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Sequential Competitive Location on Networks

PL ACADEMIC RESEARCH

Chapter 1

Introduction and Preliminaries

Location problems are concerned with the location of (physical or nonphysical) resources in some given space. *Competitive location models* additionally incorporate the fact that location decisions have been or will be made by independent decision-makers who will subsequently compete with each other, e.g. for market share when we think of locating facilities such as gas stations or supermarkets (see Plastria, 2001; ReVelle and Eiselt, 2005, for similar definitions). The location space under consideration does not necessarily need to be of geographical nature: political parties, for example, are concerned with locating in issue spaces; products may be positioned in characteristics spaces.

The practical relevance of analyzing competitive location models arises from the fact that, when locating resources in competitive environments, enterprises typically make strategic decisions that play a crucial role for their long-term success or failure. In the context of an industrial enterprise, for example, the choice of locations for production facilities influences the long-term structure of the logistics network, as relocating will, in general, induce high cost. After making its locational decisions, the enterprise will have face up to the competition.

This thesis takes a mathematical perspective (or, more specifically, an operations research perspective) on location theory. Well known non-competitive frameworks in this field are, for example, concerned with the choice of optimal points in networks according to the minimization or maximization

of the sum of weighted distances to the vertices (median or antimedian problems) or the minimization of the maximum or maximization of the minimum weighted distance to any vertex (center or anticenter problems). Classical competitive location problems include the $(r|p)$-centroid or the $(r|X_p)$-medianoid problem. This research analyzes extensions to and variations of these latter problems. It focuses on incorporating proportional choice rules, non-discrete demand, or additional pricing decisions of firms. Furthermore, it provides insights into the computational complexity of some of the resulting problems and proposes adequate solution methods.

1.1 Basic Notation and Definitions

In this thesis we will denote the set of natural numbers including zero by $\mathbb{N}$, the set of positive natural numbers by $\mathbb{N}^+$, the set of rational (real) numbers by $\mathbb{Q}$ ($\mathbb{R}$), the set of positive rational (real) numbers by $\mathbb{Q}^+$ ($\mathbb{R}^+$) and the set of nonnegative rational (real) numbers by $\mathbb{Q}_0^+$ ($\mathbb{R}_0^+$).

We assume the reader to be familiar with the fundamental concepts of operations research. The basic graph-theoretic definitions, along with the corresponding notation used throughout this thesis, however, are introduced in Section 1.1.1. Most of the definitions are taken from Alstrup et al. (2004); Bandelt (1985); Bauer et al. (1993); Garbe (1995); Gross and Yellen (2004); Swamy and Thulasiraman (1981). Similarly, Sections 1.1.2 and 1.1.3 are concerned with the basics of game theory and discrete choice theory. The latter Section is based on Train (2009).

1.1.1 Graphs and Networks

A *graph* $G = (V, E)$ is composed of two (finite) sets V ($n := |V|$) and E ($m := |E|$). The elements of V are called *vertices*. The elements of E are the *edges*. Each edge is associated (incident) to one or two vertices, which are called the edge's end points. The end points are said to be joined by the edge. An edge $e \in E$ joining vertex $u \in V$ and vertex $v \in V$ is denoted by $e = [u, v]$. Unless otherwise stated, we assume that there is

no direction associated with an edge of a graph, hence $[u, v] = [v, u]$. Two vertices $u, v \in V$, $u \neq v$, are called *neighbored*, if there exists an edge $[u, v]$ in E. An edge $[u, u]$ is a *loop*. The sum of the number of edges that join a vertex $v \in V$ with other vertices of the network and twice the number of loops at v is the *degree* of v, $deg(v)$.

A *network* $N = (V, E, \lambda)$ consists of a graph with vertex set V and edge set E and an additional mapping $\lambda : E \to \mathbb{R}^+$. For each $e = [u, v] \in E$, $\lambda(e) = \lambda(uv) = \lambda(u, v)$ defines the *length* of e. We define $\hat{D} := \max_{e \in E} \lambda(e)$. The elements of the edges of a network N, including all the vertices, are the *points* x of N ($x \in N$). A *subedge* $[x, y]$ (or xy) of an edge $e \in E$ is determined by two points x and y on e ($x, y \in e$). The length of a subedge $[x, y]$ is denoted by $\lambda([x, y]) = \lambda(xy) = \lambda(x, y)$. A subedge defined by all points of an edge $[u, v] \in E$ without including the vertices u and v is denoted by (u, v).

Let $N = (V, E, \lambda)$ and $V' \subseteq V$. Then $N' = (V', E', \lambda')$ is the *subnetwork* of N on the vertex set V', if $E' \subseteq E$ consists of all those edges of N whose end points are in V'. The mapping λ' is the restriction of λ to E'.

A *path* $P(x, y)$ joining two points x, y, $x \neq y$, of a network N is either a subedge $[x, y]$ or an alternating sequence of (sub-) edges and vertices, $[x, v_0], v_0, e_1, v_1, ..., e_k, v_k, [v_k, y]$, such that $e_j = [v_{j-1}, v_j] \in E$ for $j = 1, ..., k$, no edge and no vertex of the sequence occurs more than once, and $[x, v_0]$ as well as $[v_k, y]$ define point-disjoint subedges of N. x and y are called end points of $P(x, y)$. The length of a path is defined by the sum of the lengths of its edges and subedges. A path $P(x, y)$ is a shortest path, if there exists no path $P'(x, y) \neq P(x, y)$ of smaller length; The length of a shortest path $P(x, y)$ is the *distance* $d(x, y) = d_{xy}$ between x and y. We define $d(x, y) := \infty$ if there exists no path $P(x, y)$ for $x, y \in N$, $x \neq y$. Furthermore, we define $d(x, x) := 0$ for any $x \in N$ and $D(p, Z) := \min\{d(p, z) | z \in Z\}$ for a point $p \in N$ and a set of points $Z \subseteq N$. A *cycle* consists of an edge $e = [u, v] \in E$, and some path $P(u, v) \neq e$ connecting u and v. In a *connected network* there exists a path joining x and y for any two points $x \neq y$ of the network. We additionally define any network with $n = 1$ to be connected. A connected network is a *tree network* if there

exists no cycle or loop. A tree network where every vertex is end point of at most two edges is a *chain network*. A network without loops where every pair of vertices is joined by a unique edge is a *complete network*.

Let u and v be vertices of a network N. Then $I(u,v) := \{x \in N | d(u,v) = d(u,x)+d(x,v)\}$ is called the *interval* $I(u,v)$ between u and v.[1] The interval is *ported* if for any pair of points $x \in I(u,v)$ and $y \notin I(u,v)$ every shortest path $P(x,y)$ passes through u or v.

A *path-decomposition* of a graph $G = (V,E)$ is a sequence $V_1, ..., V_r$ of subsets of the vertex set V, such that

1. $\bigcup_{1 \leq i \leq r} V_i = V$,
2. there exists a V_i, $i \in \{1, ..., r\}$, such that $u \in V_i$ and $v \in V_i$ for all $[u,v] \in E$, and
3. $V_i \cap V_k \subseteq V_j$ holds for all $1 \leq i < j < k \leq r$.

The *width* of a path-decomposition $V_1, ..., V_r$ is defined as $\max_{1 \leq i \leq r}\{|V_i|-1\}$. The *pathwidth* of a graph $G = (V,E)$ is the minimum width over all path-decompositions of G.

Let a tree network $N = (V, E, \lambda)$ be *rooted* at some distinguished vertex $r \in V$. For each pair of vertices $i \in V$ and $j \in V$, we call i a *descendant* of j, if j is on the unique path that connects i to the root r. If i is a descendant of j, we call j an *ancestor* of i. A vertex $v \in V$ is a *common ancestor* of two vertices $x, y \in V$, if it is an ancestor of both, x and y. A common ancestor of two vertices $x, y \in V$ is the *nearest common ancestor*, $nca(x,y)$, of these very vertices, if its distance to the root is the largest among all common ancestors. If $i \in V$ is a descendant of $j \in V$ and $[i,j] \in E$, then i is said to be a *child* of j and j is called the *father* of i. A vertex without children is a *leaf* of the tree network. For any vertex $v \in V$ we denote the subnetwork (subtree) of N on the vertex set $V_{T_v} := \{v\} \cup \{i \in V | i \text{ is a descendant of } v\}$ by T_v and the subnetwork on the vertex set $V'_{T_v} := V_{T_v} \setminus \{v\}$ by T'_v.

Unless otherwise stated, we assume that the networks considered in this thesis are connected and that there are no multiple edges. Moreover, we

1 Note that we denote open intervals of real space by (a,b), $a, b \in \mathbb{R}$, $a \leq b$. Similarly, closed intervals of real space are denoted by $[a,b]$, $a, b \in \mathbb{R}$, $a \leq b$.

assume that there are no loops at the vertices. We associate a (local) coordinate $x_{uv} \in [0, \lambda(uv)]$ with every edge $[u, v]$ of a network N. Thus, we are able to define any point of the network. The direction of counting can be defined arbitrarily. For a point p on edge $[u, v]$, we refer to the local coordinate's value at p by p as well, i.e. $x_{uv} = p$ at point p. It will always become clear from the context wether we refer to the point itself or to its corresponding coordinate's value. Finally, we assume the distances between all pairs of vertices of a network to be input data of our algorithms. These distances can, for instance, be computed in $\mathcal{O}(n^3)$ time with the Floyd-Warshall algorithm (see, for example, Aho, 2004).

1.1.2 Game Theory

Noncooperative game theory deals with the behavior of agents (or players) in situations where each agent's optimal choice depends on the (forecasted) choices of his opponents and where each agent is motivated solely by self-interest (Fudenberg and Tirole, 1991).

We will consider games in *strategic form* that have three basic elements (Fudenberg and Tirole, 1991): a set of players Θ which we assume to be finite ($\Theta = \{1, ..., \theta\}$), a (pure) strategy space Ψ_i for each player $i \in \Theta$, and a payoff function $u_i(\boldsymbol{\psi})$ for each player $i \in \Theta$ that assigns a utility level to every vector of strategies $\boldsymbol{\psi} = (\psi_1, ..., \psi_\theta)$, $\psi_i \in \Psi_i$.

A strategy vector $\boldsymbol{\psi}^N = (\psi_1^N, ..., \psi_\theta^N)$ is said to be a *Nash equilibrium* in pure strategies, if no player can unilaterally increase his utility, i.e. $u_i(\boldsymbol{\psi}^N) \geq u_i(\psi_i, \boldsymbol{\psi}^N_{\Theta\setminus\{i\}})$ for all $\psi_i \in \Psi_i$, where $\boldsymbol{\psi}^N_{\Theta\setminus\{i\}} = (\psi_j^N | j \in \Theta, j \neq i)$ (cf. Gabay and Moulin, 1980).

The following theorem is well known. We refer to Fudenberg and Tirole (1991) and the references therein for a proof.

Theorem 1.1.1. *Let Θ be a nonempty set of players and consider a strategic-form game whose strategy spaces Ψ_i, $i \in \Theta$, are nonempty, convex and compact subsets of an Euclidean space. If the payoff functions u_i are continuous in $\boldsymbol{\psi}$ and quasiconcave in ψ_i, then there exists a pure strategy Nash equilibrium.*

1.1.3 Discrete Choice Theory

Consider the choices of decision makers (for example customers) among a set of alternatives (for instance a set of facilities), the so called choice set. If the alternatives are mutually exclusive and the choice set is finite and exhaustive, we call this framework a *discrete choice model*. If one assumes the decision makers to be utility maximizing and derives a specific discrete choice model from this assumption, the model is said to be a *random utility model* (RUM). For a detailed overview of RUMs and their features we refer to Anderson et al. (1992a); Train (2009).

To derive a RUM, one takes the point of view of a researcher and assumes the utility u_{ij} of a decision maker $i \in I$ (I denotes the set of decision makers) from choosing alternative $j \in J$ (J refers to the choice set) to be composed of a deterministic (i.e. observable) component v_{ij} and a stochastic component ϵ_{ij}, the latter being related to unobservable, utility affecting factors:

$$u_{ij} = v_{ij} + \epsilon_{ij}. \tag{1.1}$$

As the decision makers are assumed to be utility maximizing, the probability pr_{ij} for a decision maker $i \in I$ to choose alternative $j \in J$ is

$$pr_{ij} = Prob(u_{ij} > u_{ik} \,\forall\, k \in J, k \neq j). \tag{1.2}$$

Now, the RUM is derived by additionally providing a specific distribution of the random components. One of the most prominent RUMs which is well established in the economics, marketing and operations research literature (see also Hensher et al., 2005), the *multinomial logit model*, assumes the stochastic components to be independently and identical extreme value distributed (Gumbel distributed) with density

$$f(\epsilon_{ij}) = \left(e^{-\epsilon_{ij}/\sigma} e^{-e^{-\epsilon_{ij}/\sigma}}\right)/\sigma \tag{1.3}$$

and the cumulative distribution

$$F(\epsilon_{ij}) = e^{-e^{-\epsilon_{ij}/\sigma}}. \tag{1.4}$$

The variance is $\sigma^2\pi^2/6$, where σ is a scaling parameter. As σ approaches zero, the decision makers' choices become deterministic. The mean is $\sigma\gamma$, where γ is Euler's constant. Then, as shown in Appendix A, one derives a closed form expression of the choice probabilities of the multinomial logit model,

$$pr_{ij} = \frac{e^{v_{ij}/\sigma}}{\sum\limits_{k \in J} e^{v_{ik}/\sigma}} \quad \forall\, i \in I, j \in J. \tag{1.5}$$

1.2 Foundations of Competitive and Voting Location

The study of competitive location is rooted in the work of Hotelling (1929), who studied the location choice and pricing decision of two competitors on a finite line with uniformly spread consumers. It has been a prosperous field of research, leading to a whole variety of models, ever since. Those models can be classified with respect to multiple components (Eiselt, 1993; Eiselt and Laporte, 1989, 1996; Eiselt et al., 1993; Friesz et al., 1988; Hamacher and Nickel, 1998; Plastria, 2001; ReVelle and Eiselt, 2005). In this section we describe the most significant of these classification criteria as in Kress and Pesch (2012d).

Most important, the *representation of the underlying location space* traditionally gives rise to three classes: d-dimensional real space, network and discrete space. Distances need to be calculated by some metric in each of these classes. We follow ReVelle and Eiselt (2005) in differentiating only between d-dimensional real space and network location problems, each of which further being subdivided into continuous and discrete problems (see Figure 1.1). A *discrete* problem arises, when the set of candidate locations is assumed to be finite and known a priori. In a *continuous* problem, any point of the network or the d-dimensional space is a potential location site. By identifying finite dominating sets – finite sets of points to which at least one of the optimal solutions must belong – we are able to transform some special classes of continuous location problems into equivalent discrete prob-

lem classes a posteriori (Hooker et al., 1991). Moreover, as discrete sets of potential facility sites may easily become very large, one may consider treating those sets as continuous entities (see, for instance, Dasci and Laporte, 2005). Note, however, that there is an overlap of d-dimensional real spaces and networks. Hotelling's line, for instance, can be interpreted as a network with two vertices and a connecting edge or, alternatively, as an interval of $\mathbb{R}^1$. Other models are somewhere in between network and d-dimensional models. Suárez-Vega et al. (2011), for example, consider a "buffer" around a network to represent the space of potential locations.

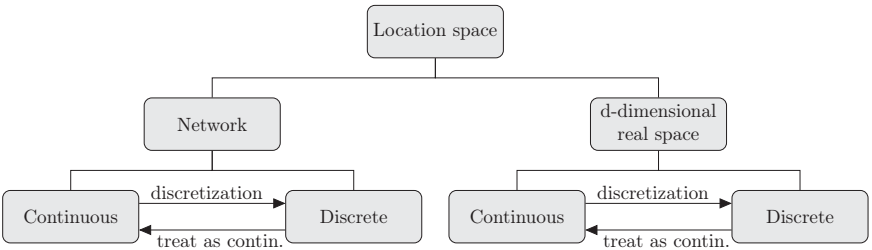

Fig. 1.1: Location spaces

Other fundamental categories of competitive location theory are related to game theoretic aspects. Competition itself may be static (present and fixed), competitors may enter in a simultaneous or sequential fashion, or we can think of dynamic competition, i.e. players who repeatedly reoptimize their locations (see Figure 1.2; Eiselt et al., 1993; Plastria, 2001). *Sequential locational competition*, dating back to Hay (1976) and Prescott and Visscher (1977), is characterized by two types of players: leaders, who choose locations at given instants, anticipating the subsequent actions of later entrants, and followers, who make their location decisions based on the past decisions of the leaders. The solution concept generally employed in sequential location problems is the Stackelberg equilibrium (von Stackelberg, 1934): assuming rational players, the location of each player is determined by backward induction. *Simultaneous locational games* (as the one of Hotelling, 1929), in contrast, usually use the concept of a Nash equilibrium (cf. Section 1.1.2). Note that the *number of players* may be exogenously given or determined

endogenously, e.g. by incorporating fixed location cost. The same holds for the *sequence* of location and the *numbers of resources* to be located by each player.

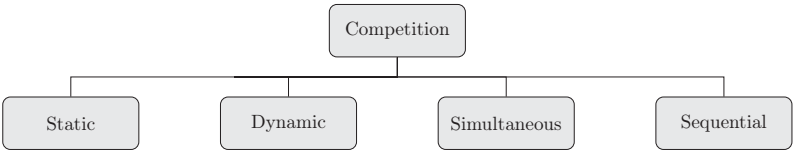

Fig. 1.2: Competition

It is generally agreed that the work on competitive location problems on (general) networks is rooted in the work of Slater (1975) and Hakimi (1983) (see Eiselt and Laporte, 1996; Smith et al., 2009, for details). Hakimi (1983) formally introduced the terms $(r|X_p)$-medianoid problem and $(r|p)$-centroid problem for sequential games with one leader (L) and one follower (F) locating p and r facilities, respectively. Note that r and p are arbitrary input parameters. Knowing the p locations of L, denoted by $X_p = (x_1, ..., x_p)$, F faces the problem of optimally locating r facilities (with respect to some objective function): the $(r|X_p)$-*medianoid problem*. We denote a feasible location decision of F by $Y_r = (x_{p+1}, ..., x_{p+r})$ and an optimal location decision by $Y_r^* = (x_{p+1}^*, ..., x_{p+r}^*)$. L's problem, the $(r|p)$-*centroid problem*, is to locate p facilities, anticipating F's subsequent behavior. An optimal solution to this latter problem is denoted by $X_p^* = (x_1^*, ..., x_p^*)$. Note that, differing from other authors as Spoerhase and Wirth (2010), we use the terms $(r|X_p)$-medianoid and $(r|p)$-centroid problem in a rather broad sense, subsuming a whole variety of choice rules and player objectives under these terms.

Another category that is related to game theory is the incorporation of pricing in competitive facility or product location models (Anderson et al., 1992b; Eiselt and Laporte, 1996; Eiselt et al., 1993; García Pérez et al., 2004). Traditional spatial pricing policies include *mill pricing* (all customers are charged the same price for the good itself, all transport costs are passed to the customers), *uniform delivered pricing* (the facility pays for the transportation; all customers of a facility are charged the same price, no matter where they are located at) and *spatial price discrimination* (the facility pays

for the transportation; prices for different customers are customer-location-specific). Prices may be set simultaneously to the location decisions or in a separate stage, either sequentially or simultaneously. Alternatively, one can incorporate *parametric prices*. The equilibrium concepts used for combined location price games depend on which of these situations is implemented (Eiselt et al., 1993).

Other ingredients of competitive location models include *characteristics of the targeted group*, as, for example, customers or voters. They may be distributed over the representation of the location space according to some density function or we may consider discrete locations (cf. Section 1.3). Demand may be *deterministic* or *stochastic*, *elastic* or *inelastic* (dependent on or independent of the conditions of its supply). Furthermore, we may take different types of choice rules into consideration (cf., for example, Hakimi, 1990). A choice rule is said to be *binary* (or *deterministic*), when it is deterministic from the perspective of the players with the total demand of a customer (voter etc.) being served by a single located resource; it is said to be *probabilistic*, if the researcher can only derive probabilities of customer (voter etc.) behavior or if demand is assumed to be split over multiple located resources. Probabilistic choice rules include *partially binary* (splitting only over one of the locations of each player) and *proportional* (splitting over all locations) behavior. Note that, at least in the case of a binary choice rule, one has to make assumptions concerning the location of two facilities in the same point of the network (*co-location*). A common assumption in the field of competitive location problems is to break ties in favor of the leader (see, for example, Hakimi, 1990; Hansen and Labbé, 1988; Hansen and Thisse, 1981) or, similarly, not allowing co-location at all (see, for example, Granot et al., 2010; Shiode and Drezner, 2003). Alternatively, ties may be broken equally as, for instance, in Dasci et al. (2002). Both, the existence and nature of equilibria in competitive location models, may vary according to different assumptions concerning co-location: Hakimi (1990) designs the above-mentioned tie breaking rule to "avoid [...] trivial solutions". Similarly, Granot et al. (2010) analyze the effect of allowing or not allowing co-location in their model in detail.