

Preface

Optimization is a rich and thriving mathematical discipline. Properties of minimizers and maximizers of functions rely intimately on a wealth of techniques from mathematical analysis, including tools from calculus and its generalizations, topological notions, and more geometric ideas. The theory underlying current computational optimization techniques grows ever more sophisticated—duality-based algorithms, interior point methods, and control-theoretic applications are typical examples. The powerful and elegant language of convex analysis unifies much of this theory. Hence our aim of writing a concise, accessible account of convex analysis and its applications and extensions, for a broad audience.

For students of optimization and analysis, there is great benefit to blurring the distinction between the two disciplines. Many important analytic problems have illuminating optimization formulations and hence can be approached through our main variational tools: subgradients and optimality conditions, the many guises of duality, metric regularity and so forth. More generally, the idea of convexity is central to the transition from classical analysis to various branches of modern analysis: from linear to nonlinear analysis, from smooth to nonsmooth, and from the study of functions to multifunctions. Thus, although we use certain optimization models repeatedly to illustrate the main results (models such as linear and semidefinite programming duality and cone polarity), we constantly emphasize the power of abstract models and notation.

Good reference works on finite-dimensional convex analysis already exist. Rockafellar's classic *Convex Analysis* [167] has been indispensable and ubiquitous since the 1970s, and a more general sequel with Wets, *Variational Analysis* [168], appeared recently. Hiriart-Urruty and Lemaréchal's *Convex Analysis and Minimization Algorithms* [97] is a comprehensive but gentler introduction. Our goal is not to supplant these works, but on the contrary to promote them, and thereby to motivate future researchers. This book aims to make converts.

We try to be succinct rather than systematic, avoiding becoming bogged down in technical details. Our style is relatively informal; for example, the text of each section creates the context for many of the result statements. We value the variety of independent, self-contained approaches over a single, unified, sequential development. We hope to showcase a few memorable principles rather than to develop the theory to its limits. We discuss no algorithms. We point out a few important references as we go, but we make no attempt at comprehensive historical surveys.

Optimization in infinite dimensions lies beyond our immediate scope. This is for reasons of space and accessibility rather than history or application: convex analysis developed historically from the calculus of variations, and has important applications in optimal control, mathematical economics, and other areas of infinite-dimensional optimization. However, rather like Halmos's *Finite Dimensional Vector Spaces* [90], ease of extension beyond finite dimensions substantially motivates our choice of approach. Where possible, we have chosen a proof technique permitting those readers familiar with functional analysis to discover for themselves how a result extends. We would, in part, like this book to be an entrée for mathematicians to a valuable and intrinsic part of modern analysis. The final chapter illustrates some of the challenges arising in infinite dimensions.

This book can (and does) serve as a teaching text, at roughly the level of first year graduate students. In principle we assume no knowledge of real analysis, although in practice we expect a certain mathematical maturity. While the main body of the text is self-contained, each section concludes with an often extensive set of optional exercises. These exercises fall into three categories, marked with zero, one, or two asterisks, respectively, as follows: examples that illustrate the ideas in the text or easy expansions of sketched proofs; important pieces of additional theory or more testing examples; longer, harder examples or peripheral theory.

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Preface to the Second Edition

Since the publication of the First Edition of this book, convex analysis and nonlinear optimization has continued to flourish. The “interior point revolution” in algorithms for convex optimization, fired by Nesterov and Nemirovski’s seminal 1994 work [148], and the growing interplay between convex optimization and engineering exemplified by Boyd and Vandenberghe’s recent monograph [47], have fuelled a renaissance of interest in the fundamentals of convex analysis. At the same time, the broad success of key monographs on general variational analysis by Clarke, Ledyaev, Stern and Wolenski [56] and Rockafellar and Wets [168] over the last decade testify to a ripening interest in nonconvex techniques, as does the appearance of [43].

The Second Edition both corrects a few vagaries in the original and contains a new chapter emphasizing the rich applicability of variational analysis to concrete examples. After a new sequence of exercises ending Chapter 8 with a concise approach to monotone operator theory via convex analysis, the new Chapter 9 begins with a presentation of Rademacher’s fundamental theorem on differentiability of Lipschitz functions. The subsequent sections describe the appealing geometry of proximal normals, four approaches to the convexity of Chebyshev sets, and two rich concrete models of nonsmoothness known as “amenability” and “partial smoothness”. As in the First Edition, we develop and illustrate the material through extensive exercises.

Convex analysis has maintained a Canadian thread ever since Fenchel’s original 1949 work on the subject in Volume 1 of the *Canadian Journal of Mathematics* [76]. We are grateful to the continuing support of the Canadian academic community in this project, and in particular to the Canadian Mathematical Society, for their sponsorship of this book series, and to the Canadian Natural Sciences and Engineering Research Council for their support of our research endeavours.

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