Preface

During the last two or three centuries, most of the developments in science (in particular in Physics and Applied Mathematics) have been founded on the use of classical algebraic structures, namely groups, rings and fields. However many situations can be found for which those usual algebraic structures do not necessarily provide the most appropriate tools for modeling and problem solving. The case of arithmetic provides a typical example: the set of nonnegative integers endowed with ordinary addition and multiplication does not enjoy the properties of a field, nor even those of a ring.

A more involved example concerns *Hamilton–Jacobi* equations in Physics, which may be interpreted as optimality conditions associated with a *variational principle* (for instance, the Fermat principle in Optics, the 'Minimum Action' principle of Maupertuis, etc.). The discretized version of this type of variational problems corresponds to the well-known *shortest path problem in a graph*. By using Bellmann's optimality principle, the equations which define a solution to the shortest path problem, which are *nonlinear* in usual algebra, may be written as a *linear system* in the algebraic structure ($\mathbb{R} \cup \{+\infty\}$, Min, $+$), i.e. the set of reals endowed with the operation Min (minimum of two numbers) in place of addition, and the operation $+$ (sum of two numbers) in place of multiplication.

Such an algebraic structure has properties quite different from those of the field of real numbers. Indeed, since the elements of $E = \mathbb{R} \cup \{+\infty\}$ do not have inverses for $\oplus = \text{Min}$, this internal operation does not induce the structure of a group on E. In that respect $(E, \oplus, \otimes)$ will have to be considered as an example of a more primitive algebraic structure as compared with fields, or even rings, and will be referred to as a *semiring*.

But this example is also representative of a particular class of semirings, for which the monoid $(E, \oplus)$ is *ordered* by the order relation $\propto$ (referred to as 'canonical') defined as:

$$a \propto b \Leftrightarrow \exists\, c \in E \quad \text{such that} \quad b = a \oplus c.$$

In view of this, $(E, \oplus, \otimes)$ has the structure of a *canonically ordered semiring* which will be called, throughout this book, a *dioid*.

More generally, it is to be observed here that the operations Max and Min, which give the set of the reals a structure of canonically ordered monoid, come rather naturally into play in connection with algebraic models for many problems, thus leading to as many applications of dioid structures. Among some of the most characteristic examples, we mention:

– The dioids ($\mathbb{R}$, Min, $+$) and ($\mathbb{R}$, Max, Min) which provide natural models for the *shortest path problem* and for the *maximum capacity path problem* respectively (the latter being closely related to *the maximum weight spanning tree* problem). Many other path-finding problems in graphs, corresponding to other types of dioids, will be studied throughout the book;
– The dioid ({0,1}, Max, Min) or *Boolean Algebra*, which is the algebraic structure underlying *logic*, and which, among other things, is the basis for modeling and solving *connectivity problems* in graphs;
– The dioid ($P(A^*)$, $\cup$, o), where $P(A^*)$ is the set of all languages on the alphabet A, endowed with the operations of union $\cup$ and concatenation o, which is at the basis of the theory of languages and automata.

One of the primary objectives of this volume is precisely, on the one hand, to emphasize the deep relations existing between the semiring and dioid structures with graphs and their combinatorial properties; and, on the other hand, to show the capability and flexibility of these structures from the point of view of *modeling and solving problems* in extremely diverse situations. If one considers the many possibilities of constructing new dioids starting from a few reference dioids (vectors, matrices, polynomials, formal series, etc.), it is true to say that the reader will find here an almost unlimited source of examples, many of which being related to applications of major importance:

– Solution of a wide variety of optimal path problems in graphs (Chap. 4, Sect. 6);
– Extensions of classical algorithms for shortest path problems to a whole class of nonclassical path-finding problems (such as: shortest paths with time constraints, shortest paths with time-dependent lengths on the arcs, etc.), cf. Chap. 4, Sect. 4.4;
– Data Analysis techniques, hierarchical clustering and preference analysis (cf. Chap. 6, Sect. 6);
– Algebraic modeling of fuzziness and uncertainty (Chap. 1, Sect. 3.2 and Exercise 2);
– Discrete event systems in automation (Chap. 6, Sect. 7);
– Solution of various nonlinear partial differential equations, such as: Hamilton–Jacobi, and Bürgers equations, the importance of which is well-known in Physics (Chap. 7).

And, among all these examples, the alert reader will recognize the most widely known, and the most elementary mathematical object, the dioid of natural numbers: ***At the start, was the dioid N!***

Besides its emphasis on models and illustration by examples, the present book is also intended as an extensive overview of the mathematical properties enjoyed by these “nonclassical” algebraic structures, which either extend usual algebra (as for the case of pre-semirings or semirings), or (as for the case of dioids) correspond to a new branch of algebra, clearly distinct from the one concerned with the classical structures of groups, rings and fields.

Indeed, a simple, though essential, result (which will be discussed in the first chapter) states that a monoid cannot simultaneously enjoy the properties of being a group and of being canonically ordered. Hence the algebra for sets endowed with two internal operations turns out to split into two disjoint branches, according to which of the following two (incompatible) assumptions holds:

- The "additive group" property, which leads to the structures of ring and of field;
- The "canonical order" property, which leads to the structures of dioid and of lattice.

For dioids, one of the immediate consequences of dropping the property of invertibility of addition to replace it by the canonical order property, is the need of considering *pairs of elements* instead of individual elements, to avoid the use of "negative" elements. Modulo this change in perspective, it will be seen how many basic results of usual algebra can be transposed. Consider, for instance, the properties involving the determinant of a square $n \times n$ matrix. In dioids (as well as in general semirings), the standard definition of the determinant cannot be used anymore, but we can define the *bideterminant* of $A = (a_{i,j})$ as the pair $(\det^+(A), \det^-(A))$, where $\det^+(A)$ denotes the sum of the weights of even permutations, and $\det^-(A)$ the sum of the weights of odd permutations of the elements of the matrix. For a matrix with a set of linearly dependent columns, the condition of zero determinant is then replaced by equality of the two terms of the bideterminant:

$$\det^+(A) = \det^-(A).$$

In a similar way, the concept of characteristic polynomial $P_A(\lambda)$ of a given matrix A, has to be replaced by the *characteristic bipolynomial*, in other words, by a pair of polynomials $({P_A}^+(\lambda), {P_A}^-(\lambda))$. Among other remarkable properties, it is then possible to transpose and generalize in dioids and in semirings, the famous Cayley–Hamilton theorem, $P_A(A) = 0$, by the matrix identity:

$${P_A}^+(A) = {P_A}^-(A).$$

Another interesting example concerns the classical Perron–Frobenius theorem. This result, which states the existence on $\mathbb{R}_+$ of an eigenvalue and an eigenvector for a nonnegative square matrix, may be viewed as a property of the dioid $(\mathbb{R}_+, +, \times)$, thus opening the way to extensions to many other dioids. Incidentally we observe that it is precisely this dioid $(\mathbb{R}_+, +, \times)$ which forms the truly appropriate underlying structure for measure theory and probability theory, rather than the field of real numbers $(\mathbb{R}, +, \times)$.

One of the ambitions of this book is thus to show that, as complements to usual algebra, based on the construct "Group-Ring-Field", other algebraic structures based on alternative constructs, such as "Canonically ordered monoid- dioid- distributive lattice" are equally interesting and rich, both in terms of mathematical properties and of applications.

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