Preface

The domain of ALGEBRAIC GEOMETRY is a fascinating branch of Mathematics that combines methods from ALGEBRA and GEOMETRY. In fact, it transcends the limited scope of pure ALGEBRA, in particular COMMUTATIVE ALGEBRA, by means of geometrical construction principles. Looking at its history, the theory has behaved more like an evolving process than a completed workpiece, as quite often the challenge of new problems has caused extensions and revisions. For example, the concept of schemes invented by Grothendieck in the late 1950s made it possible to introduce geometric methods even into fields that formerly seemed to be far from GEOMETRY, like algebraic NUMBER THEORY. This paved the way to spectacular new achievements, such as the proof by Wiles and Taylor of Fermat's Last Theorem, a famous problem that was open for more than 350 years.

The purpose of the present book is to explain the basics of modern ALGEBRAIC GEOMETRY to non-experts, thereby creating a platform from which one can take off towards more advanced regions. Several times I have given courses and seminars on the subject, requiring just two semesters of LINEAR ALGEBRA for beginners as a prerequisite. Usually I did one semester of COMMUTATIVE ALGEBRA and then continued with two semesters of ALGEBRAIC GEOMETRY. Each semester consisted of a combination of traditional lectures together with an attached seminar where the students presented additional material by themselves, extending the theory, supplying proofs that were skipped in the lectures, or solving exercise problems. The material covered in this way corresponds roughly to the contents of the present book. Just as for my students, the necessary prerequisites are limited to basic knowledge in LINEAR ALGEBRA, supplemented by a few facts from classical Galois theory of fields.

Explaining ALGEBRAIC GEOMETRY from scratch is not an easy task. Of course, there are the celebrated *Éléments de Géométrie Algébrique* by Grothendieck and Dieudonné, four volumes of increasing size that were later continued by seven volumes of *Séminaire de Géométrie Algébrique*. The series is like an extensive encyclopaedia where the basic facts are dealt with in striving generality, but which is hard work for someone who has not yet acquired a certain amount of expertise in the field. To approach ALGEBRAIC GEOMETRY from a more economic point of view, I think it is necessary to learn about its basic principles. If these are well understood, many results become easier to digest, including proofs, and getting lost in a multitude of details can be avoided.

Therefore it is not my intention to cover as many topics as possible in my book. Instead I have chosen to concentrate on a certain selection of main themes that are explained with all their underlying structures and without making use of any artificial shortcuts. In spite of thematic restrictions, I am aiming at a self-contained exposition up to a level where more specialized literature comes into reach.

Anyone willing to enter ALGEBRAIC GEOMETRY should begin with certain basic facts in COMMUTATIVE ALGEBRA. So the first part of the book is concerned with this subject. It begins with a general chapter on rings and modules where, among other things, I explain the fundamental process of localization, as well as certain finiteness conditions for modules, like being Noetherian or coherent. Then follows a classical chapter on Noetherian (and Artinian) rings, including the discussion of primary decompositions and of Krull dimensions, as well as a classical chapter on integral ring extensions. In another chapter I explain the process of coefficient extension for modules by means of tensor products, as well as its reverse, descent. In particular, a complete proof of Grothendieck's fundamental theorem on faithfully flat descent for modules is given. Moreover, as it is quite useful at this place, I cast a cautious glimpse on categories and their functors, including functorial morphisms. The first part of the book ends by a chapter on Ext and Tor modules where the general machinery of homological methods is explained.

The second part deals with ALGEBRAIC GEOMETRY in the stricter sense of the word. Here I have limited myself to four general themes, each of them dealt with in a chapter by itself, namely the construction of affine schemes, techniques of global schemes, étale and smooth morphisms, and projective and proper schemes, including the correspondence between ample and very ample invertible sheaves and its application to abelian varieties. There is nothing really new in these chapters, although the style in which I present the material is different from other treatments. In particular, this concerns the handling of smooth morphisms via the Jacobian Condition, as well as the definition of ample invertible sheaves via the use of quasi-affine schemes. This is the way in which M. Raynaud liked to see these things and I am largely indebted to him for these ideas.

Each chapter is preceded by an introductory section where I motivate its contents and give an overview. As I cannot deliver a comprehensive account already at this point, I try to spotlight the main aspects, usually illustrating these by a typical example. It is recommended to resort to the introductory sections at various times during the study of the corresponding chapter, in order to gradually increase the level of understanding for the strategy and point of view employed at different stages. The latter is an important part of the learning process, since Mathematics, like ALGEBRAIC GEOMETRY, consists of a well-balanced combination of philosophy on the one hand and detailed argumentation or even hard computation on the other. It is necessary to develop a reliable feeling for both of these components. The selection of exercise problems at the end of each section is meant to provide additional assistance for this.

Preliminary versions of my manuscripts on Commutative Algebra and Algebraic Geometry were made available to several generations of students. It was a great pleasure for me to see them getting excited about the subject, and I am very grateful for all their comments and other sort of feedback, including lists of typos, such as the ones by David Krumm and Claudius Zibrowius. Special thanks go to Christian Kappen, who worked carefully on earlier versions of the text, as well as to Martin Brandenburg, who was of invaluable help during the final process. Not only did he study the whole manuscript meticulously, presenting an abundance of suggestions for improvements, he also contributed to the exercises and acted as a professional coach for the students attending my seminars on the subject. It is unfortunate that the scope of the book did not permit me to put all his ingenious ideas into effect. Last, but not least, let me thank my young colleagues Matthias Strauch and Clara Löh, who run seminars on the material of the book together with me and who helped me setting up appropriate themes for the students. In addition, Clara Löh suggested numerous improvements for the manuscript, including matters of typesetting and language. Also the figure for gluing schemes in the beginning of Section 7.1 is due to her.

Münster, December 2011 Siegfried Bosch