

Cambridge University Press  
978-1-108-05032-6 - Theory of Groups of Finite Order  
William Burnside  
Frontmatter  
[More information](#)

---

## CAMBRIDGE LIBRARY COLLECTION

*Books of enduring scholarly value*

### Mathematics

From its pre-historic roots in simple counting to the algorithms powering modern desktop computers, from the genius of Archimedes to the genius of Einstein, advances in mathematical understanding and numerical techniques have been directly responsible for creating the modern world as we know it. This series will provide a library of the most influential publications and writers on mathematics in its broadest sense. As such, it will show not only the deep roots from which modern science and technology have grown, but also the astonishing breadth of application of mathematical techniques in the humanities and social sciences, and in everyday life.

### Theory of Groups of Finite Order

The British mathematician William Burnside (1852–1927) and Ferdinand Georg Frobenius (1849–1917), Professor at Zurich and Berlin universities, are considered to be the founders of the modern theory of finite groups. Not only did Burnside prove many important theorems, but he also laid down lines of research for the next hundred years: two Fields Medals have been awarded for work on problems suggested by him. *Theory of Groups of Finite Order*, originally published in 1897, was the first major textbook on the subject. The 1911 second edition (reissued here) contains an account of Frobenius's character theory, and remained the standard reference for many years.

Cambridge University Press  
978-1-108-05032-6 - Theory of Groups of Finite Order  
William Burnside  
Frontmatter  
[More information](#)

---

Cambridge University Press has long been a pioneer in the reissuing of out-of-print titles from its own backlist, producing digital reprints of books that are still sought after by scholars and students but could not be reprinted economically using traditional technology. The Cambridge Library Collection extends this activity to a wider range of books which are still of importance to researchers and professionals, either for the source material they contain, or as landmarks in the history of their academic discipline.

Drawing from the world-renowned collections in the Cambridge University Library and other partner libraries, and guided by the advice of experts in each subject area, Cambridge University Press is using state-of-the-art scanning machines in its own Printing House to capture the content of each book selected for inclusion. The files are processed to give a consistently clear, crisp image, and the books finished to the high quality standard for which the Press is recognised around the world. The latest print-on-demand technology ensures that the books will remain available indefinitely, and that orders for single or multiple copies can quickly be supplied.

The Cambridge Library Collection brings back to life books of enduring scholarly value (including out-of-copyright works originally issued by other publishers) across a wide range of disciplines in the humanities and social sciences and in science and technology.

Cambridge University Press

978-1-108-05032-6 - Theory of Groups of Finite Order

William Burnside

Frontmatter

[More information](#)

# Theory of Groups of Finite Order

WILLIAM BURNSIDE



CAMBRIDGE  
UNIVERSITY PRESS

Cambridge University Press  
978-1-108-05032-6 - Theory of Groups of Finite Order  
William Burnside  
Frontmatter  
[More information](#)

---

CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town,  
Singapore, São Paulo, Delhi, Mexico City

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9781108050326](http://www.cambridge.org/9781108050326)

© in this compilation Cambridge University Press 2012

This edition first published 1911

This digitally printed version 2012

ISBN 978-1-108-05032-6 Paperback

This book reproduces the text of the original edition. The content and language reflect the beliefs, practices and terminology of their time, and have not been updated.

Cambridge University Press wishes to make clear that the book, unless originally published by Cambridge, is not being republished by, in association or collaboration with, or with the endorsement or approval of, the original publisher or its successors in title.

The original edition of this book contains a number of colour plates, which have been reproduced in black and white. Colour versions of these images can be found online at [www.cambridge.org/9781108050326](http://www.cambridge.org/9781108050326)

Cambridge University Press

978-1-108-05032-6 - Theory of Groups of Finite Order

William Burnside

Frontmatter

[More information](#)

# THEORY OF GROUPS OF FINITE ORDER

Cambridge University Press

978-1-108-05032-6 - Theory of Groups of Finite Order

William Burnside

Frontmatter

[More information](#)

---

CAMBRIDGE UNIVERSITY PRESS

London: FETTER LANE, E.C.

C. F. CLAY, MANAGER

Edinburgh: 100, PRINCES STREET

Berlin: A. ASHER AND CO.

Leipzig: F. A. BROCKHAUS

New York: G. P. PUTNAM'S SONS

Bombay and Calcutta: MACMILLAN AND CO., LTD.

*All rights reserved*

Cambridge University Press  
978-1-108-05032-6 - Theory of Groups of Finite Order  
William Burnside  
Frontmatter  
[More information](#)

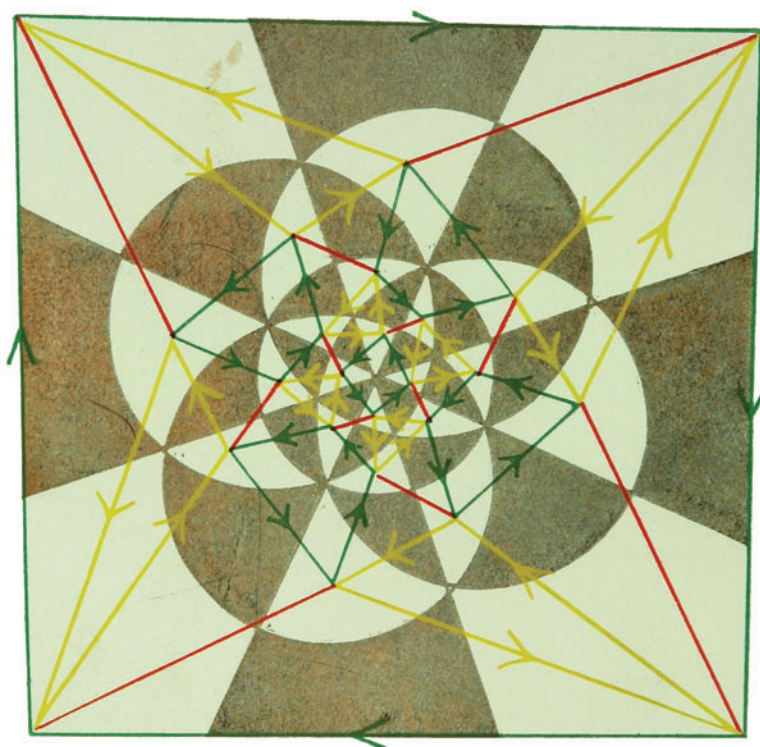
---

Cambridge University Press

978-1-108-05032-6 - Theory of Groups of Finite Order

William Burnside

Frontmatter

[More information](#)

The octohedral group represented as a colour-group

Cambridge University Press

978-1-108-05032-6 - Theory of Groups of Finite Order

William Burnside

Frontmatter

[More information](#)

# THEORY OF GROUPS OF FINITE ORDER

BY

W. BURNSIDE, M.A., F.R.S.

D.Sc. (Dublin), LL.D. (Edinburgh)

HONORARY FELLOW OF PEMBROKE COLLEGE, CAMBRIDGE

PROFESSOR OF MATHEMATICS AT THE ROYAL NAVAL COLLEGE, GREENWICH

SECOND EDITION

Cambridge  
at the University Press  
1911

Cambridge University Press

978-1-108-05032-6 - Theory of Groups of Finite Order

William Burnside

Frontmatter

[More information](#)

---

**Cambridge:**

PRINTED BY JOHN CLAY, M.A.

AT THE UNIVERSITY PRESS.

## PREFACE TO THE SECOND EDITION

VERY considerable advances in the theory of groups of finite order have been made since the appearance of the first edition of this book. In particular the theory of groups of linear substitutions has been the subject of numerous and important investigations by several writers; and the reason given in the original preface for omitting any account of it no longer holds good.

In fact it is now more true to say that for further advances in the abstract theory one must look largely to the representation of a group as a group of linear substitutions. There is accordingly in the present edition a large amount of new matter. Five Chapters, XIII to XVII, are devoted to the theory of groups of linear substitutions, including their invariants. In Chapter IV, which is also new, certain properties of abstract groups, to which no reference was made in the first edition, are dealt with; while Chapter XII develops more completely the investigation of the earlier sections of Chapter IX of the first edition.

All the chapters dealing with the abstract theory, including that of the group of isomorphisms, have been brought together in the earlier part of the book; while from Chapter X onwards various special modes of representing a group are investigated. The last Chapter of the first edition has none to correspond to it in the present, but all results of importance which it contained are given in connections in which they naturally occur. With this exception there are no considerable changes in the matter of the first edition though there is some re-arrangement, and in places additions have been made.

A number of special questions, most of which could not have been introduced in the text without somewhat marring the scheme of the work, have been dealt with in the notes.

Some of the examples, especially in the earlier part of the book, are suitable exercises for those to whom the subject is new. The examples as a whole, however, have not been inserted with this object, but rather (i) to afford further illustration of points dealt with in the text, (ii) where references are given, to call attention to points of importance not mentioned in the text, and (iii) to suggest subjects of investigation.

A separate index to the definitions of all technical terms has been prepared which it is hoped may be of considerable service to readers.

I owe my best thanks to the Rev. Alfred Young, M.A., Rector of Birdbrook, Essex, and formerly Fellow of Clare College, Cambridge, who read the whole of the book as it passed through the press. His careful criticism has saved me from many errors and his suggestions have been of great help to me. Mr Harold Hilton, M.A., Lecturer in Mathematics at Bedford College, University of London, and formerly Fellow of Magdalen College, Oxford, gave me great assistance by reading and criticising the chapters on groups of linear substitutions; and Dr Henry Frederick Baker, F.R.S., Fellow of St John's College, Cambridge, helped me with most valuable suggestions on the chapter dealing with invariants. To both these gentlemen I offer my sincere thanks. I must further not omit to thank correspondents, both English and American, for pointing out to me errors in the first edition. All these have, I hope, been corrected.

Finally I would again express my gratitude to the officers and staff of the University Press for their courtesy and for the care with which the printing has been carried out.

W. BURNSIDE

*March 1911*

## PREFACE TO THE FIRST EDITION

THE theory of groups of finite order may be said to date from the time of Cauchy. To him are due the first attempts at classification with a view to forming a theory from a number of isolated facts. Galois introduced into the theory the exceedingly important idea of a self-conjugate sub-group, and the corresponding division of groups into simple and composite. Moreover, by shewing that to every equation of finite degree there corresponds a group of finite order on which all the properties of the equation depend, Galois indicated how far reaching the applications of the theory might be, and thereby contributed greatly, if indirectly, to its subsequent developement.

Many additions were made, mainly by French mathematicians, during the middle part of the century. The first connected exposition of the theory was given in the third edition of M. Serret's "*Cours d'Algèbre Supérieure*," which was published in 1866. This was followed in 1870 by M Jordan's "*Traité des substitutions et des équations algébriques*." The greater part of M. Jordan's treatise is devoted to a development of the ideas of Galois and to their application to the theory of equations.

No considerable progress in the theory, as apart from its applications, was made till the appearance in 1872 of Herr Sylow's memoir "*Théorèmes sur les groupes de substitutions*" in the fifth volume of the *Mathematische Annalen*. Since the date of this memoir, but more especially in recent years, the theory has advanced continuously.

In 1882 appeared Herr Netto's "*Substitutionentheorie und ihre Anwendungen auf die Algebra*," in which, as in M. Serret's and M. Jordan's works, the subject is treated entirely from the point of view of groups of substitutions. Last but not least among the works which give a detailed account of the subject must be mentioned Herr Weber's "*Lehrbuch der Algebra*," of which the first volume appeared in 1895 and the second in 1896. In the last section of the first volume some of the more important properties of substitution groups are given. In the first section of the second volume, however, the subject is approached from a more general point of view, and a theory of finite groups is developed which is quite independent of any special mode of representing them.

The present treatise is intended to introduce to the reader the main outlines of the theory of groups of finite order apart from any applications. The subject is one which has hitherto attracted but little attention in this country; it will afford me much satisfaction if, by means of this book, I shall succeed in arousing interest among English mathematicians in a branch of pure mathematics which becomes the more fascinating the more it is studied.

Cayley's dictum that "a group is defined by means of the laws of combination of its symbols" would imply that, in dealing purely with the theory of groups, no more concrete mode of representation should be used than is absolutely necessary. It may then be asked why, in a book which professes to leave all applications on one side, a considerable space is devoted to substitution groups; while other particular modes of representation, such as groups of linear transformations, are not even referred to. My answer to this question is that while, in the present state of our knowledge, many results in the pure theory are arrived at most readily by dealing with properties of substitution groups, it would be difficult to find a result that could be most directly obtained by the consideration of groups of linear transformations.

The plan of the book is as follows. The first Chapter has been devoted to explaining the notation of substitutions. As

Cambridge University Press

978-1-108-05032-6 - Theory of Groups of Finite Order

William Burnside

Frontmatter

[More information](#)

## PREFACE TO THE FIRST EDITION

ix

this notation may not improbably be unfamiliar to many English readers, some such introduction is necessary to make the illustrations used in the following chapters intelligible. Chapters II to VII deal with the more important properties of groups which are independent of any special form of representation. The notation and methods of substitution groups have been rigorously excluded in the proofs and investigations contained in these chapters; for the purposes of illustration, however, the notation has been used whenever convenient. Chapters VIII to X deal with those properties of groups which depend on their representation as substitution groups. Chapter XI treats of the isomorphism of a group with itself. Here, though the properties involved are independent of the form of representation of the group, the methods of substitution groups are partially employed. Graphical modes of representing a group are considered in Chapters XII and XIII. In Chapter XIV the properties of a class of groups, of great importance in analysis, are investigated as a general illustration of the foregoing theory. The last Chapter contains a series of results in connection with the classification of groups as simple, composite, or soluble.

A few illustrative examples have been given throughout the book. As far as possible I have selected such examples as would serve to complete or continue the discussion in the text where they occur.

In addition to the works by Serret, Jordan, Netto and Weber already referred to, I have while writing this book consulted many original memoirs. Of these I may specially mention, as having been of great use to me, two by Herr Dyck published in the twentieth and twenty-second volumes of the *Mathematische Annalen* with the title "*Gruppentheoretische Studien*"; three by Herr Frobenius in the *Berliner Sitzungsberichte* for 1895 with the titles, "*Ueber endliche Gruppen*," "*Ueber auflösbare Gruppen*," and "*Verallgemeinerung des Sylow'schen Satzes*"; and one by Herr Hölder in the forty-sixth volume of the *Mathematische Annalen* with the title "*Bildung zusammengesetzter Gruppen*." Whenever a result

Cambridge University Press  
978-1-108-05032-6 - Theory of Groups of Finite Order  
William Burnside  
Frontmatter  
[More information](#)

---

x

## PREFACE TO THE FIRST EDITION

is taken from an original memoir I have given a full reference ; any omission to do so that may possibly occur is due to an oversight on my part.

To Mr A. R. Forsyth, Sc.D., F.R.S., Fellow of Trinity College, Cambridge, and Sadlerian Professor of Mathematics, and to Mr G. B. Mathews, M.A., F.R.S., late Fellow of St John's College, Cambridge, and formerly Professor of Mathematics in the University of North Wales, I am under a debt of gratitude for the care and patience with which they have read the proof-sheets. Without the assistance they have so generously given me, the errors and obscurities, which I can hardly hope to have entirely escaped, would have been far more numerous. I wish to express my grateful thanks also to Prof. O. Hölder of Königsberg who very kindly read and criticized parts of the last chapter. Finally I must thank the Syndics of the University Press of Cambridge for the assistance they have rendered in the publication of the book, and the whole Staff of the Press for the painstaking and careful way in which the printing has been done.

W. BURNSIDE

*July 1897*

CONTENTS

CHAPTER I.

ON PERMUTATIONS.

§§		PAGE
1	Object of the chapter . . . . .	1
2	Definition of a permutation . . . . .	1
3—6	Notation for permutations; cycles; products of per- mutations . . . . .	1—4
7, 8	Identical permutation; inverse permutations; order of a permutation . . . . .	4—6
9, 10	Circular, regular, similar and permutable permutations	7, 8
11	Transpositions; representation of a permutation as a product of transpositions; odd and even permuta- tions; Examples . . . . .	9, 10

CHAPTER II.

THE DEFINITION OF A GROUP.

12	Definition of a group . . . . .	11, 12
13	The identical operation . . . . .	12
14	Continuous, mixed, and discontinuous groups . . . .	13
15, 16	Order of an operation; product of operations; every operation of order $mn$ , $m$ and $n$ relatively prime, can be expressed in just one way as the product of permutable operations of orders $m$ and $n$ . . . .	14—16
17	Examples of groups of operations; multiplication table of a group . . . . .	17—19
18, 19	Generating operations of a group; defining relations; simply isomorphic groups . . . . .	20—22
20	Representation of a group of order $N$ as a group of regular permutations of $N$ symbols . . . . .	22—24
21	Various modes of representing groups . . . . .	24
		<i>b 2</i>

CHAPTER III.

ON THE SIMPLER PROPERTIES OF A GROUP WHICH ARE  
INDEPENDENT OF ITS MODE OF REPRESENTATION.

§§		PAGE
22	Sub-groups; the order of a sub group is a factor of the order of the group containing it; various notations connected with a group and its sub-groups . . .	25—27
23	Common sub-group of two groups; further notations.	27, 28
24	Transforming one operation by another; conjugate operations and sub-groups; self-conjugate operations and self-conjugate sub-groups; Abelian groups; simple and composite groups . . .	29, 20
25	The operations of a group which are permutable with a given operation or sub-group form a group . . .	31, 32
26	Complete conjugate sets of operations and sub-groups	33, 34
27	Theorems concerning self-conjugate sub-groups; maximum self-conjugate sub-groups; maximum sub-groups . . . . .	34—36
28—31	Multiply isomorphic groups; factor-groups; direct product of two groups . . . . .	37—40
32	General isomorphism between two groups . . . .	41
33, 34	Permutable groups; the group generated by two self-conjugate sub-groups of a given group; Examples	42—45

CHAPTER IV.

FURTHER PROPERTIES OF A GROUP WHICH ARE INDEPENDENT  
OF ITS MODE OF REPRESENTATION.

35	If $p^m$ ( $p$ prime) divides the order of a group, there is a sub-group of order $p^m$ . . . . .	46, 47
36	Groups of order $p^2$ and $pq$ . . . . .	48
37	The number of operations of a group of order $N$ whose $n$ th powers are conjugate to a given operation is zero or a multiple of the highest common factor of $N$ and $n$ . . . . .	49—53
38—40	Commutators; commutator sub-group or derived group; series of derived group; soluble groups; metabelian groups . . . . .	54—57
41—43	Multiplication of conjugate sets; inverse sets; self-inverse set . . . . .	57—60
44—47	Multiplication table of conjugate sets; deductions from it . . . . .	60—63

CHAPTER V.

ON THE COMPOSITION-SERIES OF A GROUP.

§§		PAGE
48	The composition-series, composition-factors and factor-groups of a given group . . . . .	64, 65
49, 50	Invariance of the factor-groups for different composition-series . . . . .	65—68
51	Chief composition-series; invariance of its factor-groups	68, 69
52, 53	Nature of the factor-groups of a chief-series; minimum self-conjugate sub-groups . . . . .	69—71
54	Construction of a composition-series to contain a given chief-series . . . . .	71
55, 56	Examples of composition-series . . . . .	72, 73
57, 58	Theorems concerning composition-series . . . . .	74, 75
59	Groups of order $p^2q$ . . . . .	76—80

CHAPTER VI.

ON THE ISOMORPHISM OF A GROUP WITH ITSELF.

60	Object of the chapter . . . . .	81
61	Definition of an isomorphism; identical isomorphism .	82
62	The group of isomorphisms of a group . . . . .	82, 83
63	Inner and outer isomorphisms; the inner isomorphisms constitute a self-conjugate sub-group of the group of isomorphisms . . . . .	84, 85
64	The holomorph of a group . . . . .	86—88
65, 66	Isomorphisms which permute the conjugate sets .	88—90
67	Permutation of sub-groups by the group of isomorphisms . . . . .	91, 92
68	Definition of a characteristic sub-group; nature of a group with no characteristic sub-group . . . . .	92
69	Characteristic-series of a group . . . . .	93
70	Definition of a complete group; a group with a complete group as a self-conjugate sub-group must be a direct product . . . . .	93, 94
71, 72	Theorems concerning complete groups . . . . .	95—97
73	The orders of certain isomorphisms; Examples . . .	97, 98

CHAPTER VII.

ON ABELIAN GROUPS.

§§		PAGE
74	Introductory . . . . .	99
75	Every Abelian group is the direct product of Abelian groups whose orders are powers of primes . . . . .	100
76	Limitation of the discussion to Abelian groups whose orders are powers of primes . . . . .	101
77	Existence of a set of independent generating operations for such a group . . . . .	101—103
78	The orders of certain sub-groups of such a group	103, 104
79, 80	Invariance of the orders of the generating opera- tions; simply isomorphic Abelian groups; symbol for Abelian group of given type . . . . .	104—106
81	Determination of all types of sub-group of a given Abelian group . . . . .	106, 107
82	Characteristic series of an Abelian group . . . . .	108, 109
83, 84	Properties of an Abelian group of type $(1, 1, \dots, 1)$	110, 111
85	The group of isomorphisms and the holomorph of such a group . . . . .	111, 112
86	The orders of the isomorphisms of an Abelian group . . . . .	112
87	The group of isomorphisms and the holomorph of any Abelian group . . . . .	113
88	The group of isomorphisms and the holomorph of a cyclical group . . . . .	114, 115
89, 90	The linear homogeneous group; Examples . . . . .	116—119

CHAPTER VIII.

ON GROUPS WHOSE ORDERS ARE THE POWERS OF PRIMES.

91	Object of the chapter . . . . .	119
92	Every group whose order is the power of a prime contains self-conjugate operations . . . . .	119
93	The series of self-conjugate sub-groups $H_1, H_2, \dots,$ $H_n, E$ , such that $H_i/H_{i+1}$ is the central of $G/H_{i+1}$ . . . . .	120
94, 95	The series of derived groups . . . . .	120, 121

CONTENTS xv

§§		PAGE
96	Every sub-group is contained self-conjugately in a sub-group of greater order . . . . .	122
97	The operations conjugate to a given operation . . . . .	123, 124
98, 99	Illustrations of preceding paragraphs . . . . .	124—126
100	Operations conjugate to powers of themselves . . . . .	126, 127
101—103	Number of sub-groups of given order is congruent to 1, mod. $p$ . . . . .	128, 129
104	Groups of order $p^m$ with a single sub-group of order $p^s$ are cyclical, where $p$ is an odd prime . . . . .	130, 131
105	Groups of order $2^m$ with a single sub-group of order $2^s$ are cyclical unless $s$ is 1, in which case there is just one other type . . . . .	131, 132
106	The quaternion group . . . . .	132, 133
107	Some characteristic sub-groups . . . . .	133, 134
108, 109	Groups of order $p^m$ with a self-conjugate cyclical sub-group of order $p^{m-1}$ . . . . .	134, 135
110, 111	Groups of order $p^m$ with a self-conjugate cyclical sub-group of order $p^{m-2}$ . . . . .	136—139
112	Distinct types of groups of orders $p^2$ and $p^3$ . . . . .	139, 140
113—116	Distinct types of groups of order $p^4$ . . . . .	140—144
117, 118	Tables of groups of orders $p^2$ , $p^3$ and $p^4$ . . . . .	144—146
119	Examples . . . . .	146—148

CHAPTER IX.

ON SYLOW'S THEOREM.

120	Proof of Sylow's theorem . . . . .	149—151
121	Generalisation of Sylow's theorem . . . . .	152
122	Theorem concerning the maximum common sub-group of two Sylow sub-groups . . . . .	153, 154
123—125	Further theorems concerning Sylow sub-groups . . . . .	154—157
126	Determination of all distinct types of group of order 24 . . . . .	157—161
127	Determination of the only group of order 60 with no self-conjugate sub-group of order 5 . . . . .	161, 162
128, 129	Groups whose Sylow sub-groups are all cyclical; their defining relations . . . . .	163—166
130	Groups with properties analogous to those of groups whose orders are powers of primes; Examples . . . . .	166, 167

CHAPTER X.

ON PERMUTATION-GROUPS: TRANSITIVE AND INTRANSITIVE  
GROUPS: PRIMITIVE AND IMPRIMITIVE GROUPS.

§§		PAGE
131	The degree of a permutation-group . . . .	168
132	The symmetric and the alternating groups . .	169
133	Transitive and intransitive groups; the degree of a transitive group is a factor of the order .	170, 171
134	Transitive groups whose permutations, except identity, permute all or all but one of the symbols . . . . .	171—173
135	Conjugate permutations are similar; self-conjugate operations and self-conjugate sub-groups of a transitive group . . . . .	173, 174
136	Transitive groups of which the order is equal to the degree. . . . .	174—176
137	Multiply transitive groups; the order of a $k$ -ply transitive group of degree $n$ is divisible by $n(n-1)\dots(n-k+1)$ . . . . .	176—178
138	Groups of degree $n$ , which do not contain the alternating group, cannot be more than $(\frac{1}{3}n+1)$ -ply transitive . . . . .	178—180
139	The alternating group of degree $n$ is simple except when $n$ is 4 . . . . .	180, 181
140, 141	Examples of doubly and triply transitive groups .	181—185
142—144	Intransitive groups; transitive constituents; the general isomorphisms between two groups .	186—189
145	Tests of transitivity . . . . .	189—191
146	Definition of primitivity and imprimitivity; im- primitive systems . . . . .	191, 192
147	Test of primitivity . . . . .	192, 193
148	Properties of imprimitive systems. . . . .	194, 195
149	Self-conjugate sub-groups of transitive groups; a self-conjugate sub-group of a primitive group must be transitive . . . . .	195, 196
150	Self-conjugate sub-groups of $k$ -ply transitive groups are in general $(k-1)$ -ply transitive . . . .	197, 198
151—154	Further theorems concerning self-conjugate sub- groups of multiply transitive groups; a group which is at least doubly transitive must, in general, either be simple or contain a simple group as a self-conjugate sub-group . . . .	198—202
155	Construction of a primitive group with an im- primitive self-conjugate sub-group . . . .	202, 203
156	Examples. . . . .	203, 204

CHAPTER XI.

ON PERMUTATION-GROUPS: TRANSITIVITY AND PRIMITIVITY:  
(CONCLUDING PROPERTIES).

§§		PAGE
157—160	Primitive groups with transitive sub-groups of smaller degree; limit to the order of a primitive group of given degree . . . . .	205—207
161	Property of the symmetric group . . . . .	208, 209
162	The symmetric group of degree $n$ is a complete group except when $n$ is 6; the group of isomorphisms of the symmetric group of degree 6 . . . . .	209, 210
163—165	Further limitations on the order of a primitive group; examples of the same . . . . .	210—214
166	Determination of all primitive groups whose degrees do not exceed 8. . . . .	214—221
167—169	Sub-groups of doubly transitive groups which leave two symbols unchanged; complete sets of triplets . . . . .	221—224
170, 171	The most general permutation-group each of whose operations is permutable with a given permutation, or with every permutation of a given group . . . . .	224—227
172	The most general transitive group whose order is the power of a prime . . . . .	227—229
173	Examples . . . . .	229, 230

CHAPTER XII.

ON THE REPRESENTATION OF A GROUP OF FINITE ORDER AS  
A PERMUTATION-GROUP.

174	Definition of representation; equivalent and distinct representations . . . . .	231, 232
175	The two representations of a group as a regular permutation-group given by pre- and post-multiplication are equivalent . . . . .	232
176	The imprimitive systems in the representation of a group as a regular permutation-group . . . . .	232, 233

xviii	CONTENTS	
§§		PAGE
177—179	To each conjugate set of sub-groups there corresponds a transitive representation; every transitive representation arises in this way .	233—236
180, 181	The mark of a sub-group in a representation; table of marks; there are just $s$ distinct representations, where $s$ is the number of distinct conjugate sets of sub-groups . . .	236—238
182, 183	The same set of permutations may give two or more distinct representations; connection with outer isomorphisms . . . . .	239
184, 185	Composition of representations; number of representations of given degree . . . . .	240, 241
186	A more general definition of equivalence . . . . .	241, 242
187	Alternative process for setting up representations	242

CHAPTER XIII.

ON GROUPS OF LINEAR SUBSTITUTIONS; REDUCIBLE AND IRREDUCIBLE GROUPS.

188, 189	Linear substitutions; their determinants; groups of linear substitutions . . . . .	243—245
190	Transposed groups of linear substitutions; conjugate groups of linear substitutions; generalisation . . . . .	245—247
191	Composition of isomorphic groups of linear substitutions . . . . .	247—249
192	Characteristic equation of a substitution; characteristic of a substitution . . . . .	249, 250
193	Canonical form of a linear substitution of finite order . . . . .	251, 252
194	Definition of an Hermitian form; definite forms; properties of a definite form . . . . .	253—255
195	Existence of a definite Hermitian form which is invariant for a group and its conjugate . . . . .	255, 256
196	Standard form for a group of linear substitutions of finite order . . . . .	256, 257
197	Reducible and irreducible groups of linear substitutions; completely reducible groups . . . . .	258
198—200	A group of linear substitutions of finite order is either irreducible or completely reducible . . . . .	259—263

CONTENTS xix

§§		PAGE
201	Proof of the preceding result when the coefficients are limited to a given algebraic field . . . . .	264, 265
202	Substitutions permutable with every substitution of an irreducible group . . . . .	265, 266
203	The group of linear substitutions permutable with every substitution of a given group of linear substitutions; Examples; Note . . . . .	266—268

CHAPTER XIV.

ON THE REPRESENTATION OF A GROUP OF FINITE ORDER AS  
A GROUP OF LINEAR SUBSTITUTIONS.

204	Definition of a representation; distinct and equivalent representations; Examples . . . . .	269—271
205	The identical representation; irreducible components of a representation; reduced variables . . . . .	271, 272
206	The number of linearly independent invariant Hermitian forms for a representation and its conjugate . . . . .	272, 273
207	The completely reduced form of any representation of a group as a transitive permutation-group . . . . .	273—275
208	The completely reduced form of the representation of a group as a regular permutation-group; all the irreducible representations occur in it; the number of distinct irreducible representations is equal to the number of conjugate sets . . . . .	276—278
209, 210	Irreducible representations with which the group is multiply isomorphic; irreducible representations in a single symbol . . . . .	278, 279

CHAPTER XV.

ON GROUP-CHARACTERISTICS.

211	Explanation of the notation . . . . .	280, 281
212	Set of group-characteristics; in conjugate representations corresponding group-characteristics are conjugate imaginaries . . . . .	281, 282

xx	CONTENTS	
§§		PAGE
213—215	Proof of relations between the sets of group-characteristics . . . . .	283—287
216	Two representations of a group are equivalent if, and only if, they have the same group-characteristics . . . . .	287, 288
217	Relations between the representation of a group as a transitive permutation-group when the more general definition of equivalence is used . . . . .	288, 289
218	Further relations between the group-characteristics; table of relations . . . . .	290, 291
219, 220	Composition of the irreducible representations . . . . .	291—293
221	Two distinct conjugate sets cannot have the same characteristic in every representation . . . . .	293
222	Case of groups of odd order . . . . .	294, 295
223, 224	Determination of the characteristics from the multiplication table of the conjugate sets; Example . . . . .	295—297
225	The number of variables operated on by an irreducible representation is a factor of the order of the group . . . . .	297, 298
226	Property of set of irreducible representations which combine among themselves by composition . . . . .	298—300
227	Completely reduced form of the group on the homogeneous products of the variables operated on by a group of linear substitutions . . . . .	300, 301
228	The irreducible representation and conjugate sets of a factor-group . . . . .	301, 302
229—231	The reduction of a regular permutation-group; the complete reduction of the general group $\{G, G'\}$ of § 136 . . . . .	302—307
232	The representation of the simple groups of orders 60 and 168 as irreducible groups in 3 variables . . . . .	307—311
233	Nature of the coefficients in a group of linear substitutions of finite order . . . . .	311
234	Families of irreducible representations; the number of families is equal to the number of distinct conjugate sets of cyclical sub-groups . . . . .	311—314
235	The characteristics of a family of representations . . . . .	314, 315
236	Invariant property of the multiplication table of conjugate sets . . . . .	316, 317
237	Similar invariant property of the composition table of the irreducible representations . . . . .	317, 318
238	Examples . . . . .	318—320

CHAPTER XVI.

SOME APPLICATIONS OF THE THEORY OF GROUPS OF LINEAR  
SUBSTITUTIONS AND OF GROUP-CHARACTERISTICS.

§§		PAGE
239	Introductory . . . . .	321
240, 241	Groups of order $p^a q^b$ are soluble . . . . .	321—323
242	Representation of a group as a group of monomial substitutions . . . . .	324, 325
243	Application of this representation to obtain con- ditions for the existence of self-conjugate sub-groups . . . . .	325, 326
244	Particular cases; a group whose order is not divisible by 12 or by the cube of a prime is soluble . . . . .	327, 328
245	Further particular cases; the order, if even, of a simple group is divisible by 12, 16 or 56 .	328—330
246	Relations between the characteristics of a group and those of any sub-group . . . . .	330, 331
247	A transitive permutation-group whose operations permute all or all but one of the symbols has a regular self-conjugate sub-group . .	331—334
248	Groups of isomorphisms which leave $E$ only un- changed . . . . .	334—336
249	Isomorphisms which change each conjugate set into itself . . . . .	336—338
250	The irreducible components of a transitive per- mutation-group . . . . .	338, 339
251	Simply transitive groups of prime degree are soluble . . . . .	339—341
252	Generalisation of preceding theorem . . . . .	341—343
253	On the result of compounding an irreducible group with itself; some properties of groups of odd order . . . . .	343—345
254	Criterion for the existence of operations of com- posite order . . . . .	346, 347
255	On certain Abelian sub-groups of irreducible groups . . . . .	348
256, 257	Congruences between characteristics which indicate the existence of self-conjugate sub-groups; illustrations . . . . .	349—351
258	Every irreducible representation of a group whose order is the power of a prime can be expressed as a group of monomial substitutions . .	351, 352
259	Examples . . . . .	353, 354

CHAPTER XVII.

ON THE INVARIANTS OF GROUPS OF LINEAR SUBSTITUTIONS.

§§		PAGE
260, 261	Definition of invariants and relative invariants; condition for existence of relative invariants; invariant in the form of a rational fraction .	355—357
262	Existence of an algebraically independent set of invariants . . . . .	357
263	Formation, for a group in $n$ variables, of a set of $n+1$ invariants in terms of which all in- variants are rationally expressible . . . .	357—359
264	On the possibility of replacing the above set of $n+1$ invariants by a set of $n$ . . . .	360
265	The group of linear substitutions for which each of a given set of functions is invariant . .	360, 361
266—268	Examples of sets of invariants for certain special groups . . . . .	362—366
269	Property of invariants of an irreducible group .	366, 367
270	Condition that an irreducible group may have a quadratic invariant . . . . .	367, 368
271	General remarks on the relation of a group to its invariants . . . . .	369
272	Examples . . . . .	370, 371

CHAPTER XVIII.

ON THE GRAPHICAL REPRESENTATION OF A GROUP.

273	Introductory remarks . . . . .	372
274, 275	The most general discontinuous group that can be generated by a finite number of opera- tions; the relation of this group to the special group that arises when one or more relations hold between the generating operations . .	373—376
276	Graphical representation of a cyclical group .	376—379
277—280	Graphical representation of the general group, when no relations connect the generating operations . . . . .	379—384
281, 282	Graphical representation of the special group when relations connect the generating operations .	384—386
283	Illustration of the preceding paragraphs . .	386—389
284—286	Graphical representation of the special group when the generating operations are of finite order .	389—394

CONTENTS xxiii

§§		PAGE
287, 288	Graphical representation of a group of finite order	394—396
289	The genus of a group . . . . .	397
290, 291	Limitation on the order and on the number of defining relations of a group of given genus: Examples; Note . . . . .	398—401

CHAPTER XIX.

ON THE GRAPHICAL REPRESENTATION OF GROUPS: GROUPS OF  
GENUS ZERO AND UNITY: CAYLEY'S COLOUR-GROUPS.

292	The diophantine relation connecting the order, the genus, and the number and orders of the generating operations . . . . .	402, 403
293—296	Groups of genus zero: their defining relations and graphical representation; the dihedral, tetra- hedral, octahedral and icosahedral groups . .	403—409
297—302	Groups of genus unity: their defining relations and graphical representation; groups of genus two . . . . .	410—419
303	The graphical representation of the simple group of order 168; deduction of its defining re- lations . . . . .	419—422
304—307	Cayley's colour-groups . . . . .	423—427

CHAPTER XX.

ON CONGRUENCE GROUPS.

308	Object of the chapter: the homogeneous linear group . . . . .	428, 429
309, 310	Its sub-group constituted by the operations of determinant unity; its self-conjugate opera- tions . . . . .	429—431
311—313	Its self-conjugate sub-groups; its composition- factors; the simple group defined by it . .	431—434
314	The case $n=2$ ; the fractional linear group . .	434—436
315—320	The distribution of its operations in conjugate sets	436—442
321—324	Tetrahedral, octahedral and icosahedral sub-groups contained in it. . . . .	442—447
325—327	Enumeration of all types of sub-groups contained in it . . . . .	447—451
328	Generalisation of the fractional linear group . .	451, 452

xxiv	CONTENTS	
§§		PAGE
329, 330	Representation of the simple group defined by the linear homogeneous group as a doubly transitive permutation-group . . . . .	452—455
331	Special cases of the linear homogeneous group; simple isomorphism between the alternating group of degree 8 and the group of isomorphisms of an Abelian group of order 16 and type (1, 1, 1, 1). . . . .	455—457
332, 333	Generalisation of the homogeneous linear group; Examples . . . . .	457—459
NOTE A.	On the equation $N = h_1 + h_2 + \dots + h_r$ . . . . .	461
NOTE B.	On the group of isomorphisms of a group . . . . .	463
NOTE C.	On the symmetric group . . . . .	464
NOTE D.	On the completely reduced form of a group of monomial substitutions . . . . .	470
NOTE E.	On the irreducible representations of a group which has a self-conjugate sub-group of prime index . . . . .	472
NOTE F.	On groups of finite order which are simply isomorphic with irreducible groups of linear substitutions . . . . .	476
NOTE G.	On the representation of a group of finite order as a group of linear substitutions with rational coefficients . . . . .	479
NOTE H.	On the group of the twenty-seven lines on a cubic surface . . . . .	485
NOTE I.	On the conditions of reducibility of a group of linear substitutions of finite order . . . . .	489
NOTE J.	On conditions for the finiteness of the order of a group of linear substitutions . . . . .	491
NOTE K.	On the representation of a group of finite order as a group of birational transformations of an algebraic curve . . . . .	496
NOTE L.	On the group-characteristics of the fractional linear group . . . . .	499
NOTE M.	On groups of odd order . . . . .	503
NOTE N.	On the orders of simple groups . . . . .	504
NOTE O.	On algebraic numbers . . . . .	505
	INDEX OF TECHNICAL TERMS . . . . .	507
	INDEX OF AUTHORS QUOTED . . . . .	508
	GENERAL INDEX . . . . .	509