Lecture Notes in Mathematics 2052

A1-Algebraic Topology over a Field

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Preface

This work should be considered as a natural sequel to the foundational paper [59] where the $\mathbb{A}^1$-homotopy category of smooth schemes over a base scheme was defined and its first properties studied. In this text the base scheme will always be the spectrum of a perfect field k.

One of our first motivations is to emphasize that, contrary to the first impression, the relationship between the $\mathbb{A}^1$-homotopy theory over k and the category Sm_k of smooth k-schemes is of the same nature as the relationship between the classical homotopy theory and the category of differentiable manifolds. This explains the title of this work; we hope to convince the reader in this matter. This slogan was already discussed in [55], see also [4].

This text is the result of the compilation of two preprints "$\mathbb{A}^1$-algebraic topology over a field" and "$\mathbb{A}^1$-homotopy classification of vector bundles over smooth affine schemes" to which we added some recent new stuff, consisting of the two sections: "Geometric *versus* canonical transfers" (Chap. 4) and "The Rost–Schmid complex of a strongly $\mathbb{A}^1$-invariant sheaf" (Chap. 5).

The main objective of these new sections was primarily to correctly establish the equivalence between the notions of "strongly $\mathbb{A}^1$-invariant" and "strictly $\mathbb{A}^1$-invariant" for sheaves of abelian groups, see Theorem 1.16. These new sections appear also to be interesting in their own. As the reader will notice, the introduction of the notion of *strictly* $\mathbb{A}^1$*-invariant sheaves with generalized transfers* in our work on the Friedlander–Milnor conjecture [56–58] is directly influenced from these.

Our treatment of transfers in Chap. 4, which is an adaptation of the original one of Bass and Tate for Milnor K-theory [9], is, we think, clarifying. Besides reaching the structure of strictly $\mathbb{A}^1$-invariant sheaves with generalized transfers, we obtain on the way a new proof of the fact [38] that the transfers in Milnor K-theory do not depend on choices of a sequence of generators, see Theorem 4.27 and Remark 4.32. Our proof is in spirit different from the one of Kato [38], as we prove directly by geometric means the independence in the case of two generators.

The construction in Chap. 5 for a general strongly $\mathbb{A}^1$-invariant sheaf of abelian groups of its "Rost–Schmid" complex is directly influenced by the work of Rost [68] and its adaptation to Witt groups in [70]. Our philosophy on transfers in this work was to use them as less as possible, and to only use them when they really show up by themselves. For instance we will define in Chap. 3 the sheaves of unramified Milnor K-theory (as well as Milnor–Witt K-theory) without using any transfers. The proof of Lemma 5.24 contains the geometric explanation of the formula for the differential of the "Gersten–Rost–Schmid" complex (Corollary 5.44) and the appearance of transfers in it.

The results and ideas in this work have been used and extended in several different directions. In [16] some very concrete computations on $\mathbb{A}^1$-homotopy classes of rational fractions give a nice interpretation of our sheaf theoretic computations of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$. The structure and property of the $\mathbb{A}^1$-fundamental group sheaves as well as the associated theory of $\mathbb{A}^1$-coverings have been used in [2,3,86]. It is also the starting point of [4]. Our result (Chap. 9) concerning the Suslin–Voevodsky construction $Sing_\bullet^{\mathbb{A}^1}(SL_n)$, $n \geq 3$, has been generalized to a general split semi-simple group of type not SL_2 in [87] and to the case of SL_2 in [60].

The present work plays also a central role in our approach to the Friedlander–Milnor conjecture [56–58].

Conventions and notations. Everywhere in this work, k denotes a fixed perfect field and Sm_k denotes the category of smooth finite type k-schemes.

We denote by $\mathcal{F}_k$ the category of field extensions $k \subset F$ of k such that F is of finite transcendence degree over k. By a discrete valuation v on $F \in \mathcal{F}_k$ we will always mean one which is trivial on k. We let $\mathcal{O}_v \subset F$ denote its valuation ring, $m_v \subset \mathcal{O}_v$ its maximal ideal, and $\kappa(v)$ its residue field.

For any scheme X and any integer i we let $X^{(i)}$ denote the set of points in X of codimension i. Given a point $x \in X$, $\kappa(x)$ will denote its residue field, and $\mathcal{M}_x$ will denote the maximal ideal of the local ring $\mathcal{O}_{X,x}$.

The category Sm_k is always endowed with the Nisnevich topology [59,62], unless otherwise explicitly stated. Thus for us "sheaf" always means sheaf in the Nisnevich topology.

We will let $\mathcal{S}et$ denote the category of sets, $\mathcal{A}b$ that of abelian groups. A *space* is a simplicial object in the category of sheaves of sets on Sm_k [59]. We will also assume the reader is familiar with the notions and results of *loc. cit.*

We denote by Sm'_k the category of *essentially smooth k-schemes*. For us, an *essentially smooth k-scheme* is a noetherian k-scheme X which is the inverse limit of a left filtering system $(X_\alpha)_\alpha$ with each transition morphism $X_\beta \to X_\alpha$ being an étale affine morphism between smooth k-schemes (see [29]).

For any $F \in \mathcal{F}_k$ the k-scheme $Spec(F)$ is essentially k-smooth. For each point $x \in X \in Sm_k$, the local scheme $X_x := Spec(\mathcal{O}_{X,x})$ of X at x as well as its henselization $X_x^h := Spec(\mathcal{O}_{X,x}^h)$ are essentially smooth k-schemes.

In the same way the complement of the closed point in $Spec(\mathcal{O}_{X,x})$ or X_x^h is essentially smooth over k. We will sometime make the abuse of writing “smooth k-scheme” instead of “essentially smooth k-scheme”, if no confusion can arise.

Given a presheaf of sets on Sm_k, that is to say a functor $F:(Sm_k)^{op} \to Sets$, and an essentially smooth k-scheme $X = lim_\alpha X_\alpha$ we set $F(X) := colimit_\alpha F(X_\alpha)$. From the results of [29] this is well defined. When $X = Spec(A)$ is affine we will also simply denote this set by $F(A)$.

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