Preface

Calculus of quantum probabilities is one of the most important sectors of the mathematical formalism of quantum mechanics. This calculus is based on the representation of probabilities by *complex probability amplitudes* (or normalized complex vectors in the abstract Hilbert space formalism). This algebraic representation was created in the process of the development of quantum mechanics. Since that time it has been successfully applied to the statistical analysis of data obtained in various experiments with quantum systems.

Applications to quantum physics play an extremely stimulating role in the development of quantum probability mathematics. However, the original appearance of this mathematical apparatus in the framework of a special physical theory—quantum mechanics—created a barrier (in any event, a psychological barrier) on the way to generalizations and applications of quantum probability calculus outside quantum physics.

The quantum mechanical origin of this mathematical formalism induced the impression that the quantum probabilistic behavior can only be found in very special (often regarded as even mystical) systems, namely, quantum particles and fields.[1]

[1] “Quantum mechanics is magic,” Daniel Greenberger; “ Everything we call real is made of things that cannot be regarded as real,” Niels Bohr; “Those who are not shocked when they first come across quantum theory cannot possibly have understood it,” Niels Bohr; “If you are not completely

The aim of this book is *to demystify quantum probability.*[2] The only possibility to do this is to derive the quantum probability calculus without starting with the conventional Hilbert space formalism directly. We recall the standard scheme leading to quantum probability: *complex Hilbert state space, Born's postulate, the derivation of interference of probabilities* via the transition from one basis of eigenvectors to another (for two noncommuting operators $\hat{a}$ and $\hat{b}$), see, e.g., Dirac [84] and von Neumann [313] or for the modern presentations Holevo [125], Busch, Grabowski and Lahti [44]. The interference of probabilities is interpreted as the exhibition of quantum (and hence nonclassical) probabilistic laws (as at least Dirac [84] and von Neumann [313] as well as Feynman [91] believed).

We would like to invert the scheme. We are going to start with a very general scheme of probabilistic description of experimental statistical data. We call this scheme the contextual probabilistic model (*Växjö model*) [180]. The origin of the data (i.e., the context of observation) does not play any role. It could be statistical data from quantum mechanics as well as from classical statistical mechanics, biology, sociology, economics, or meteorology. Then we will show that the interference of probabilities (even more general than in quantum mechanics) can be found for any kind of data [145, 159–163, 167–170, 173, 181–183].

confused by quantum mechanics, you do not understand it," John Wheeler; "It is safe to say that nobody understands quantum mechanics," Richard Feynman; "If [quantum theory] is correct, it signifies the end of physics as a science," Albert Einstein; "I do not like [quantum mechanics], and I am sorry I ever had anything to do with it;" Erwin Schrödinger; " Quantum mechanics makes absolutely no sense," Roger Penrose.

[2] Cf. Mackey [239], Accardi [2–6], Aerts et al. [12], Bohm and Hiley [38], Ballentine [29–31], Boyer [41], Cole [61], Davidson [64], De Baere [69], De la Pena and Cetto [71–75], De Muynck [79, 80], Gudder [106–108], Helland [115], Ludwig [237], Khrennikov [139–142, 144–210], Klyshko [220], Kirkpatrick [213], Manko et al. [240–243], Nelson, [252], Nieuwenhuizen [253], Santos [276, 277], Svozil [294, 295], 't Hooft [297–299], Adenier et al. [11–178], Haven [111, 112], Choustova [52–58], Busemeyer [46, 47].

Moreover, one can easily obtain the linear space representation of probabilities from the interference of probabilities and then recover Born's rule (which will not be a postulate anymore). Thus, in our approach the quantum probabilistic calculus is just a special linear space representation of given probabilistic data [145].

One of our main contributions to this sphere is the creation of a special representation algorithm. Since our algorithm can be applied not only to physical statistical data, we will speak about *quantum-like (QL) representation* rather than about a quantum one. The latter is reserved for quantum physics. We call our algorithm for the representation of probabilistic data by complex amplitudes *quantum-like representation algorithm* or QLRA. We repeat again that QLRA can work in any domain of science.

One of the advantages of the QL representation of probabilistic data is an essential simplification of calculation of probabilities. We emphasize that linear algebra in complex Hilbert space is essentially simpler than the theory of Lebesgue integral and measurable functions! We stress this point, because quantum mechanics is commonly considered as much more mathematically complicated than classical statistical mechanics. However, it is not the case.

The basic notion in our approach is that of *context*—a complex of conditions under which the measurement is performed. We could consider contexts of different kinds: physical, biological or even political. Our approach to the subject of probability is contextual. It is meaningless to consider a probability not specifying the context of consideration (we remark that this point was already discussed by Kolmogorov [222]).

We will make a few remarks regarding the terminology in this book. The notion of context can be related to the notion of the *preparation procedure* which is widely used in quantum measurement theory [44, 125, 237, 257]. Of course, preparation procedures—devices preparing systems for subsequent measurements—give a wide class of contexts. However, the context is a more general concept. For example, we can develop models operating with social, psychological or financial contexts, [46, 47, 52–58, 90, 102–105, 111, 112, 162, 166, 167, 175, 199, 261–266].

The notion of *contextual probability* can be coupled to the notion of *conditional probability* which is used in classical (e.g., Kolmogorovian) probability models. However, once again the direct identification can be rather misleading, since the conventional meaning of the conditional probability $\mathbf{P}(B|A)$ is the probability that *event* B occurs under the condition that *event* A has occurred [222]. Thus, the conventional conditioning is the *event-conditioning.* Our conditioning is a *context-conditioning:* $\mathbf{P}(b = \beta|C)$ is the probability that observable b takes the value β in the process of measurement under context C. In principle, we are not against the term "conditional probability" if it is used in the contextual sense. We also remark that typically the definition of conditional probability is associated with the one given by Bayes' formula. In particular, such an approach preassum the possibility of operating with the joint probability distribution. However, in the general contextual probabilistic model the latter need not be defined.

The main terminological problem is related to the notion of *contextuality.* The use of the term "contextual" is characterized by a huge diversity of meanings, see Bell [34], Svozil [296] or Beltrametti and Cassinelli [35] for the notion of contextuality in quantum physics as well as Light and Butterworth [235], Bernasconi and Gustafson [36] for the notion of contextuality in cognitive science and AI. In quantum physics the contextuality is typically reduced to a rather specific contextuality—"Bell contextuality." J. Bell invented this notion in the framework of the EPR-Bohm experiment [34, 35]. We recall that such *quantum contextuality* ("Bell's contextuality") is defined as follows.

The result of the measurement of an observable a depends on another measurement of observable b, although these two observables commute with each other.

It should be emphasized that the *nonlocality* in the framework of the EPR-Bohm experiment is a special case of quantum contexuality.

Our contextuality is essentially more general than Bell's. In a very special case one can determine the context for the a-measurement by fixing an observable b which is compatible with a. However, in the general case there is nothing about a mutual dependence or compatibility of observables. The context is simply a complex of conditions (e.g., physical or biological). In particular, our description of the EPR-Bohm experiment is contextual, but there is no direct coupling with nonlocality.

Our approach to contextuality is closer to the one used in cognitive science and AI, cf. [36, 235].

The basis of linear representations of probabilities is a generalization of the well-known formula of *total probability* [286]. We recall that in the case of two dichotomous variables $a = \alpha_1, \alpha_2$ and $b = \beta_1, \beta_2$ this basic formula has the form

$$\mathbf{P}(b = \beta) = \mathbf{P}(a = \alpha_1)\mathbf{P}(b = \beta | a = \alpha_1) + \mathbf{P}(a = \alpha_2)\mathbf{P}(b = \beta | a = \alpha_2), \quad (0.1)$$

where $b = \beta_1$ or $b = \beta_2$.

Starting with a general *contextual probabilistic model M* (Växjö model) we shall obtain a generalization of the conventional formula, (0.1), which is characterized by the appearance of an additional term, an *interference term*

$$\mathbf{P}(b = \beta) = \mathbf{P}(a = \alpha_1)\mathbf{P}(b = \beta | a = \alpha_1) + \mathbf{P}(a = \alpha_2)\mathbf{P}(b = \beta | a = \alpha_2) \\ + \delta(b = \beta | a). \quad (0.2)$$

Depending on the magnitude of this term (relatively to magnitudes of probabilities in the right-hand side of (0.1)), we obtain either the conventional *trigonometric interference* which is well known in classical wave mechanics as well as in quantum mechanics or a *hyperbolic interference* which was not predicted by conventional physical theories, neither by the classical wave theory nor by quantum mechanics. Such a new type of interference arose naturally in the Växjö model, see Part II and especially Part V.

We recall that all probabilities in (0.2) are contextual. They depend on a complex of conditions, context C, for measurements of a and b. Starting with the formula of total probability with an interference term and applying QLRA we obtain two types of representations of contexts, $C \to \psi_C$, in linear spaces:

(a) representation of some special collection of contexts $\mathscr{C}^{\rm tr}$ ("trigonometric contexts")[3] in complex Hilbert space, see Part II;

(b) representation of some special collection of contexts $\mathscr{C}^{\rm hyp}$ ("hyperbolic contexts")[4] in the so-called hyperbolic Hilbert space, see Part V.

We emphasize that in general the collections of trigonometric and hyperbolic contexts, $\mathscr{C}^{\rm tr}$ and $\mathscr{C}^{\rm hyp}$, are just proper subsets of the complete collection of contexts $\mathscr{C}$ of a Växjö model M (contextual probabilistic model). Depending on model M there can exist contexts which can not be represented algebraically: neither in complex nor in hyperbolic Hilbert spaces.

Thus quantum probabilities are demystified. In the Växjö-approach the quantum probabilistic calculus appears via the application of QLRA: the linear space representation of a special collection of contexts. The origin of those contexts is not important. In any event they need not have any relation to the microworld. In our approach the quantum probabilistic model is incomplete in the following sense: there can exist prequantum models M having essentially larger collections of contexts $\mathscr{C}$ than those that can be represented in complex Hilbert space. Of course, quantum contexts are special. Such "speciality" is characterized not by the properties of systems comprising the contexts, but by the mutual relations between probabilities which are used for the linear space representation of contexts.

Through the demystification of quantum probabilities we have closed the gap between the classical Kolmogorov and the quantum probabilistic models. These

[3] They produce the ordinary cos-interference.

[4] They produce hyperbolic cosh-interference.

models are simply different partial representations of the general Växjö model, cf. Mackey [239]. The classical Kolmogorov model is a representation of the Växjö model such that contexts are given by elements of a σ-algebra $\mathscr{F}$ of subsets of some set Ω (sample set, or space of elementary events). The quantum probabilistic model is a representation of the Växjö model such that contexts are given by normalized vectors of complex Hilbert space (or equivalent classes of such vectors). These mentioned representations are not the only possible representations of the Växjö model. We have discovered new representations: hyperbolic and hyper-trigonometric.

Any Kolmogorov model can be considered as a contextual statistical model (with the set-representation of contexts). Hence we can apply our general algorithm of the linear space representation—QLRA—to the Kolmogorov measure-theoretic model, see Part II, Chap. 6. Here trigonometric contexts $\mathscr{C}^{\mathrm{tr}}$ form a special subcollection of σ-algebra $\mathscr{F}$. In general $\mathscr{C}^{\mathrm{tr}}$ is a proper subset of σ-algebra $\mathscr{F}$. There exist Kolmogorov "classical contexts" which can not be represented by quantum states. Contrary to rather common opinion, the quantum probabilistic model does not cover the classical Kolmogorov model. Neither can the Kolmogorov model cover the quantum one. The image of the collection of Kolmogorov's contexts in complex Hilbert space does not coincide with the whole set of quantum states—the unit sphere in complex Hilbert space. Nevertheless, all basic QL effects and structures (interference of probabilities, Born's rule, complex probabilistic amplitudes, Hilbert state space, the representation of observables by operators) are present in the classical Kolmogorov model, but in a latent form. The discovery [181, 182] of those hidden QL structures in the classical probabilistic model was very important for demystification of the quantum theory.

As was already noted, the QL representation of the Kolmogorov model can not cover the conventional quantum probabilistic model. One can be curious about a possibility of finding a classical probabilistic model which would completely cover the quantum one. Such a classical probabilistic model really exists [139]. It is the von Mises frequency model [309–311] completed by the contextual interpretation of frequency probability.[5] I am well aware of a prejudice against the von Mises approach—especially among mathematicians. (On the other hand, von Neumann used von Mises' approach to probability as the basis for quantum probability [313].) Roots of such prejudice are related to the attempt of Richard von Mises to formalize the notion of randomness through the so-called "place selections." We agree that the original attempt of von Mises was rather misleading.

Still, the crucial point is not the improvement of von Mises theory, but the fact that in quantum physics people are not interested in mathematical study of randomness of sequences produced by experiments (at least at the moment). It is assumed that these sequences are intrinsically random (thus a mathematical proof is replaced by a "physical proof"). The only thing from von Mises theory that is used in applications is the *frequency definition of probability.* This definition can work really fruitfully (as we have already shown in [139] and as we are going to confirm in the present book).

We hope that after reading this book people will pay more attention to the role of context in physics as well as in biology, cognitive and social sciences, psychology, and economics. All considerations of probabilities would be with respect to the context. All known probabilistic models, the classical ones (Kolmogorov measure-theoretic and von Mises frequency models) as well as the quantum model and more general quantum-like models, would be interpreted as special partial representations of the Växjö model, the contextual probabilistic model for description of statistical data from any domain of science.

By starting with contextual probabilities one can escape such mysterious notions as irreducible quantum randomness or quantum nonlocality. Born's rule, interfer-

[5] Probability is defined as the limit of relative frequencies if it exists. Existence of the limit-probability is also known in statistics as *the principle of statistical stabilization* of relative frequencies.

ence of probabilities, origins of complex probability amplitudes and the operator representation can be realistically explained. The mathematical formalism of quantum mechanics as well as its generalizations can be applied outside physics and provide useful, and also verifiable, results.

Växjö-Sevastopol-Copenhagen *Andrei Khrennikov*

2002–2007