

Homological Algebra of Semimodules and Semicontramodules

Semi-infinite Homological Algebra of Associative Algebraic Structures

Bearbeitet von
Leonid Positselski

1. Auflage 2010. Buch. xxiv, 352 S. Hardcover
ISBN 978 3 0346 0435 2
Format (B x L): 15,5 x 23,5 cm
Gewicht: 793 g

[Weitere Fachgebiete > Mathematik > Algebra > Homologische Algebra](#)

schnell und portofrei erhältlich bei


DIE FACHBUCHHANDLUNG

Die Online-Fachbuchhandlung beck-shop.de ist spezialisiert auf Fachbücher, insbesondere Recht, Steuern und Wirtschaft. Im Sortiment finden Sie alle Medien (Bücher, Zeitschriften, CDs, eBooks, etc.) aller Verlage. Ergänzt wird das Programm durch Services wie Neuerscheinungsdienst oder Zusammenstellungen von Büchern zu Sonderpreisen. Der Shop führt mehr als 8 Millionen Produkte.

Preface*

The subject of this book is Semi-Infinite Algebra, or more specifically, Semi-Infinite Homological Algebra. The term “semi-infinite” is loosely associated with objects that can be viewed as extending in both a “positive” and a “negative” direction, with some natural position in between, perhaps defined up to a “finite” movement. Geometrically, this would mean an infinite-dimensional variety with a natural class of “semi-infinite” cycles or subvarieties, having always a finite codimension in each other, but infinite dimension and codimension in the whole variety [37]. (For further instances of semi-infinite mathematics see, e.g., [38] and [57], and references below.)

Examples of algebraic objects of the semi-infinite type range from certain infinite-dimensional Lie algebras to locally compact totally disconnected topological groups to ind-schemes of ind-infinite type to discrete valuation fields. From an abstract point of view, these are ind-pro-objects in various categories, often endowed with additional structures. One contribution we make in this monograph is the demonstration of another class of algebraic objects that should be thought of as “semi-infinite”, even though they do not at first glance look quite similar to the ones in the above list. These are *semialgebras* over coalgebras, or more generally over corings – *the* associative algebraic structures of semi-infinite nature.

The subject lies on the border of Homological Algebra with Representation Theory, and the introduction of semialgebras into it provides an additional link with the theory of corings [23], as the semialgebras are the natural objects dual to corings. The author’s main interests belong to Homological Algebra, and so the main body of the monograph consists of the formal development of the homological theory of corings and semialgebras, while the representation-theoretic (and other) examples and applications are relegated to appendices.

One such application worth mentioning here is related to the duality between complexes of representations of an infinite-dimensional Lie algebra with the complementary central charges, e.g., c and $26 - c$ for the Virasoro algebra [39, 77]. We interpret it as a particular case of a very general homological phenomenon related to coalgebras, which we call the *comodule-contramodule correspondence*. The latter is a coalgebra version of the Serre–Grothendieck duality – covariant, noncommutative, and not depending on any finiteness assumptions (the coalgebra \mathcal{C} itself plays the role of the dualizing complex; cf. [65, 71]). This allows us to formulate the duality for infinite-dimensional Lie algebra representations as a (covariant) equivalence of triangulated categories.

On a less ambitious level, with the formal neighborhood of a closed subgroup in an algebraic group one can associate a semialgebra of (roughly speaking) dis-

*What follows is very speculative and should be taken with a grain of salt.

tributions on it, and the category of Harish-Chandra modules over an algebraic Harish-Chandra pair is the category of semimodules over this semialgebra. For further applications to Representation Theory, see [16], [17], and [45].

Another important area that Semi-Infinite Algebra and Geometry are related to is Mathematical Physics. The author of this monograph stands at the receiving end of a long chain of interpretative work through which the ideas originating in the interaction of Mathematics with Quantum Field Theory or String Theory are transferred to the heart of Algebra. We are not in a position to comment here on the possibilities of applications of the content of this book to Mathematical Physics, so we will restrict ourselves to a couple of references and some very general remarks. The semi-infinite homology of Lie algebras are closely related to what the physicists call the BRST construction [10, 11]; for a discussion of the significance of the semi-infinite homology in String Theory, see [41] and the introduction to [42].

The field of functions on the formal circle, which is the field of Laurent power series, is a very simple example of a semi-infinite algebraic object; and much more complicated algebraic or geometric objects built on the basis of the formal circle often have very visible semi-infinite structures. This includes the Virasoro and affine Kac-Moody Lie algebras, the varieties [58] and groups of formal loops, the semi-infinite flag variety [37], etc. The formal circle is obviously important for Conformal Field Theory [12], hence the significance of such objects of study as the semi-infinite homology [10, 40], the semi-infinite de Rham complex [58, 8], or the chiral differential operators [10, 59] in Mathematical Physics.

Things semi-infinite play a role in Class Field Theory [83] and the Langlands Program [11, 40] for the very same reason. A much more detailed discussion of the links between the semi-infinite cohomology and various other mathematical and physical disciplines can be found in the introduction to [84].

Another class of algebraic objects prominently featured in this monograph is that of *contramodules*. Their definition, introduced originally in the case of coalgebras or corings [33], can be extended to certain topological rings, topological Lie algebras, certain topological groups . . . These are modules with infinite summation operations, but of the kind that cannot be interpreted as any sort of limit with respect to a topology. Here one finds an approach to the infinite summation entirely different from the one most common to Analysis.

Typically, for an abelian category of “discrete”, “smooth”, or “torsion” modules there is an accompanying abelian category of contramodules. The latter contains all kinds of objects “dual” to the objects of the former, and some other objects in addition. For example, the category of “weakly l -complete abelian groups” appearing in the continuous étale cohomology theory [54] is simply the category of contramodules over the l -adic integers. While not “semi-infinite” in themselves, contramodules always come up whenever one wishes to pass from a semi-infinite *homology* to a semi-infinite *cohomology* theory.

One area where our approach is inspired by, still essentially different from, the classical one is Relative Homological Algebra. While the classical theory [47, 33, 80, 35] emphasizes relative derived functors of nonexact functors that may be quite conventional and not necessarily “relative” in themselves, here we are mainly interested in *absolute* derived functors, but the nonexact functors that we derive and the categories where they are defined are essentially relative by their nature. We always want our derived functors to assign long exact sequences of cohomology to arbitrary short exact sequences of complexes in the arguments, not only to short exact sequences that are split over some base. Still the base (or even two bases, one over the other) are built into the definitions of the categories and functors we work with.

One thing we cannot pretend to explain, still cannot avoid mentioning here, are the *exotic derived categories*. These are variations on the theme of the unbounded derived category. Their names are the (conventional unbounded) derived, the *coderived*, the *contraderived*, and the mixed, or *semiderived* categories. Historically, these first occurred in the derived nonhomogeneous Koszul duality theory [61], but from a wider point of view, the coderived and contraderived categories appear to be intrinsic to the comodules and contra-modules (while occasionally useful for modules, too). For a definitive treatment of the exotic derived categories and their role in Koszul duality, we refer the reader to the long paper [76]. As to the nonhomogeneous Koszul duality itself, it is developed and used in this book as a strong technical tool.

An object of the contraderived category can be thought of as a complex having, in addition to the conventional cohomology at finite degrees, some kind of “cohomology in the degree $+\infty$ ”. Analogously, a complex in the coderived category can be viewed as having a “cohomology in the degree $-\infty$ ”. This is essential, in particular, for the construction of the comodule-contra-module correspondence, as the latter can well transform irreducible modules into acyclic complexes (i.e., those with no cohomology anywhere but “at infinity”) and back. For example, an acyclic, but nontrivial object in the contraderived category of contra-modules can be represented by an acyclic, unbounded complex of projective contra-modules, and the latter thought of as a “left projective resolution of something living in the degree $+\infty$ ”.

We also propose a very simple, bordering on self-evident, still apparently not widely known, approach to derived functors of two arguments, which allows us to obtain double-sided derived functors for free. It wouldn’t get one too far without the exotic derived categories, though. The concrete double-sided derived functors we are interested in are the *SemiExt* and *SemiTor* over semias-sociative semialgebras, and the semi-infinite (co)homology of Tate Lie algebras and locally compact totally disconnected topological groups. The semimodule-semicontra-module correspondence connects these with the more conventional one-sided *Ext* and *CtrTor*.

The functors to be derived are the *semitensor product* and *seminhomomorphisms* in the semiassociative case, and the *semiinvariants* and *semicontrainvariants* in the Lie algebra or topological group case. These neither left, nor right exact functors are naturally associated with certain semi-infinite algebraic structures, and particularly with semialgebras. Still they are nontrivial enough even for finite-dimensional Lie algebras and finite groups.

To end these preparatory notes, let us say a few words about the state of the subject after this monograph. It appears that the question of *defining* the semi-infinite homology and cohomology generally, and in the case of associative algebraic structures specifically, has been now worked out and understood to a very significant extent. Compared to this development, our knowledge of the ways of *computing* the semi-infinite cohomology is next to nonexistent outside of the classical Lie algebra case. The only example where the semi-infinite cohomology of associative algebras has been computed as of now is that of the small quantum group with its triangular decomposition [4, 17]. The methods used for this computation have so far resisted, essentially, all attempts of transfer to other situations or generalization. Computing the semi-infinite cohomology remains a challenge for future researchers to take on.

It is our special pleasure to finish these most cursory remarks with a reference to B. Feigin's paper [36] that introduced the semi-infinite homology and the very term *semi-infinite*.