Preface

A more accurate title for these notes would be: "The Hahn–Banach–Lagrange theorem, Convex analysis, Symmetrically self–dual spaces, Fitzpatrick functions and monotone multifunctions".

The Hahn–Banach–Lagrange theorem is a version of the Hahn–Banach theorem that is admirably suited to applications to the theory of monotone multifunctions, but it turns out that it also leads to extremely short proofs of the standard existence theorems of functional analysis, a minimax theorem, a Lagrange multiplier theorem for constrained convex optimization problems, and the Fenchel duality theorem of convex analysis.

Another feature of the Hahn–Banach–Lagrange theorem is that it can be used to transform problems on the existence of continuous linear functionals into problems on the existence of a single real constant, and then obtain a sharp lower bound on the norm of the linear functional satisfying the required condition. This is the case with both the Lagrange multiplier theorem and the Fenchel duality theorem applications mentioned above.

A multifunction from a Banach space into the subsets of its dual can, of course, be identified with a subset of the product of the space with its dual. Simon Fitzpatrick defined a convex function on this product corresponding with any such multifunction. So part of these notes is devoted to the rather special convex analysis for the product of a Banach space with its dual.

The product of a Banach space with its dual is a special case of a "symmetrically self–dual space". The advantage of going to this slightly higher level of abstraction is not only that it leads to more general results but, more to the point, it cuts the length of each proof approximately in half which, in turn, gives a much greater insight into the nature of the processes involved. Monotone multifunctions then correspond to subsets of the symmetrically self–dual space that are "positive" with respect to a certain quadratic form.

We investigate a particular kind of convex function on a symmetrically self–dual space, which we call a "BC–function". Since the Fitzpatrick function of a maximally monotone multifunction is always a BC–function, these BC–functions turn out to be very successful for obtaining results on maximally monotone multifunctions on reflexive spaces.

The situation for nonreflexive spaces is more challenging. Here, it turns out that we must consider two symmetrically self–dual spaces, and we call the corresponding convex functions "$\widetilde{\mathrm{BC}}$–functions". In this case, a number of different subclasses of the maximally monotone multifunctions have been introduced over the years — we give particular attention to those that are "of type (ED)". These have the great virtue that all the common maximally monotone multifunctions are of type (ED), and maximally monotone multifunctions of type (ED) have nearly all the properties that one could desire. In order to study the maximally monotone multifunctions of type (ED), we have to introduce a weird topology on the bidual which has a number of very nice properties, despite that fact that it is not normally compatible with its vector space structure.

These notes are somewhere between a sequel to and a new edition of [99]. As in [99], the essential idea is to reduce questions on monotone multifunctions to questions on convex functions. In [99], this was achieved using a "big convexification" of the graph of the multifunction and the "minimax technique" for proving the existence of linear functionals satisfying certain conditions. The "big convexification" is a very abstract concept, and the analysis is quite heavy in computation. The Fitzpatrick function gives another, more concrete, way of associating a convex functions with a monotone multifunction. The problem is that many of the questions on convex functions that one obtains require an analysis of the special properties of convex functions on the product of a Banach space with its dual, which is exactly what we do in these notes. It is also worth noting that the minimax theorem is hardly used here.

We envision that these notes could be used for four different possible courses/seminars:

- An introductory course in functional analysis which would, at the same time, touch on minimax theorems and give a grounding in convex Lagrange multiplier theory and the main theorems in convex analysis.
- A course in which results on monotonicity on general Banach spaces are established using symmetrically self–dual spaces and Fitzpatrick functions.
- A course in which results on monotonicity on reflexive Banach spaces are established using symmetrically self–dual spaces and Fitzpatrick functions.
- A seminar in which the the more technical properties of maximal monotonicity on general Banach spaces that have been established since 1997 are discussed.

We give more details of these four possible uses at the end of the introduction.

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Of course, despite all the excellent efforts of the people mentioned above, these notes doubtless still contain errors and ambiguities, and also doubtless have other stylistic shortcomings. At any rate, I hope that there are not too many of these. Those that do exist are entirely my fault.

Stephen Simons
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Santa Barbara
California