arithmetic upstairs” with “arithmetic downstairs” via the transcendental period map Φ . The particular case when the Noether-Lefschetz locus consists of isolated points was alluded to in the discussion of CM Hodge structures.

A related observation [A2] is that one may formulate a variant of the “Grothendieck conjecture” in the setting of period maps and period domains. Informally stated, Grothendieck’s conjecture is that for a smooth projective variety X defined over a number field, all of the \mathbb{Q} -relations on the period matrix obtained by comparing bases for the algebraic de Rham cohomology group $\mathbb{H}_{Zar}^n(\Omega_{X(k)/k}^\bullet)$ and the Betti cohomology group $H_B^n(X(\mathbb{C}), \mathbb{Q})$ are generated by algebraic cycles on self-products of X . Again, informally stated the period domain analog of this is (cf. (VIII.A.8))

Let $p \in S(k)$ and let $\varphi \in D$ be any point lying over $\Phi(p)$. Then φ is a very general point in the variety \check{D}_{M_φ} .

Here, very general means that φ is a point of maximal transcendence degree in \check{D}_{M_φ} , which is a subvariety of \check{D} defined over a number field. The relation between this and the above formulation in terms of period matrices arises when $\mathcal{X} \rightarrow S$ is a moduli space with the property that the fields of definition of X_p and of p are both number fields exactly when one of them is.

A final conjecture (VIII.B.1) may be informally stated as follows:

Let $\mathcal{E} \subset D$ be a set of CM points and assume that the image $\rho(\mathcal{E})$ of \mathcal{E} in $\Gamma \backslash D$ lies in $\Phi(S(\mathbb{C}))$. Then the \mathbb{C} -Zariski closure of $\rho(\mathcal{E})$ in $\Phi(S(\mathbb{C}))$ is a union of unconstrained Hermitian symmetric Mumford-Tate domains.

A geometric consequence of this would be that if $S(\mathbb{C})$ contains a Zariski dense set of CM Hodge structures, then the corresponding Mumford-Tate domain in the structure theorem is *Hermitian symmetric* whose IPR is trivial.

Before turning to some concluding remarks in this introduction, we observe that Mumford-Tate groups may be said to lie at the confluence of several subjects:

- (A) **Geometry**; specifically, algebraic geometry and variation of Hodge structure as realized by mappings to Mumford-Tate domains;

- (B) **Representation theory**; specifically, the introduction and analysis of Hodge representations, leading among other things to the observation that Hodge groups turn out to be exactly the reductive \mathbb{Q} -algebraic groups whose associated semi-simple real Lie groups have discrete series representations; and
- (C) **Arithmetic**; specifically, the rich honeycomb of arithmetically defined Noether-Lefschetz loci in a period domain.

It is certainly our sense that very interesting work should be possible at the interfaces among these areas.

We would like to mention several topics that are *not* discussed in this work and may be worth further study.

*Extensions of moduli spaces of Γ -equivalence classes of polarized Hodge structures whose generic Mumford-Tate group is M .*³¹

This means the following: Let M be a reductive \mathbb{Q} -algebraic group such that the pair (M, φ) gives a Hodge group together with a circle $\varphi : \mathbb{U}(\mathbb{R}) \rightarrow M(\mathbb{R})$, and let D_{M_φ} be the corresponding Mumford-Tate domain. Let $\Gamma \subset M$ be an arithmetic group and set

$$D_{M_\varphi}(\Gamma) = \Gamma \backslash D_{M_\varphi}.$$

We may think of $D_{M_\varphi}(\Gamma)$ as the Γ -equivalence classes of polarized Hodge structures whose generic Mumford-Tate group is M . A subtlety here is that D_{M_φ} may be realized in many different ways as a Mumford-Tate domain for polarized Hodge structures of different weights and different sets of Hodge numbers and, if $n > 1$, different Hodge orientations in the same period domain. What is common among all these realizations is that the infinitesimal period relations given by the Pfaffian system $I \subset T^*D_{M_\varphi}$ all coincide.

We let $S = (\Delta^*)^k \times \Delta^l$ be a punctured polycylinder and

$$\Phi : S \rightarrow D_{M_\varphi}(\Gamma)$$

a variation of Hodge structure whose monodromies are unipotent; i.e., if $\gamma_i \in \pi_1(S)$ is the circle around the origin in the i^{th} factor of $(\Delta^*)^k$, then

$$\Phi_*(\gamma_i) = T_i \in \Gamma$$

is a unipotent element of M . Given any variation of Hodge structure as above, by a result of Borel the unipotency of monodromies may be achieved by passing

³¹This topic was already briefly alluded to above.

to a finite covering of S . By an *extension* we mean (i) a log-analytic variety $D_{M_\varphi}^e$ in which $D_{M_\varphi} \subset D_{M_\varphi}^e$ is a Zariski open set and to which the action of Γ extends to give a log-analytic quotient variety $D_{M_\varphi}^e(\Gamma) = \Gamma \backslash D_{M_\varphi}^e$; and (ii) setting $S^e = \Delta^k \times \Delta^l$, the above variation of Hodge structure extends to give

$$\begin{array}{ccc} S & \xrightarrow{\Phi} & D_{M_\varphi}(\Gamma) \\ \cap & & \cap \\ S^e & \xrightarrow{\Phi^e} & D_{M_\varphi}^e(\Gamma) \end{array}$$

where $\Phi^e : S^e \rightarrow D_{M_\varphi}^e(\Gamma)$ is a morphism of log-analytic varieties. The issue is:

Can the Kato-Usui theory [KU] be extended to the above situation?

The point is that since Hodge domains parametrize many different types of polarized Hodge structures, focusing on them may serve to isolate the essential algebraic group aspects of the Kato-Usui construction: the “input data” is (M, φ) , and not any particular Hodge representation of M .³²

Ordinary cohomology. This begins with the study of

$$H^*(D_{M_\varphi}(\Gamma), \mathbb{Q}).$$

There are a number of refinements to this.

(i) For a local system $\mathbb{V} \rightarrow D_{M_\varphi}(\Gamma)$ given by a Hodge representation of M , one may study $H^*(D_{M_\varphi}(\Gamma), \mathbb{V})$.

(ii) Tensoring with \mathbb{C} and using

$$H^*(D_{M_\varphi}(\Gamma), \mathbb{V}_{\mathbb{C}}) \cong H_{DR}^*(D_{M_\varphi}(\Gamma), \mathbb{V}_{\mathbb{C}}),$$

one may study the *characteristic cohomology* (cf. [CGG])

$$H_I^*(D_{M_\varphi}(\Gamma), \mathbb{V}_{\mathbb{C}})$$

computed from the de Rham complex of C^∞ forms modulo the differential ideal generated by the C^∞ sections of $I \oplus \bar{I}$ where \bar{I} is the complex conjugate sub-bundle to I in the complexification of the real tangent bundle. This is the cohomology that is relevant for variations of Hodge structure.

³²This is carried out in a different context and for $SU(2, 1)$ in [C3]. Also, the example at the end of Section IV.A suggests a positive indication that the above question may be feasible.

(iii) We assume that Γ is *neat*, meaning that Γ acts freely on the Riemannian symmetric space X_M .³³ The fibering

$$D_M \rightarrow X_M$$

has rational, homogeneous projective varieties F as fibres, and therefore *additively*, but *not* multiplicatively, we have (for any coefficients, including a local system as above)

$$H^*(D_{M_\varphi}(\Gamma)) \cong H^*(\Gamma \backslash X_M) \otimes H^*(F).$$

Now two important points arise:

(a) Using that M is a \mathbb{Q} -algebraic group, for a suitable open compact subgroup $U \subset M(\mathbb{A})$ the *adelization*

$$M(\mathbb{Q}) \backslash M(\mathbb{A}) / U$$

of $\Gamma \backslash X_M$ may be defined and its cohomology is the subject of considerable arithmetic and representation-theoretic interest (see [Schw] for a recent survey). As a set, the adelization is the limit of $\Gamma \backslash D_M$'s or $\Gamma' \backslash X_M$'s over the congruence subgroups Γ' of Γ , or some variant of these constructions — e.g., taking

$$M(\mathbb{Q}) \backslash M(\mathbb{A}) / K_\varphi$$

where

$$K_\varphi = H_\varphi \times \prod K_p$$

is the restricted product where an element $\prod K_p$ is in $M(\mathbb{Z}_p)$ for almost all p .

(b) In the case discussed in [Schw], the cohomology that is of interest for the study of cuspidal automorphic representation is $H_{\text{cusp}}^*(M(\mathbb{Q}) \backslash M(\mathbb{A}) / K_\varphi, \mathbb{V})$. By virtue of the characterization of semi-simple Mumford-Tate groups as those semi-simple \mathbb{Q} -algebraic groups M such that $M(\mathbb{R})$ contains a compact maximal torus, which by Harish-Chandra [HC1], [HC2] is equivalent to $L^2(M(\mathbb{R}))$ containing non-trivial discrete series representations; it is essentially known that cuspidal cohomology may be defined in the context of Hodge domains and have a Lie algebra cohomological description of the sort

$$\bigoplus_{\pi} H^*(\mathfrak{m}_{\mathbb{R}}, H_\varphi; H_{\pi_\infty} \otimes V) \otimes H_{\pi_f}$$

(see [Schw, p. 258] for explanation of notations).

³³This is always possible by passing to a subgroup of finite index in Γ . Here $X_M = M(\mathbb{R})/K$ where K is the maximal compact subgroup of $M(\mathbb{R})$ containing the isotropy group H_φ .

(iv) Combining (ii) and (iii), one may at least formally define the *cuspidal characteristic cohomology* as

$$\bigoplus_{\pi} H^*(\mathfrak{m}_{\mathbb{R}}, H_{\varphi}; \mathfrak{w}, H_{\pi_{\infty}} \otimes V) \otimes H_{\pi_f}$$

where $\mathfrak{m}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{w} \subset \mathfrak{p}$ defines the infinitesimal period relation (see [CGG] for an explanation of the notation). Especially if it could be related to variations of Hodge structure defined over \mathbb{Q} (or over a number field), this may be an object of interest to study.

Coherent cohomology. Here one begins with a holomorphic, homogeneous vector bundle $\mathcal{E} \rightarrow D_{M_{\varphi}}$ and considers the L^2 -cohomology groups

$$H_{(2)}^*(D_{M_{\varphi}}, \mathcal{E})$$

and

$$H_{(2)}^*(\Gamma \backslash D_{M_{\varphi}}, \mathcal{E}).$$

Again it is exactly for Hodge domains that $H_{(2)}^*(D_{M_{\varphi}}, \mathcal{E})$ is well-understood, and varying φ and \mathcal{E} realizes all the irreducible discrete series representations (see [Schm4] for an excellent overview). The groups $H_{(2)}^*(\Gamma \backslash D_{M_{\varphi}}, \mathcal{E})$ are less well-understood, although as shown in [WW] Poincaré series do give a non-trivial map

$$H_{(1)}^*(D_{M_{\varphi}}, \mathcal{E}) \rightarrow H_{(2)}^*(\Gamma \backslash D_{M_{\varphi}}, \mathcal{E}).$$

In the classical case when $D_{M_{\varphi}}$ is Hermitian symmetric and the quotient $\Gamma \backslash D_{M_{\varphi}}$ is a component of the complex points of a Shimura variety, the adelization of $H^*(\Gamma \backslash D_{M_{\varphi}}, \mathcal{E})$ has been, and continues to be, the object of intense study (cf. [H], [Mi1], [Mi2], [BHR], and [Mor]). Once again there is a Lie algebra cohomological description of the cuspidal coherent cohomology ([H], [BHR]), and whose existence also is related to $M(\mathbb{R})$ having discrete series representations. This Lie algebra cohomological formulation makes sense for Hodge domains with H_{φ} replacing K .

In an interesting series of papers [C1], [C2], [C3] this automorphic cohomology is studied in detail for the Hodge domain

$$D_{M_{\varphi}} = \mathrm{SU}(2, 1)/T$$

where the complex structure on $D_{M_{\varphi}}$ is the one that does not fibre holomorphically or anti-holomorphically over an Hermitian symmetric space. There the automorphic cohomology in degrees one and two is studied and shown to correspond to automorphic representations whose archimedean component is

a degenerate limit of discrete series, a phenomenon that cannot happen in the classical Shimura variety case.

Two further cases that may merit further detailed study are the period domains

$$\begin{cases} D_1 &= \mathrm{SO}(4, 1, \mathbb{R})/\mathcal{U}(2) \\ D_2 &= \mathrm{Sp}(4, \mathbb{R})/\mathcal{U}(1) \times \mathcal{U}(1), \end{cases}$$

where in the second the complex structure is again the one that does not fibre holomorphically or anti-holomorphically over an Hermitian symmetric domain. In the first case there is no Hermitian symmetric domain with symmetry group $\mathrm{SO}(4, 1)$, so the methods of [C1] and [C2] would not seem to apply. Each of these has algebro-geometric interest, the second because it arises in mirror symmetry.

Cycle spaces (cf. [FHW]). Cycle spaces are the following: In a Mumford-Tate domain $D_{M_\varphi} = M_\varphi(\mathbb{R})/H_\varphi$, the orbit $\mathcal{O}_\varphi := K \cdot \varphi$ of a point $\varphi \in D$ under the maximal compact subgroup $K \subset G(\mathbb{R})$ with $H_\varphi \subset K$ is a maximal compact, complex analytic subvariety of D_{M_φ} . The *cycle space*

$$\mathcal{U} = \{g \cdot \mathcal{O}_\varphi : g \in M_\varphi(\mathbb{C}), g \cdot \mathcal{O}_\varphi \subset D_{M_\varphi}\}$$

is the set of translates of \mathcal{O}_φ by those elements in the complex group that leaves the translate in D_{M_φ} .³⁴ The interest in cycle spaces began with the observation ([G1], [G2]) that $\dim \mathcal{O}_\varphi := d$ is the degree in which the cohomology $H^d(D_{M_\varphi}, \mathcal{L})$ of suitable homogeneous line bundles was expected to occur, which in fact turned out to be the case [Schm5]. This suggested interpreting the somewhat mysterious group $H^d(D_{M_\varphi}, \mathcal{L})$ as holomorphic sections of a vector bundle over \mathcal{U} — a sort of Radon transform. For a discrete subgroup $\Gamma \subset M$ there is an “automorphic version” of this construction [WW], a variant of which has been used effectively in recent years (cf. [EGW], [Gi] and [C1], [C2], [C3]). The study of the cycle spaces themselves has turned out to be a rich subject with applications to representation theory; a recent comprehensive account is given in [FHW].

In concluding this introduction, we would like to observe that when we began this project we were of the view that Mumford-Tate groups and Mumford-Tate domains were primarily of interest because of their use in the period mappings arising from algebraic geometry. As our work has evolved, we have come to the point of view that the objects in the title of this monograph are arguably of equal interest in their own right, among other things for the rich arithmetic and representation-theoretic structure that they reveal.

³⁴It may be shown that \mathcal{U} is an open set in the Hilbert scheme associated to $\mathcal{O}_\varphi \subset \check{D}_{M_\varphi}$.

Finally, we would like to first thank the authors of [FH], [GW], and [K] — their books on Lie groups and representation theory were invaluable to this work. Secondly, we would like to thank Colleen Robles, J. M. Landsberg, and Paula Cohen for valuable suggestions and helpful edits. Lastly, we would like to express our enormous appreciation to Sarah Warren for her work in typing the manuscript through seemingly innumerable iterations and revisions.³⁵

Notations (the terms to be explained in the text).

- Throughout, V will denote a rational vector space. For $k = \mathbb{R}$ or \mathbb{C} we set $V_k = V \otimes_{\mathbb{Q}} k$.
- \mathbb{S} and \mathbb{U} will denote the *Deligne torus* and its maximal compact subgroup as defined in Section I.A.
- A *Hodge structure* on V will be denoted by $(V, \tilde{\varphi})$, or frequently simply by $V_{\tilde{\varphi}}$. A *polarized Hodge structure* will be denoted by (V, Q, φ) or (V_{φ}, Q) .
- $E_{\tilde{\varphi}}$ will denote the endomorphism algebra $\text{End}(V, \tilde{\varphi})$ of $(V, \tilde{\varphi})$.
- *Polarized Hodge structures* on a \mathbb{Q} -vector space V with polarizing form Q will be denoted by φ . The space of all such, with given Hodge number $h^{p,q}$, is a *period domain* D . The period domain is acted on transitively by the real Lie group $G(\mathbb{R}) := \text{Aut}(V_{\mathbb{R}}, Q)$, with isotropy group of $\varphi \in D$ denoted by H_{φ} , or just H if no confusion is possible.
- The *compact dual* \check{D} is given by all filtrations F^{\bullet} on $V_{\mathbb{C}}$ with $\dim F^p = f^p = \sum_{p' \geq p} h^{p',q'}$ and satisfying the first Hodge-Riemann bilinear relations. It is a rational, homogeneous variety $G(\mathbb{C})/P$ defined over \mathbb{Q} and with *Plücker embedding* $\check{D} \subset \mathbb{P}^N$. The *Plücker coordinate* of $F^{\bullet} \in \check{D}$ will be denoted by $[F^{\bullet}]$.³⁶
- For $\varphi \in D$, F_{φ}^{\bullet} will denote the corresponding point in \check{D} .
- The *Mumford-Tate group* of $\varphi \in D$ will be denoted by M_{φ} ; that of $F^{\bullet} \in \check{D}$ will be denoted by $M_{F^{\bullet}}$. Sometimes, when no confusion is possible and to minimize notational clutter, we will set $M = M_{\varphi}$ or $M = M_{F^{\bullet}}$.

³⁵The third author would also like to acknowledge partial support from EPSRC through First Grant EP/H021159/1.

³⁶In a few circumstances we will omit the \bullet in F^{\bullet} , as including it would entail distracting notational clutter. Examples are $\text{Hg}_{(F,W)}^{\bullet,\bullet}$, $\text{Hg}_{(\widehat{F,W})}^{\bullet,\bullet}$ and $\text{Hg}_{(F,E)}^{\bullet,\bullet}$.

- *Integral elements* of the canonical exterior differential system on \check{D} will be denoted by $E \subset T_{F^\bullet} \check{D}$. The corresponding Mumford-Tate group will be denoted by $M_{(F^\bullet, E)}$.
- Mumford-Tate domains will be denoted by D_{M_φ} or by just D_M if confusion is unlikely, for $\varphi \in D$, and by $\check{D}_{M_{F^\bullet}}$ or again \check{D}_M for $F^\bullet \in \check{D}$.
- A *Hodge representation* is denoted

$$(M, \rho, \varphi)$$

where

$$\rho : M \rightarrow \text{Aut}(V, Q)$$

is a representation and

$$\varphi : \mathbb{U}(\mathbb{R}) \rightarrow M(\mathbb{R})$$

is a circle such that $(V, Q, \rho \circ \varphi)$ gives a polarized Hodge structure.

- A *Hodge domain* is a homogeneous complex manifold $D_{\mathfrak{m}} = M_{\mathfrak{a}}(\mathbb{R})/H_\varphi$ where M is a reductive \mathbb{Q} -algebraic group and $\varphi : \mathbb{U}(\mathbb{R}) \rightarrow M(\mathbb{R})$ is a circle such that $(\mathfrak{m}, B, \text{Ad} \circ \varphi)$ is a Hodge representation.
- $\mathcal{Z} \subset \check{D}$ will be the set of $F^\bullet = \{F^p\}$ such that some $F^p \otimes \bar{F}^{n-p+1} \rightarrow V_{\mathbb{C}}$ fails to be an isomorphism. The remaining points $F^\bullet \in \check{D} \setminus \mathcal{Z}$ will all give indefinitely polarized Hodge structures and in Chapters V–VII will frequently be denoted by φ where $\varphi : \mathbb{S}(\mathbb{U}) \rightarrow \text{Aut}(V_{\mathbb{R}}, Q)$ defines the Hodge structure.
- We denote by G the \mathbb{Q} -algebraic group $\text{Aut}(V, Q)$, and for any field $k \supseteq \mathbb{Q}$, $G(k)$ denotes the k -valued points of G . Similar notations will be used for other \mathbb{Q} -algebraic groups.
- K will denote a maximal compact subgroup of $G(\mathbb{R})$. As will be seen, a $\varphi \in D$ will determine a unique K with $H_\varphi \subset K$, sometimes denoted by K_φ .
- A Lie or algebraic group A is an *almost direct product* of subgroups A_i if the intersections $A_i \cap A_j$ are finite and the map $A_1 \times \cdots \times A_m \rightarrow A$ is finite and surjective. We denote by A^0 the identity components in either the Lie or algebraic group sense.
- We set $T^{k,l} = T^{k,l}(V) = (\otimes^k V) \otimes (\otimes^l \check{V})$ and $T^{\bullet,\bullet} = \bigoplus_{k,l} T^{k,l}$.

- For either $\varphi \in D$ or $F^\bullet \in \check{D}$, we denote by $\text{Hg}_\varphi^{k,l}$ or $\text{Hg}_{F^\bullet}^{k,l}$ the set of *Hodge classes* in $T^{k,l}$. $\text{Hg}_\varphi^{\bullet,\bullet} = \bigoplus_{k,l} \text{Hg}_\varphi^{k,l}$, and similarly $\text{Hg}_{F^\bullet}^{\bullet,\bullet}$, will be the *algebra of Hodge tensors*. For M a group, we denote by $\text{Hg}_M^{\bullet,\bullet}$ the algebra of M -Hodge tensors, as defined in Section IV.A.
- We denote by $\text{NL}_\varphi, \check{\text{NL}}_\varphi, \text{NL}_{F^\bullet}, \text{NL}_M, \check{\text{NL}}_M$ the various *Noether-Lefschetz loci*, to be defined in Section II.C and Chapter VI.
- For a linear algebraic group $A \subset \text{GL}(W)$, we denote by $A' \subset \text{GL}(W)$ the subgroup defined by

$$A' = \left\{ a \in \text{GL}(W) : \begin{array}{l} a \text{ fixes all } w \in T^{\bullet,\bullet}W \\ \text{that are fixed by } A \end{array} \right\}.$$

- For an algebraic torus T , we denote by $X^*(T), X_*(T)$ the groups of characters, respectively co-characters of T .
- For $M \subset G$ a \mathbb{Q} -algebraic subgroup and $k = \mathbb{R}$ or \mathbb{C} , we denote by $N_G(M, k)$ the normalizer of $M(k)$ in $G(k)$.
- $W_M(T, \mathbb{R}), W_M(T, \mathbb{R})^0$, and $W_M(T, \mathbb{C})$ will denote various Weyl groups, defined in Section VI.D.
- $\text{End}(V, \varphi)$ or $\text{End}(V_\varphi)$ will denote the algebra of endomorphisms of a Hodge structure V_φ .
- CM-Hodge structure will be the standard abbreviation for a *complex multiplication Hodge structure*.
- Given a number field F , for k equal to \mathbb{R} or \mathbb{C} , $S_F(k)$ will denote the set of embeddings of F in k .
- F^c will denote the Galois closure of a field F . For (F, Π) an oriented CM field associated to a polarized CM-Hodge structure (see Section V.A), F' will denote the generalized *reflex field* (which depends on Π), and $R(F, \Pi)$ will denote the generalized *Kubota rank*.
- The notations $\text{OIF}(n), \text{subOIF}(n), \text{OCMF}(n), \widetilde{\text{OCMF}}(n), \text{WCMHS}$, and SCMHS all represent concepts associated to CM-Hodge structure; see Section VI.B.
- $\text{Corr}(A, \mathbb{Q})$ will denote the \mathbb{Q} -correspondences of an abelian variety A ; see Section V.E.

- A Q -quasi-unitary basis for a polarized Hodge structure $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ is given by bases $\omega_{p,i}$ for $V^{p,q}$ satisfying

(i) $\omega_{n-p,i} = \bar{\omega}_{p,i}$;

(ii) $(\sqrt{-1})^{2p-n} Q(\omega_{p,i}, \bar{\omega}_{p,j}) = \delta_j^i$.

- Nilpotent orbits will be denoted by Φ^{nilp} .
- The notations and terminology from the theory of Lie groups and Lie algebras and their representations are collected in the appendix to Chapter IV, where most of the discussion involving Lie theory takes place.
- The canonical exterior systems generated by the *infinitesimal period relation* (IPR) will be denoted by $I \subset T\check{D}$.
- *Variations of Hodge structure* will be denoted by

$$\Phi : S \rightarrow \Gamma \backslash D.$$

- The *Noether-Lefschetz locus of a variation of Hodge structure* will be denoted by

$$\text{NL}_{s_0}(S) \subset S.$$