METHODS OF APPLIED MATHEMATICS FOR ENGINEERS AND SCIENTISTS

Based on course notes from more than twenty years of teaching engineering and physical sciences at Michigan Technological University, Tomas B. Co's engineering mathematics textbook is rich with examples, applications, and exercises. Professor Co uses analytical approaches to solve smaller problems to provide mathematical insight and understanding, and numerical methods for large and complex problems. The book emphasizes applying matrices with strong attention to matrix structure and computational issues such as sparsity and efficiency. Chapters on vector calculus and integral theorems are used to build coordinate-free physical models, with special emphasis on orthogonal coordinates. Chapters on ordinary differential equations and partial differential equations cover both analytical and numerical approaches. Topics on analytical solutions include similarity transform methods, direct formulas for series solutions, bifurcation analysis, Lagrange-Charpit formulas, and shocks/rarefaction. Topics on numerical methods include stability analysis, differential algebraic equations, high-order finite-difference formulas, and Delaunay meshes. MATLAB implementations of the methods and concepts are fully integrated.

Tomas B. Co is an associate professor of chemical engineering at Michigan Technological University. After completing his PhD in chemical engineering at the University of Massachusetts at Amherst, he was a postdoctoral researcher at Lehigh University, a visiting researcher at Honeywell Corp., and a visiting professor at Korea University. He has been teaching applied mathematics to graduate and advanced undergraduate students at Michigan Tech for more than twenty years. His research areas include advanced process control, including plantwide control, nonlinear control, and fuzzy logic. His journal publications span broad areas in such journals as *IEEE Transactions in Automatic Control*, *Automatica*, *AIChE Journal*, *Computers in Chemical Engineering*, and *Chemical Engineering Progress*. He has been nominated twice for the Distinguished Teaching Award at Michigan Tech and is a member of the Michigan Technological University Academy of Teaching Excellence.

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Preface

This book was written as a textbook on applied mathematics for engineers and scientist, with the expressed goal of merging both analytical and numerical methods more tightly than other textbooks. The role of applied mathematics has continued to grow increasingly important with advancement of science and technology, ranging from modeling and analysis of natural phenomenon to simulation and optimization of man-made systems. With the huge and rapid advances of computing technology, larger and more complex problems can now be tackled and analyzed in a very timely fashion. In several cases, what used to require supercomputers can now be solved using personal computers. Nonetheless, as the technological tools continue to progress, it has become even more imperative that the results can be understood and interpreted clearly and correctly, as well as the need for a deeper knowledge behind the strengths and limitations of the numerical methods used. This means that we cannot forgo the analytical techniques because they continue to provide indispensable insights on the veracity and meaning of the results. The analytical tools continue to be of prime importance for basic understanding for building mathematical models and data analysis. Still, when it comes to solving large and complex problems, numerical methods are needed.

The level of exposition in this book is aimed at graduate students, advanced undergraduate students, and researchers in the engineering and science field. Thus the topics were mostly chosen to continue several topics found in most undergraduate textbooks in applied mathematics. We have focused on advanced concepts and implementation of various mathematical tools to solve the problems that most graduate students are likely to face in their research work and other advanced courses.

The contents of the book can be divided into four main parts: matrix theory, vectors and tensors, ordinary differential equations, and partial differential equations. We begin the book with matrix theory because the tools developed in matrix theory form the crucial foundations used in the rest of the book. The next part centers on the concepts used in vector and tensor theory, including the application of tensor calculus and integral theorems to develop mathematical models of physical systems, often resulting in several differential equations. The last two parts focus on the solution of ordinary and partial differential equations. It can be argued that the primary needs of applied mathematics in engineering and the physical sciences are to obtain models for a system or phenomena in the form of differential equations and then to be able to "solve" them to predict and understand the effects of changes in model parameters, boundary conditions, or initial conditions.

Although the methods of applied mathematics are independent of computing platform and programs, we have chosen to use MATLAB as a particular platform under which we investigate the mathematical methods, techniques, and ideas so that the approaches can be tested and the results can be visualized. The supplied MATLAB codes are all included on the book's website, and the reader can modify the codes for their own use. There are several excellent MATLAB toolboxes supplied by third-party software developers, and they have been optimized for speed, efficiency, and user-friendliness. However, the unintended consequences of user-friendly tools can sometimes render the users to be "button pushers." We contend that students in applied mathematics still need to discover the mechanism and ideas behind the full-blown programs – at least to apply them to simple test problems and gain some basic understanding of the various approaches. The links to the supplemental MATLAB programs and files can be accessed through the link: `www.cambridge.org/Co`.

The appendices are collected as chapter fortifications. They include proofs, advanced topics, additional tables, and examples. The reader should be able to access these materials through the web via the link: `www.cambridge.org/Co`. The index also contains topics that can be found in the appendices, and they are given page numbers that continue the count from the main text.

Several colleagues and students have helped tremendously in the writing of this textbook. Mostly, I want to thank my best friend and wife, Faith Morrison, for the support and encouragement and the sacrifice she's made so that I could finish this extended and personally significant project. I hope the textbook will contain useful information for the readers, enough for them to share in the continued exploration of the methods and applications of mathematics to further improve the understanding and conditions of our world.

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