INTRODUCTION

The world described by the natural and the physical sciences is a concrete and perceptible one: in the first approximation through the senses, and in the second approximation through their various extensions provided by technology. The world described by mathematics is instead an abstract world, made up of ideas that can only be perceived through the mind's eyes. With time and practice, abstract concepts such as numbers and points have nevertheless acquired enough objectivity to allow even an ordinary person to picture them in an essentially concrete way, as though they belonged to a world of objects as concrete as those of the physical world.

Modern science has nonetheless undermined the naive vision of the external world. Scientific research has extended its reach to the vastness of the cosmos as well as to the infinitesimally small domain of the particles, making a direct sensorial perception of galaxies and atoms impossible—or possible only indirectly, through technological means—and thus reducing them in effect to mathematical representations. Likewise, modern mathematics has also extended its domain of inquiry to the rarefied abstractions of structures and the meticulous analysis of the foundations, freeing itself completely from any possible visualization.

Twentieth-century science and mathematics thus share a common difficulty to explain their achievements in terms of classical concepts. But these difficulties can be overcome: often it is only the superficial and futile abstractions that are difficult to justify, while the profound and fruitful ones are rooted in concrete problems and intuitions. In other words, a good abstraction is never an end in itself, an art-for-art's-sake conception, but it is always a necessity, an art-for-humans creation.

A second difficulty in any attempt to survey twentieth-century science and mathematics is the production explosion. Mathematicians, once a small group that often had to earn their living by means other than their trade, are today legion. They survive by producing research that too often has neither interest nor justification, and the university circles in which the majority of mathematicians work unwisely encourage them to "publish or perish," according to an unfortunate American motto. As a result of all this, there are now hundreds of specialized journals in which year after year hundreds of thousands of theorems are published, the majority of them irrelevant.

A third kind of difficulty is due to the fragmentation of mathematics that began in the 1700s, and which became pathological in the 1900s. The production explosion is one of its causes, but certainly not the only one. Another, perhaps even more significant cause is the very progress of mathematical knowledge. The problems that are simple and easy to solve are few, and once they have been solved a discipline can only grow by tackling complex and difficult problems, requiring the development of specific techniques, and hence of specialization. This is indeed what happened in the twentieth century, which has witnessed a hyperspecialization of mathematics that resulted in a division of the field into subfields of ever narrower and strictly delimited borders.

The majority of these subfields are no more than dry and atrophied twigs, of limited development in both time and space, and which die a natural death. But the branches that are healthy and thriving are still numerous, and their growth has produced a unique situation in the history of mathematics: the extinction of a species of universal mathematicians, that is, of those individuals of an exceptional culture who could thoroughly dominate the entire landscape of the mathematics of their time. The last specimen of such a species appears to have been John von Neumann, who died in 1957.

For all these reasons, it is neither physically possible nor intellectually desirable to provide a complete account of the activities of a discipline that has clearly adopted the typical features of the prevailing industrial society, in which the overproduction of low-quality goods at low cost often takes place by inertia, according to mechanisms that pollute and saturate, and which are harmful for the environment and the consumer.

The main problem with any exposition of twentieth-century mathematics is, therefore, as in the parable, to separate the wheat from the chaff, burning up the latter and storing the former away in the barn. The criteria that might guide us in a selection of results are numerous and not at all unambiguous: the historical interest of the problem, the seminal or final nature of a result, the intrinsic beauty of the proposition or of the techniques employed, the novelty or the difficulty of the proof, the mathematical consequences or the practical usefulness of the applications, the potential philosophical implications, and so on.

The choice we propose to the reader can only be a subjective one, with both its positive and its negative aspects. On the one hand, this choice must be made within the bounds of a personal knowledge that is inevitably restricted from a general point of view. And on the other hand, the choice must result from a selection dictated by the author's particular preferences and taste.

The subjective aspects of our choice can nevertheless be minimized by trying to conform to criteria that are in some sense "objective." In the present case, our task has been facilitated by two complementary factors that have marked the development of mathematics throughout the century. Both are related, as we shall see, to the International Congresses of Mathematicians. As in the case of the Olympic Games, these meetings take place every four years, and those invited to present their work are the ones whom the mathematical community considers to be its most distinguished representatives.

The first official congress took place in Zurich in 1897, and the opening address was given by Henri Poincaré, who devoted it to the connections between mathematics and physics. Paris hosted the second congress in 1900, and this time David Hilbert was chosen to open the meeting. The numerological factor prevailed over his desire to reply, three years later in time, to Poincaré's speech, and Hilbert chose rather to "indicate probable directions for mathematics in the new century."

In his inspired address, he gave, first of all, certain implicit clues that shall guide our choice of topics: the important results are those that exhibit a historical continuity with the past, bring together different aspects of mathematics, throw a new light on old knowledge, introduce profound simplifications, are not artificially complicated, admit meaningful examples, or are so well understood that they can be explained to the person in the street.

But Hilbert's address became famous above all for his explicit list of twenty-three open problems that he considered crucial for the development of mathematics in the new century. As if to confirm his lucid foresight, many of those problems really turned out to be fruitful and stimulating, especially during the first half of the century—and we shall examine some of these in detail. In the second half of the century, the thrust from Hilbert's problems petered out, and mathematics often followed paths that did not even exist at the beginning of the century.

To guide us during this period it is useful to turn our attention to a prize created in 1936 and awarded at the Interna-

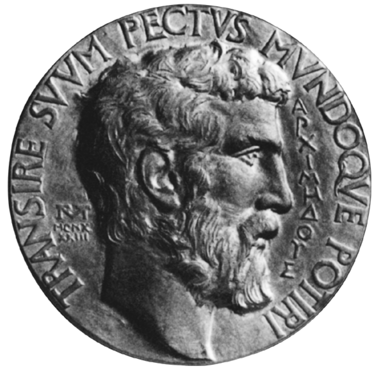

Fig. I.1 The Fields Medal.

tional Congress to mathematicians under age forty who have obtained the most important results in the past few years. The age limit is not particularly restrictive, given that most significant results are in fact obtained during a mathematician's youth. As Godfrey Hardy put it in *A Mathematician's Apology*: "No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game."

The prize was established in memory of John Charles Fields, the mathematician who came up with the idea and obtained the necessary funds. It consists of a medal bearing an engraving of Archimedes' head and the inscription *Transire suum pectus mundoque potiri*, "to transcend human limitations and to master the universe" (fig. I.1). For this reason the prize is nowadays known as the Fields Medal.

This award is considered the equivalent of the Nobel Prize in mathematics, which does not exist. What does exist is a story, widely circulated in mathematical circles, according to which the absence of a Nobel Prize in mathematics would have been due to Alfred Nobel's intention to prevent the Swedish mathematician Gösta Mittag-Leffler from obtaining it. In fact, the two men hardly knew each other, and the latter was certainly not the lover of the former's wife, as the story goes, since Nobel was not married. The real reason is simply that the five original prizes (physics, chemistry, medicine, literature, and peace) were dedicated to the disciplines in which Nobel had had a lifelong interest, and mathematics was not one of them.

In the twentieth century, forty-two Fields medals were awarded, two in 1936 and the rest between 1950 and 1998. Since the winners include some of the best mathematicians of the second half of the century, and the results for which the medals were granted are among the top mathematical achievements of the time, we shall often come back to the subject.

A complement to the Fields Medal is the Wolf Prize, a kind of Oscar for life achievement in a field established in 1978 by Ricardo Wolf, a Cuban philanthropist of German origin who was ambassador to Israel from 1961 to 1973. As is the case for the Nobel Prize, the Wolf Prize has no age restriction, is awarded in various fields (physics, chemistry, medicine, agriculture, mathematics, and art), is presented by the head of state in the awarding country's capital (the king of Sweden in Stockholm in one case, and the president of Israel in Jerusalem in the other) and involves a substantial sum of money ($100,000, compared to $10,000 for a Fields Medal, and $1 million for a Nobel Prize).

To prevent any misunderstanding, I wish to emphasize that the solutions to Hilbert's problems, and the results for which the Fields Medal or the Wolf Prize were awarded, are only significant landmarks and do not exhaust the landscape of twentieth-century mathematics. It will thus be necessary to go beyond them in order to give as complete an account as possible, within the limits previously established, of the variety and depth of contemporary mathematics.

The decision to focus on the great results which, furthermore, constitute the essence of mathematics, determines the asynchronous character of the book's exposition, which will inevitably take the form of a *collage*. This approach has the advantage of allowing a largely independent reading of the various sections, and the disadvantage of resulting in a loss of unity. This inconvenience could be removed on a second reading, which would allow the reader, having already an overall view of the whole, to revisit the various parts.