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Nonlinear Partial Differential Equations with Applications

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Preface

The theoretical foundations of differential equations have been significantly developed, especially during the 20th century. This growth can be attributed to fast and successful development of supporting mathematical disciplines (such as functional analysis, measure theory, and function spaces) as well as to an ever-growing call for applications especially in engineering, science, and medicine, and ever better possibility to solve more and more complicated problems on computers due to constantly growing hardware efficiency as well as development of more efficient numerical algorithms.

A great number of applications involve distributed-parameter systems (which can be, in particular, described by *partial differential equations*[1]) often involving various nonlinearities. This book focuses on the theory of such equations with the aim of bringing it as fast as possible to a stage applicable to real-world tasks. This competition between rigor and applicability naturally needs many compromises to keep the scope reasonable. As a result (or, conversely, the reason for it) the book is primarily meant for graduate or PhD students in programs such as mathematical modelling or applied mathematics. Although some preliminary knowledge of modern methods in linear partial differential equations is useful, the book is basically self-contained if the reader consults Chapter 1 where auxiliary material is briefly presented without proofs.

The prototype tasks addressed in this book are boundary-value problems for *semilinear*[2] equations of the type

$$-\Delta u + c(u) = g, \qquad \text{or more general} \qquad -\operatorname{div}\big(\kappa(u)\nabla u\big) + c(u) = g, \tag{0.1}$$

or, still more general, for *quasilinear*[3] equations of the type

$$-\operatorname{div}\big(a(u,\nabla u)\big) + c(u,\nabla u) = g, \tag{0.2}$$

[1]The adjective "partial" refers to occurrence of partial derivatives.

[2]In this book the adjective "semilinear" will refer to equations where the highest derivatives stand linearly and the induced mappings on function spaces are weakly continuous.

[3]The adjective "quasilinear" refers to equations where the highest derivatives occur linearly but multiplied by functions containing lower-order derivatives, which means here the form $-\sum_{i,j=1}^n a_{ij}(x,u,\nabla u)\partial^2 u/\partial x_i \partial x_j + c(x,u,\nabla u) = g$. After applying the chain rule, one can see that (0.2) is only a special case, namely an equation in the so-called divergence form.

and various generalizations of those equations, in particular variational inequalities. Furthermore, systems of such equations are treated with emphasis on various real-world applications in (thermo)mechanics of solids and fluids, in electrical devices, engineering, chemistry, biology, etc. These applications are contained in Part I.

Part II addresses evolution variants of previously treated boundary-value problems like, in case of (0.2),[4]

$$\frac{\partial u}{\partial t} - \mathrm{div}\big(a(u,\nabla u)\big) + c(u,\nabla u) = g, \tag{0.3}$$

completed naturally by boundary conditions and initial or periodic conditions.

Let us emphasize that our restriction on the quasilinear equations (or inequalities) in the divergence form is not severe from the viewpoint of applications. However, in addition to fully nonlinear equations of the type $a(\Delta u) = g$ or $\frac{\partial}{\partial t}u + a(\Delta u) = g$, topics like problems on unbounded domains, homogenization, detailed qualitative aspects (asymptotic behaviour, attractors, blow-up, multiplicity of solutions, bifurcations, etc.) and, except for a few remarks, hyperbolic equations are omitted.

In particular cases, we aim primarily at formulation of a suitable definition of a solution and methods to prove existence, uniqueness, stability or regularity of the solution.[5] Hence, the book balances the presentation of general methods and concrete problems. This dichotomy results in two levels of discourse interacting with each other throughout the book:

- abstract approach – can be explained systematically and lucidly, has its own interest and beauty, but has only an auxiliary (and not always optimal) character from the viewpoint of partial differential equations themselves,
- targeted concrete partial differential equations – usually requires many technicalities, finely fitted with particular situations and often not lucid.

The addressed methods of general purpose can be sorted as follows:

- ◦ indirect in a broader sense: construction of auxiliary approximate problems easier to solve (e.g. Rothe method, Galerkin method, penalization, regularization), then a-priori estimates and a limit passage;
- ◦ direct in a broader sense: reformulation of the differential equation or inequality into a problem solvable directly by usage of abstract theoretical results, e.g. potential problems, minimization by compactness arguments;
- ◦ iterational: fixed points, e.g. Banach or Schauder's theorems;

[4] In fact, a nonlinear term of the type $c(u)\frac{\partial}{\partial t}u$ can easily be considered in (0.3) instead of $\frac{\partial}{\partial t}u$; see p. 277 for a transformation to (0.3) or Sect. 11.2 for a direct treatment. Besides, nonlinearity like $C(\frac{\partial}{\partial t}u)$ will be considered, too; cf. Sect. 11.1.1 or 11.1.2.

[5] To complete the usual mathematical-modelling procedure, this scheme should be preceded by a formulation of the model, and followed by numerical approximations, numerical analysis, with computer implementation and graphic visualization. Such, much broader ambitions are not addressed in this book, however.

We make the general observation that simple problems usually allow several approaches while more difficult problems require sophisticated combination of many methods, and some problems remain even unsolved.

The material in this book is organized in such a way that some material can be skipped without losing consistency. At this point, Table 1 can give a hint:

	steady-state	evolution
basic minimal scenario	Chapters 2,4	Chapter 7, Sect. 8.1–8.8
variational inequalities	Chapter 5	Chapter 10
accretive setting	Chapter 3	Chapter 9
systems of equations	Chapters 6	Chapter 12, Sect. 9.5
some special topics	—	Sect. 8.9–8.10, Chapter 11
auxiliary summary of general tools	Chapter 1	

Table 1. *General organization of this book.*

Except for the basic minimal scenario, the rest can be combined (or omitted) quite arbitrarily, assuming that the evolution topics will be accompanied by the corresponding steady-state part. Most chapters are equipped with exercises whose solution is mostly sketched in footnotes. Suggestions for further reading as well as some historical comments are in biographical notes at the ends of the chapters.

The book reflects both my experience with graduate classes I taught in the program "Mathematical modelling" at Charles University in Prague during 1996–2005[6] and my own research[7] and computational activity in this area during the past (nearly three) decades, as well as my electrical-engineering background and research contacts with physicists and material scientists. My thanks and deep

[6]In the usual European 2-term organization of an academic year, a natural schedule was Part I (steady-state problems) for one term and Part II (evolution problems) for the other term. Yet, only a selection of about 60% of the material was possible to expose (and partly accompanied by exercises) during a 3-hour load per week for graduate- or PhD-level students. Occasionally, I also organized one-term special "accretive-method" course based on Chapters 3 and 9 only.

[7]It concerns in particular a research under the grants 201/03/0934 (GA ČR), IAA 1075402 (GA AV ČR), and MSM 21620839 (MŠMT ČR) whose support is acknowledged.

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Praha, 2005 T.R.

Preface to the 2nd edition

Although the core of the book is identical with the 1st edition, at particular spots this new edition modifies and expands it quite considerably, reflecting partly the further research of my own[8] and my colleagues, as well as a feedback from continuation of classes in the program "Mathematical modelling" at Charles University in Prague during 2005–2012, partly executed also by my colleague, M. Bulíček.

More specifically, the main changes are as follows: On an abstract level, Rothe's method has been improved by using a finer discrete Gronwall inequality and by refining some estimates to work if the governing potential is only semiconvex, as well as various semi-implicit variants have been added. Also the presentation of Galerkin's method in Sect. 8.4 has been simplified. Morever, needless to list, some particular assertions have been strengthened or their proofs simplified.

On the level of concrete partial differential equations, boundary conditions in higher-order equations in Section 2.4.4 are now more elaborated. Interpolation by using Gagliardo-Nirenberg inequality has been applied more systematically; in particular the exponent $p^{\circledast}$ has been defined more meticuously and a new "boundary" exponent $p^{\#}$ has been introduced and used in a modified presentation of the parabolic equations in Sect. 8.6. Interpolation has also been exploited in new estimates especially in examples of thermally coupled systems which have been more elaborated or completely rewritten, cf. Sects. 6.2 and 12.1, and even some newer ones have been added, cf. Sects. 12.7–12.9. Other issues concern e.g. newly added singularly-perturbed problems, positivity of solutions (typically temperature in the heat equation), Navier's boundary conditions, etc.

Moreover, some rather local augments have been implemented. Various exercises have been expanded or added and some new applications have been involved. Also the list of references has been expanded accordingly. Of course, various typos or mistakes have been corrected, too.

Last but not least, it should be emphasized that inspiring discussions with M. Bulíček, J. Málek, J. Malý, P. Podio-Guidugli, U. Stefanelli, and G. Tomassetti have thankfully been reflected in this 2nd edition.

Praha, 2012 T.R.

[8]In particular it concerns the GA ČR-projects 201/09/0917, 201/10/0357, and 201/12/0671.