Preface

Automorphic forms on the upper half plane have been studied for a long time. Most attention has gone to the holomorphic automorphic forms, with numerous applications to number theory. Maass, [34], started a systematic study of real analytic automorphic forms. He extended Hecke's relation between automorphic forms and Dirichlet series to real analytic automorphic forms. The names Selberg and Roelcke are connected to the spectral theory of real analytic automorphic forms, see, e.g., [50], [51]. This culminates in the trace formula of Selberg, see, e.g., Hejhal, [21].

Automorphic forms are functions on the upper half plane with a special transformation behavior under a discontinuous group of non-euclidean motions in the upper half plane. One may ask how automorphic forms change if one perturbs this group of motions. This question is discussed by, e.g., Hejhal, [22], and Phillips and Sarnak, [46]. Hejhal also discusses the effect of variation of the multiplier system (a function on the discontinuous group that occurs in the description of the transformation behavior of automorphic forms). In [5]–[7] I considered variation of automorphic forms for the full modular group under perturbation of the multiplier system. A method based on ideas of Colin de Verdière, [11], [12], gave the meromorphic continuation of Eisenstein and Poincaré series as functions of the eigenvalue and the multiplier system jointly. The present study arose from a plan to extend these results to much more general groups (discrete cofinite subgroups of $\mathrm{SL}_2(\mathbb{R})$).

To carry this out I look at more general families of automorphic forms than one usually considers. In particular, I admit singularities inside the upper half plane, and relax the usual condition of polynomial growth at the cusps. This led me to reconsider a fairly large part of the theory of real analytic automorphic forms on the upper half plane.

This is done in Part I for arbitrary cofinite discrete groups. Chapters 2–6 discuss real analytic automorphic forms of a rather general type. Most results are known, or are easy extensions of known results. Chapters 7–12 consider families of these automorphic forms, with the eigenvalue and the multiplier system as the parameters. The ideas of Colin de Verdière are worked out in Chapters 8–10. The central result is Theorem 10.2.1; it gives the meromorphic continuation of Eisenstein and Poincaré series. The meromorphic continuation in the eigenvalue is well known; the meromorphy in all parameters jointly is new. Chapters 11 and 12 study singularities of the resulting families of automorphic forms. In Chapter 11 the eigenvalue is the sole variable. I summarize known results, and prepare for the study in Chapter 12 of the singularities in more than one variable. Table 1.1 on p. 18 gives a more detailed description of Part I.

The treatment in Part I is complicated. This is due to the fact that I consider general cofinite discrete groups, without restriction on the dimension of the group of multiplier systems. Chapter 1 is meant as an introduction. It explains the main ideas and results in the context of the full modular group. In the three chapters of Part II I consider three examples of cofinite discrete groups. Chapter 13 extends the discussion for the full modular group in Chapter 1. The other groups considered are the theta group and the commutator subgroup of the modular group. Although the first objective of Part II is to give the reader examples of the concepts, I have included some discussions that did not fit into the general context of Part I; see 1.7.7–1.7.10.

The reader I have had in mind has seen automorphic forms before, holomorphic as well as real analytic ones. For the latter I would suggest to have a look at Chapters IV and V of Maass's lecture notes [35], or at §3.5–7 of Terras's book [57]. The reader should also be prepared to look up facts concerning analytic functions in more than one complex variable, and be not afraid of a modest use of sheaf language when dealing with this subject.

The ideas of Colin de Verdière employed in Part I concern unbounded operators in Hilbert spaces. Kato's book, [25], is consulted for many results from functional analysis. I restrict the discussion of the spectral theory of automorphic forms to those results I need. In particular I do not mention the continuous part of the spectral decomposition.

R. Matthes visited Utrecht in 1989. The discussions we had gave me the stimulus to start this work. I am very grateful for this contribution, and also for the many comments he gave on early versions of this book. At a later stage the interest of D. Zagier has been a great encouragement. I also thank F. Beukers, J. Elstrodt, and B. van Geemen for corrections, comments, and suggestions.