Preface

We consider one-dimensional homogeneous stochastic differential equations of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0, \qquad (*)$$

where b and σ are supposed to be measurable functions and $\sigma \neq 0$.

There is a rich theory studying the existence and the uniqueness of solutions of these (and more general) stochastic differential equations. For equations of the form $(*)$, one of the best sufficient conditions is that the function $(1 + |b|)/\sigma^2$ should be locally integrable on the real line. However, both in theory and in practice one often comes across equations that do not satisfy this condition. The use of such equations is necessary, in particular, if we want a solution to be positive. In this monograph, these equations are called *singular stochastic differential equations.* A typical example of such an equation is the stochastic differential equation for a geometric Brownian motion.

A point $d \in \mathbb{R}$, at which the function $(1 + |b|)/\sigma^2$ is not locally integrable, is called in this monograph a *singular* point. We explain why these points are indeed "singular". For the *isolated singular points*, we perform a complete qualitative classification. According to this classification, an isolated singular point can have one of 48 possible types. The type of a point is easily computed through the coefficients b and σ. The classification allows one to find out whether a solution can leave an isolated singular point, whether it can reach this point, whether it can be extended after having reached this point, and so on.

It turns out that the isolated singular points of 44 types do not disturb the uniqueness of a solution and only the isolated singular points of the remaining 4 types disturb uniqueness. These points are called here the *branch points.* There exists a large amount of "bad" solutions (for instance, non-Markov solutions) in the neighbourhood of a branch point. Discovering the branch points is one of the most interesting consequences of the constructed classification.

The monograph also includes an overview of the basic definitions and facts related to the stochastic differential equations (different types of existence and uniqueness, martingale problems, solutions up to a random time, etc.) as well as a number of important examples.

We gratefully acknowledge financial support by the DAAD and by the European Community's Human Potential Programme under contract HPRN-CT-2002-00281.

Moscow, Jena,
October 2004

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