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196 The Theory of Hardy's *Z*-Function

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G. H. Hardy, 1877-1947

The Theory of Hardy's Z-Function

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Preface

This text has grown out of a mini-course held at the Arctic Number Theory School, University of Helsinki, May 18-25, 2011. The central topic is Hardy's function $Z(t)$, of great importance in the theory of the Riemann zeta-function $\zeta(s)$. It is named after Godfrey Harold ("G. H.") Hardy FRS (1877-1947), who was a prominent English mathematician, well-known for his achievements in number theory and mathematical analysis. Sometimes by Hardy function(s) one denotes the element(s) of Hardy spaces H^p, which are certain spaces of holomorphic functions on the unit disk or the upper half-plane. In this text, however, Hardy's function $Z(t)$ will always denote the function defined by (0) below. It was chosen as the object of study because of its significance in the theory of $\zeta(s)$ and because, initially, considerable material could be presented on the blackboard within the framework of six lectures. Some results, like Theorem 6.7 and the bounds in (4.25) and (4.26) are new, improving on older ones. It is "Hardy's function" which is the thread that holds this work together. I have thought it is appropriate for a monograph because the topic is not as vast as the topic of the Riemann zeta-function itself. Moreover, specialized monographs such as [Iv4], [Lau5], [Mot5] and [Ram] cover in detail other parts of zeta-function theory, but not $Z(t)$. It appears that one volume cannot suffice today to cover all the relevant material concerning $\zeta(s)$, although some of the most important problems (such as the distribution of zeros, in particular the notorious Riemann hypothesis that all complex zeros of $\zeta(s)$ have real parts 1/2), are not settled yet. On the other hand, there seems now to be enough material on $Z(t)$ to warrant the writing of a monograph dedicated solely to it, especially since there is at present a lot of research concerning $Z(t)$ going on.

In the desire to keep the size of the text compact, I had to avoid going into detail in subjects which are connected with zeta-function theory, but are already adequately covered in the literature. These include spectral theory, random matrix theory and representation theory, among others. Proofs of some well-known results such as the first and second derivative tests for exponential integrals (Lemma 2.2 and Lemma 2.3) are omitted, while the proof of, for example, Lemma 7.2 would lead us too much astray. I have tried to include most of the material of the ongoing research on $Z(t)$, thus there is much material from papers which are just published or in print. Starting with Chapter 2, the theory of Hardy's function is systematically developed, hence the title *The Theory of Hardy's Z-Function* seems appropriate. In developing the theory, some modern tools of analytic number theory are systematically used and discussed, such as approximate functional equations, exponential sums and integrals, and integral transforms, to name just a few.

The text is intended for ambitious graduate students, Ph.D. students, and researchers in the field. The prerequisites are basically standard courses in real and complex analysis, but it will help if the reader is familiar with a general text on the Riemann zeta-function, such as the monographs [Tit3], [KaVo] and [Iv1], although efforts have been made to keep the text self-contained. Besides mathematicians, the book will hopefully appeal to physicists whose work is linked to the applications of $\zeta(s)$ in Physics.

The text offers sufficient material for a course of one semester. Except for the introductory Chapter 1, other chapters are basically independent, and various shorter courses may be given based on disposable time and the interest of students.

The material is organized as follows. In Chapter 1 we present the Riemann zeta-function and discuss some of its basic properties. Hardy's function

$$Z(t) := \zeta(\tfrac{1}{2} + it)\big(\chi(\tfrac{1}{2} + it)\big)^{-1/2}, \quad \zeta(s) = \chi(s)\zeta(1-s), \tag{0}$$

is introduced and some of its properties are presented. One of the most important aspects of $Z(t)$ is that it is a real-valued function of the real valuable t, and its zeros exactly correspond to the zeros of $\zeta(s)$ on the so-called "critical line" $\mathbb{R}\mathrm{e}\, s = 1/2$. Thus it is very convenient to work with, being real-valued, both theoretically and in conjunction with the calculation of zeta-zeros. It is in Chapter 2 that we discuss the zeros of the zeta-function on the critical line. In particular, we present a variant of Hardy's original proof that there are infinitely many zeta-zeros on $\mathbb{R}\mathrm{e}\, s = 1/2$. Besides that, we discuss Lehmer's phenomenon, namely that the Riemann hypothesis fails if $Z(t)$ has a positive local minimum or a negative local maximum (except at $t = 2.475\,75\ldots$) and the unconditional estimation of gaps between the consecutive zeros of $Z(t)$.

Chapter 3 is devoted to the Selberg class $\mathcal{S}$ of L-functions, of which $\zeta(s)$ can be thought of as a prototype, since the elements of $\mathcal{S}$ inherit the intrinsic properties of $\zeta(s)$, namely the Euler product (see (1.1)) and the functional equation (see (1.5)). Basic notions and properties of functions in $\mathcal{S}$ are presented. The class $\mathcal{S}$ (introduced by A. Selberg in 1989) is a natural generalization of $\zeta(s)$, and the modern theory of L-functions often focuses on families and classes of L-functions containing $\zeta(s)$, such as $\mathcal{S}$.

Chapter 4 is dedicated to AFEs (approximate functional equations) for $\zeta^k(s)$ when $k \in \mathbb{N}$. The AFEs are an essential tool in the theory of L-functions, approximating a Dirichlet series by a finite number of Dirichlet polynomials. For the AFEs that we treat, the accent is on the methods of proof, of which there are several. From an AFE for $\zeta^k(s)$ one then deduces an AFE for $Z^k(t)$.

In Chapter 5 we provide formulas for the derivatives $Z^{(k)}(t)$, and in Chapter 6 we deal with Gram points g_n which satisfy $\theta(g_n) = \pi n$, where $Z(t) = \mathrm{e}^{i\theta(t)}\zeta(\frac{1}{2} + it)$ and $\theta(t)$ is given by (1.19). We also discuss the so-called "Gram's law" related to Gram points.

In Chapter 7 we give formulas for the kth moment of $Z(t)$, that is, for the integral $\int_0^T Z^k(t)\,\mathrm{d}t$, where $k \in \mathbb{N}$. This is achieved by employing an AFE with smooth weights.

M. Jutila's recent formula for the oscillating function $F(T) = \int_0^T Z(t)\,\mathrm{d}t$ is given in detail in Chapter 8. It furnishes the results $F(T) = O(T^{1/4})$ and $F(T) = \Omega_\pm(T^{1/4})$, thereby determining the true order of magnitude of $F(T)$. This approach is different from the author's work and from the recent work of M. Korolev [Kor3], [Kor4], who also obtained an explicit formula for $F(T)$.

The Mellin transforms ($k \in \mathbb{N}$)

$$\mathcal{Z}_k(s) = \int_1^\infty |\zeta(\tfrac{1}{2} + ix)|^{2k} x^{-s}\,\mathrm{d}x, \quad M_k(s) = \int_1^\infty Z^k(x) x^{-s}\,\mathrm{d}x$$

form the subject of Chapter 9 and Chapter 10. In general, classical integral transforms, in particular Fourier, Laplace and Mellin transforms, play an important rôle in analytic number theory. Analytic continuation, pointwise and mean square estimates of $\mathcal{Z}_k(s)$, $\mathcal{M}_k(s)$ are given in detail, by using various techniques of analytic number theory. The importance of these functions is their connections with power moments of $|\zeta(\frac{1}{2} + it)|$, one of the most important topics of zeta-function theory.

The concluding Chapter 11 focuses on some problems involving Hardy's function and zeta moments. I hope that this will be of interest to all who want to do research in areas connected to these topics. This is in the spirit of P. Erdős's motto: "Prove and conjecture!".

Each chapter is followed by "Notes", where many references, comments and remarks are given. This seemed preferable to burdening the body of the text with footnotes, parentheses, etc. On many occasions dates of birth and death are given for the mathematicians mentioned in the text.

A full bibliography of all the relevant works mentioned in the text is to be found at the end of the text.

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Notation

Owing to the nature of this text, absolute consistency in notation could not be attained, although whenever possible standard notation is used. Notation used commonly through the text is explained there, while specific notation introduced in the proof of a theorem or lemma is given at the proper place in the body of the text.

$k, l, m, n, \dots$	Natural numbers (positive integers).
p	A generic prime number.
$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$	The sets of natural numbers, integers, real and complex numbers, respectively.
$A, B, C, C_1, \dots$	Absolute, positive constants (not necessarily the same ones at each occurrence).
ε	An arbitrarily small positive number, not necessarily the same one at each occurrence.
s, z, w	Complex variables ($\mathbb{R}\mathrm{e}\, s$ and $\mathbb{I}\mathrm{m}\, s$ denote the real and imaginary part of s, respectively; common notation is $\sigma = \mathbb{R}\mathrm{e}\, s$ and $t = \mathbb{I}\mathrm{m}\, s$).
t, x, y	Real variables.
$\operatorname*{Res}_{s=s_0} F(s)$	Denotes the residue of $F(s)$ at the point $s = s_0$.
$\zeta(s)$	The Riemann zeta-function is defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\mathbb{R}\mathrm{e}\, s > 1$ and otherwise by analytic continuation.
$\Gamma(s)$	$= \int_0^\infty x^{s-1} \mathrm{e}^{-x}\, \mathrm{d}x$ for $\mathbb{R}\mathrm{e}\, s > 0$, otherwise by analytic continuation by $s\Gamma(s) = \Gamma(s+1)$. This is the Euler gamma-function.
γ	Euler's constant $\gamma = -\Gamma'(1) = 0.577\,2157\dots$.

$\chi(s)$	The function defined by $\zeta(s) = \chi(s)\zeta(1-s)$, so that by the functional equation for $\zeta(s)$ we have $\chi(s) = (2\pi)^s/(2\Gamma(s)\cos(\pi s/2))$.
$Z(t)$	Hardy's function, $Z(t) = \zeta(\frac{1}{2}+it)(\chi(\frac{1}{2}+it))^{-1/2}$.
$\theta(t)$	For real t defined as $\theta(t) = \mathbb{I}\mathrm{m}\,\left\{\log\Gamma(\frac{1}{4}+\frac{1}{2}it)\right\} - \frac{1}{2}t\log\pi$.
$\rho = \beta + i\gamma$	A complex zero of $\zeta(s)$; $\beta = \mathbb{R}\mathrm{e}\,\rho$, $\gamma = \mathbb{I}\mathrm{m}\,\rho$.
$N(T)$	The number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, counted with multiplicities, for which $0 < \gamma \le T$.
$S(T)$	$= \frac{1}{\pi}\arg\zeta(\frac{1}{2}+iT)$.
$\mu(\sigma)$	For real σ defined as $\mu(\sigma) = \limsup\limits_{t\to\infty}\frac{\log\lvert\zeta(\sigma+it)\rvert}{\log t}$.
$\exp(z)$	$= \mathrm{e}^z$.
$\mathrm{e}(z)$	$= \mathrm{e}^{2\pi iz}$.
$\log x$	$= \mathrm{Log}_{\mathrm{e}}\mathrm{x} \equiv \ln \mathrm{x}$.
$[x]$	The greatest integer not exceeding the real number x.
$\{x\}$	$= x - [x]$, the fractional part of x.
$\sum_{n\le x} f(n)$	A sum taken over all natural numbers n not exceeding x; the empty sum is defined to be equal to zero.
$\prod_j$	The product taken over all possible values of the index j; the empty product is defined to be unity.
$d_k(n)$	The number of ways n can be written as a product of $k \ge 2$ fixed factors; $d_2(n) = d(n)$ is the number of divisors of n.
$\Delta(x)$	$= \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1)$, the error term in the Dirichlet divisor problem.
$f(x) \sim g(x)$	Means $\lim_{x\to x_0} f(x)/g(x) = 1$, as $x \to x_0$, with x_0 not necessarily finite.
$f(x) = O(g(x))$	Means $\lvert f(x)\rvert \le Cg(x)$ for $x \ge x_0$ and some constant $C > 0$. Here $f(x)$ is a complex function of the real variable x and $g(x)$ is a positive function of x for $x \ge x_0$. $f(x) = O_{\alpha,\beta,\ldots}(g(x))$ means the implied constant depends on $\alpha, \beta, \ldots$.
$f(x) \ll g(x)$	Means the same as $f(x) = O(g(x))$. Likewise $f(x) \ll_{\alpha,\beta,\ldots} g(x)$ means the implied constant depends on $\alpha, \beta, \ldots$.
$f(x) \gg g(x)$	Means the same as $g(x) = O(f(x))$.
$f(x) \asymp g(x)$	Means that both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ hold.

(a, b)	Means the interval $a < x < b$.
$[a, b]$	Means the interval $a \le x \le b$.
$C^r[a, b]$	The class of functions having a continuous rth derivative in $[a, b]$.
$L^p(a, b)$	The class of measurable functions $f(x)$ such that $\int_a^b \lvert f(x)\rvert^p \, dx$ is finite.
$f(x) = o(g(x))$ as $x \to x_0$	Means $\lim_{x\to x_0} f(x)/g(x) = 0$, with x_0 possibly infinite.
$f(x) = \Omega\big(g(x)\big)$	Means that $f(x) = o(g(x))$ does not hold when $x \to \infty$.
$f(x) = \Omega_+(g(x))$	Means that there exists a suitable constant $C > 0$ such that $f(x) > Cg(x)$ holds for some arbitrarily large values of x.
$f(x) = \Omega_-(g(x))$	Means that there exists a suitable constant $C > 0$ such that $f(x) < -Cg(x)$ holds for some arbitrarily large values of x.
$f(x) = \Omega_\pm(g(x))$	Means that both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ hold.
$\int_{(c)} G(s)\, ds$	$= \lim_{T\to\infty} \int_{c-iT}^{c+iT} G(s)\, ds$.
$\mathcal{M}_k(s)$	$= \int_1^\infty Z^k(x)x^{-s}\, dx \quad (k \in \mathbb{N})$.
$\mathcal{Z}_k(s)$	$= \int_1^\infty \lvert\zeta(\tfrac12 + ix)\rvert^{2k} x^{-s}\, dx \quad (k \in \mathbb{N})$.
AFE	Means "approximate functional equation".