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Theory of Groups of Finite Order

William Burnside

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OF

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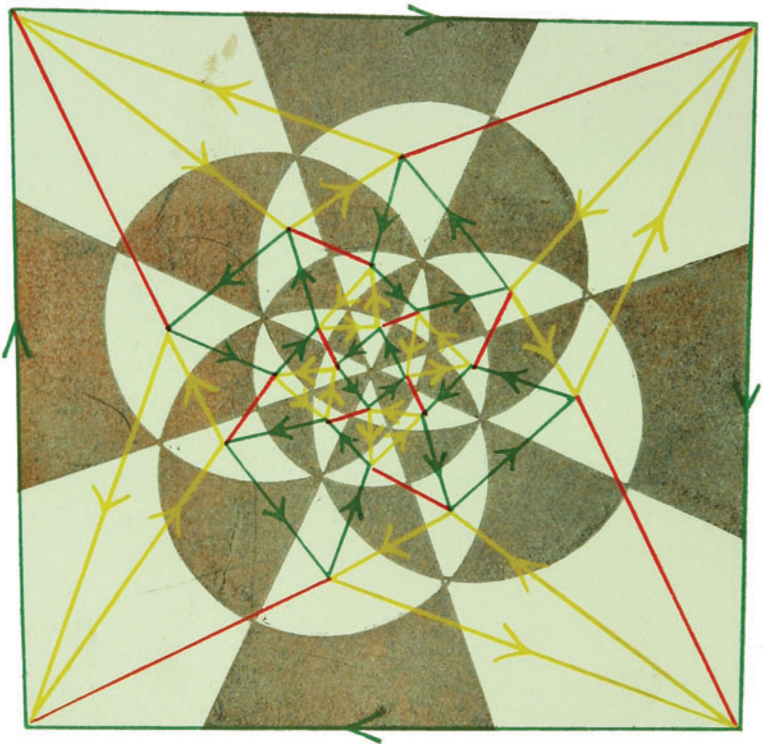

The octohedral group represented as a colour-group

THEORY OF GROUPS
OF
FINITE ORDER

BY

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SECOND EDITION

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1911

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PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is accordingly in the present edition a large amount of new matter. Five Chapters, XIII to XVII, are devoted to the theory of groups of linear substitutions, including their invariants. In Chapter IV, which is also new, certain properties of abstract groups, to which no reference was made in the first edition, are dealt with; while Chapter XII develops more completely the investigation of the earlier sections of Chapter IX of the first edition.

All the chapters dealing with the abstract theory, including that of the group of isomorphisms, have been brought together in the earlier part of the book; while from Chapter X onwards various special modes of representing a group are investigated. The last Chapter of the first edition has none to correspond to it in the present, but all results of importance which it contained are given in connections in which they naturally occur. With this exception there are no considerable changes in the matter of the first edition though there is some re-arrangement, and in places additions have been made.

A number of special questions, most of which could not have been introduced in the text without somewhat marring the scheme of the work, have been dealt with in the notes.

Some of the examples, especially in the earlier part of the book, are suitable exercises for those to whom the subject is new. The examples as a whole, however, have not been inserted with this object, but rather (i) to afford further illustration of points dealt with in the text, (ii) where references are given, to call attention to points of importance not mentioned in the text, and (iii) to suggest subjects of investigation.

A separate index to the definitions of all technical terms has been prepared which it is hoped may be of considerable service to readers.

I owe my best thanks to the Rev. Alfred Young, M.A., Rector of Birdbrook, Essex, and formerly Fellow of Clare College, Cambridge, who read the whole of the book as it passed through the press. His careful criticism has saved me from many errors and his suggestions have been of great help to me. Mr Harold Hilton, M.A., Lecturer in Mathematics at Bedford College, University of London, and formerly Fellow of Magdalen College, Oxford, gave me great assistance by reading and criticising the chapters on groups of linear substitutions; and Dr Henry Frederick Baker, F.R.S., Fellow of St John's College, Cambridge, helped me with most valuable suggestions on the chapter dealing with invariants. To both these gentlemen I offer my sincere thanks. I must further not omit to thank correspondents, both English and American, for pointing out to me errors in the first edition. All these have, I hope, been corrected.

Finally I would again express my gratitude to the officers and staff of the University Press for their courtesy and for the care with which the printing has been carried out.

W. BURNSIDE

March 1911

PREFACE TO THE FIRST EDITION

THE theory of groups of finite order may be said to date from the time of Cauchy. To him are due the first attempts at classification with a view to forming a theory from a number of isolated facts. Galois introduced into the theory the exceedingly important idea of a self-conjugate sub-group, and the corresponding division of groups into simple and composite. Moreover, by shewing that to every equation of finite degree there corresponds a group of finite order on which all the properties of the equation depend, Galois indicated how far reaching the applications of the theory might be, and thereby contributed greatly, if indirectly, to its subsequent developement.

Many additions were made, mainly by French mathematicians, during the middle part of the century. The first connected exposition of the theory was given in the third edition of M. Serret's "*Cours d'Algèbre Supérieure*," which was published in 1866. This was followed in 1870 by M Jordan's "*Traité des substitutions et des équations algébriques.*" The greater part of M. Jordan's treatise is devoted to a developement of the ideas of Galois and to their application to the theory of equations.

No considerable progress in the theory, as apart from its applications, was made till the appearance in 1872 of Herr Sylow's memoir "*Théorèmes sur les groupes de substitutions*" in the fifth volume of the *Mathematische Annalen*. Since the date of this memoir, but more especially in recent years, the theory has advanced continuously.

In 1882 appeared Herr Netto's "*Substitutionentheorie und ihre Anwendungen auf die Algebra,*" in which, as in M. Serret's and M. Jordan's works, the subject is treated entirely from the point of view of groups of substitutions. Last but not least among the works which give a detailed account of the subject must be mentioned Herr Weber's "*Lehrbuch der Algebra,*" of which the first volume appeared in 1895 and the second in 1896. In the last section of the first volume some of the more important properties of substitution groups are given. In the first section of the second volume, however, the subject is approached from a more general point of view, and a theory of finite groups is developed which is quite independent of any special mode of representing them.

The present treatise is intended to introduce to the reader the main outlines of the theory of groups of finite order apart from any applications. The subject is one which has hitherto attracted but little attention in this country; it will afford me much satisfaction if, by means of this book, I shall succeed in arousing interest among English mathematicians in a branch of pure mathematics which becomes the more fascinating the more it is studied.

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

The plan of the book is as follows. The first Chapter has been devoted to explaining the notation of substitutions. As this notation may not improbably be unfamiliar to many English readers, some such introduction is necessary to make the illustrations used in the following chapters intelligible. Chapters II to VII deal with the more important properties of groups which are independent of any special form of representation. The notation and methods of substitution groups have been rigorously excluded in the proofs and investigations contained in these chapters; for the purposes of illustration, however, the notation has been used whenever convenient. Chapters VIII to X deal with those properties of groups which depend on their representation as substitution groups. Chapter XI treats of the isomorphism of a group with itself. Here, though the properties involved are independent of the form of representation of the group, the methods of substitution groups are partially employed. Graphical modes of representing a group are considered in Chapters XII and XIII. In Chapter XIV the properties of a class of groups, of great importance in analysis, are investigated as a general illustration of the foregoing theory. The last Chapter contains a series of results in connection with the classification of groups as simple, composite, or soluble.

A few illustrative examples have been given throughout the book. As far as possible I have selected such examples as would serve to complete or continue the discussion in the text where they occur.

In addition to the works by Serret, Jordan, Netto and Weber already referred to, I have while writing this book consulted many original memoirs. Of these I may specially mention, as having been of great use to me, two by Herr Dyck published in the twentieth and twenty-second volumes of the *Mathematische Annalen* with the title "*Gruppentheoretische Studien*"; three by Herr Frobenius in the *Berliner Sitzungsberichte* for 1895 with the titles, "*Ueber endliche Gruppen,*" "*Ueber auflösbare Gruppen,*" and "*Verallgemeinerung des Sylow'schen Satzes*"; and one by Herr Hölder in the forty-sixth volume of the *Mathematische Annalen* with the title "*Bildung zusammengesetzter Gruppen.*" Whenever a result is taken from an original memoir I have given a full reference; any omission to do so that may possibly occur is due to an oversight on my part.

To Mr A. R. Forsyth, Sc.D., F.R.S., Fellow of Trinity College, Cambridge, and Sadlerian Professor of Mathematics, and to Mr G. B. Mathews, M.A., F.R.S., late Fellow of St John's College, Cambridge, and formerly Professor of Mathematics in the University of North Wales, I am under a debt of gratitude for the care and patience with which they have read the proof-sheets. Without the assistance they have so generously given me, the errors and obscurities, which I can hardly hope to have entirely escaped, would have been far more numerous. I wish to express my grateful thanks also to Prof. O. Hölder of Königsberg who very kindly read and criticized parts of the last chapter. Finally I must thank the Syndics of the University Press of Cambridge for the assistance they have rendered in the publication of the book, and the whole Staff of the Press for the painstaking and careful way in which the printing has been done.

W. BURNSIDE

July 1897

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