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Intersection Spaces, Spatial Homology Truncation, and String Theory

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Preface

The primary concern of the work presented here is Poincaré duality for spaces that are not manifolds, but are still put together from manifolds that form the strata of a stratification of the space. Goresky and MacPherson's intersection homology [GM80, GM83], see also [B$^+$84, KW06, Ban07], associates to a stratified pseudomanifold X chain complexes $IC_*^{\bar{p}}(X;\mathbb{Q})$ depending on a perversity parameter $\bar{p}$, whose homology $IH_*^{\bar{p}}(X;\mathbb{Q}) = H_*(IC_*^{\bar{p}}(X;\mathbb{Q}))$ satisfies generalized Poincaré duality across complementary perversities when X is closed and oriented. L^2-cohomology [Che80, Che79, Che83] associates to a triangulated pseudomanifold X equipped with a suitable conical Riemannian metric on the top stratum a differential complex $\Omega^*_{(2)}(X)$, the complex of differential L^2-forms ω on the top stratum of X such that $d\omega$ is L^2 as well, whose cohomology $H^*_{(2)}(X) = H^*(\Omega^*_{(2)}(X))$ satisfies Poincaré duality (at least when X has no strata of odd codimension; in more general situations one must choose certain boundary conditions). The linear dual of $IH_*^{\bar{m}}(X;\mathbb{R})$ is isomorphic to $H^*_{(2)}(X)$, by integration. In the present work, we adopt the "spatial philosophy" outlined in the announcement [Ban09], maintaining that a theory of Poincaré duality for stratified spaces benefits from being implemented on the level of spaces, with passage to coarser filters such as chain complexes, homology or homotopy groups occurring as late as possible in the course of the development. Thus we pursue here the following program. To a stratified pseudomanifold X, associate *spaces*

$$I^{\bar{p}}X,$$

the *intersection spaces of* X, such that the *ordinary* homology $\widetilde{H}_*(I^{\bar{p}}X;\mathbb{Q})$ satisfies generalized Poincaré duality when X is closed and oriented. If X has no odd-codimensional strata and $\bar{p}$ is the middle perversity $\bar{p} = \bar{m}$, then we are thus assigning to a singular pseudomanifold a (rational) Poincaré complex. The resulting homology theory $X \rightsquigarrow \widetilde{H}_*(I^{\bar{p}}X)$ is not isomorphic to intersection homology or L^2-cohomology. In fact, it solves a problem in type II string theory related to the existence of massless D-branes, which is neither solved by ordinary homology nor by intersection homology. We show that while $IH_*^{\bar{m}}(X)$ is the correct theory in the realm of type IIA string theory (giving the physically correct counts of massless particles), $\widetilde{H}_*(I^{\bar{m}}X)$ is the correct theory in the realm of type IIB string theory. In other words, the two theories $IH_*^{\bar{m}}(X), \widetilde{H}_*(I^{\bar{m}}X)$ form a mirror pair in the sense of mirror symmetry in algebraic geometry. We will return to these considerations in more detail later in this preface.

The assignment $X \rightsquigarrow I^{\bar{p}}X$ should satisfy the following requirements:

1. $\widetilde{H}_*(I^{\bar{p}}X;\mathbb{Q})$ should satisfy generalized Poincaré duality across complementary perversities.
2. $\widetilde{H}_*(I^{\bar{p}}X;\mathbb{Q})$ should be a mirror of $IH^{\bar{m}}_*(X;\mathbb{Q})$ in the sense of mirror symmetry.
3. $X \rightsquigarrow I^{\bar{p}}X$ should be as "natural" as possible.
4. X should be modified as little as possible (only near the singularities; the homotopy type away from the singularities should be completely preserved).
5. If X is a finite cell complex, then $I^{\bar{p}}X$ should again be a finite cell complex.
6. $X \rightsquigarrow I^{\bar{p}}X$ should be homotopy-theoretically tractable, so as to facilitate computations.

Note that full naturality in (3) with respect to all continuous maps is too much to expect, since a corresponding property cannot be achieved for intersection homology either. In order to demonstrate (6), we have worked out numerous examples throughout the text, giving concrete intersection spaces for pseudomanifolds ranging from toy examples to complex algebraic threefolds and Calabi–Yau conifolds arising in mathematical physics. In the present monograph, we carry out the above program for pseudomanifolds with isolated singularities as well as, more generally, for two-strata spaces with arbitrary bottom stratum but trivial link bundle. In addition, we make suggestions for how to proceed when there are more than two strata, or when the link bundle is twisted. Future research will have to determine the ultimate domain of pseudomanifolds for which an intersection space is definable. Throughout the general development of the theory, we assume the links of singular strata to be simply connected. In concrete applications, this assumption is frequently unnecessary, see also the paragraph preceding Example 2.15. In the example, we discuss the intersection space of a concrete space whose links are not simply connected. Our construction of intersection spaces is of a homotopy-theoretic nature, resting on technology for spatial homology truncation, which we develop in this book. This technology is completely general, so that it may be of independent interest.

What are the purely mathematical advantages of introducing intersection spaces? Algebraic Topology has developed a vast array of functors defined on spaces, many of which do not factor through chain complexes. For instance, let E_* be any generalized homology theory, defined by a spectrum E, such as K-theory, L-theory, stable homotopy groups, bordism and so on. One may then study the composite assignment

$$X \rightsquigarrow I^{\bar{p}}E_*(X) := E_*(I^{\bar{p}}X).$$

Section 2.7, for example, studies symmetric L-homology, where E_* is given by Ranicki's symmetric L-spectrum $E = \mathbb{L}^\bullet$. We show in Corollary 2.38 that capping with the $\mathbb{L}^\bullet$-homology fundamental class of an n-dimensional oriented compact pseudomanifold X with isolated singularities indeed induces a Poincaré duality isomorphism

$$\widetilde{H}^0(I^{\bar{m}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q} \xrightarrow{\cong} \widetilde{H}_n(I^{\bar{n}}X;\mathbb{L}^\bullet)\otimes\mathbb{Q}.$$

K-theory is discussed in Section 2.8. A *$\bar{p}$-intersection vector bundle on X* may be defined as an actual vector bundle on $I^{\bar{p}}X$. More generally, given any structure group G, one may define *principal intersection G-bundles over X* as homotopy classes of maps $I^{\bar{p}}X \to BG$. In Example 2.40, we show that there are infinitely many distinct seven-dimensional pseudomanifolds X, whose tangent bundle elements in the KO-theory $\widetilde{\mathrm{KO}}(X - \mathrm{Sing})$ of their nonsingular parts do not lift to $\widetilde{\mathrm{KO}}(X)$, but do lift to $\widetilde{\mathrm{KO}}(I^{\bar{n}}X)$, where $\bar{n}$ is the upper middle perversity. So this framework allows one to formulate the requirement that a pseudomanifold have a $\bar{p}$-intersection tangent bundle, and by varying $\bar{p}$, the severity of this requirement can be adjusted at will. Ultimately, one may want to study the Postnikov tower of $I^{\bar{p}}X$ and view it as a "$\bar{p}$-intersection Postnikov tower" of X.

A further asset of the spatial philosophy is that cochain complexes will automatically come equipped with internal multiplications, making them into differential graded algebras (DGAs). The Goresky–MacPherson intersection chain complexes $IC_*^{\bar{p}}(X)$ are generally not algebras, unless $\bar{p}$ is the zero-perversity, in which case $IC_*^{\bar{p}}(X)$ is essentially the ordinary cochain complex of X. (The Goresky–MacPherson intersection product raises perversities in general.) Similarly, the differential complex $\Omega^*_{(2)}(X)$ of L^2-forms on $X - \mathrm{Sing}$ is not an algebra under wedge product of forms because the product of two L^2-functions need not be L^2 anymore (consider for example $r^{-1/3}$ for small $r > 0$). Using the intersection space framework, the ordinary cochain complex $C^*(I^{\bar{p}}X)$ of $I^{\bar{p}}X$ *is* a DGA, simply by employing the ordinary cup product. For similar reasons, the cohomology of $I^{\bar{p}}X$ is by default endowed with internal cohomology operations, which do not exist for intersection cohomology. These structures, along with Massey triple products and other secondary and higher order operations, remain to be investigated elsewhere. Operations in intersection cohomology that weaken the perversity by a factor of two have been constructed in [Gor84].

In Section 2.6, we construct cap products of the type

$$\widetilde{H}^r(I^{\bar{m}}X) \otimes \widetilde{H}_i(X) \xrightarrow{\cap} \widetilde{H}_{i-r}(I^{\bar{n}}X). \tag{0.1}$$

These products have their applications not only in formulating and proving duality statements, but also in developing various characteristic class formulae, which may lead to extensions of the results of [BCS03, Ban06a]. An $\bar{m}$-intersection vector bundle on X has Chern classes in $H^{\mathrm{even}}(I^{\bar{m}}X)$. Characteristic classes of pseudomanifolds, such as the L-class, generally lie only in $H_*(X;\mathbb{Q})$ and do not lift to intersection homology or to $H_*(I^{\bar{m}}X;\mathbb{Q})$, see for example [GM80, Ban06b]. Consequently, the ordinary cap product $H^r(I^{\bar{m}}X) \otimes H_i(I^{\bar{m}}X) \to H_{i-r}(I^{\bar{m}}X)$ is useless in multiplying the Chern classes of the bundle and the characteristic classes of the pseudomanifold. The above product (0.1) then enables one to carry out such a multiplication. The product (0.1) seems counterintuitive from the point of view of intersection homology because an analogous product

$$IH^r(X) \otimes H_i(X) \dashrightarrow IH_{i-r}(X)$$

on intersection homology cannot exist. The motivational Section 2.6.1 explains why the desired product cannot exist for intersection homology but does exist for intersection space homology. The products themselves are constructed in Section 2.6.3.

Let us briefly indicate how intersection spaces are constructed. We are guided initially by mimicking spatially what intersection homology does algebraically. By Mayer-Vietoris sequences, the overall behavior of intersection homology is primarily controlled by its behavior on cones. If L is a closed n-dimensional manifold, $n > 0$, then

$$IH_r^{\bar{p}}(\overset{\circ}{\mathrm{cone}}(L)) \cong \begin{cases} H_r(L), & r < n - \bar{p}(n+1), \\ 0, & \text{otherwise}, \end{cases}$$

where $\overset{\circ}{\mathrm{cone}}(L)$ denotes the open cone on L and we are using intersection homology built from finite chains. Thus, intersection homology is a process of truncating the homology of a space algebraically above some cut-off degree given by the perversity and the dimension of the space. This is also apparent from Deligne's formula for the intersection chain sheaf. The task at hand is to implement this spatially. Let $\mathbf{C}$ be a category of spaces, that is, a category with a functor $i : \mathbf{C} \to \mathbf{Top}$ to the category $\mathbf{Top}$ of topological spaces and continuous maps. (For instance, $\mathbf{C}$ might be a subcategory of $\mathbf{Top}$ and i the inclusion functor, but it might also be spaces endowed with extra structure with i the forgetful functor, etc.) Let $p : \mathbf{Top} \to \mathbf{HoTop}$ be the natural projection functor to the homotopy category of spaces, sending a continuous map to its homotopy class. Suppose then that we had a functor

$$t_{<k} : \mathbf{C} \longrightarrow \mathbf{HoTop},$$

where k is a positive integer, together with a natural transformation $\mathrm{emb}_k : t_{<k} \to pi$ (think of pi as the "identity functor") such that

$$\mathrm{emb}_k(L)_* : H_r(t_{<k}(L)) \longrightarrow H_r(pi(L))$$

is an isomorphism for $r < k$, while $H_r(t_{<k}(L)) = 0$ for $r \geq k$, for all objects L in $\mathbf{C}$. We refer to such a functor as a *spatial homology truncation functor*. Let X be an n-dimensional closed pseudomanifold with one isolated singular point. Such an X is of the form

$$X = M \cup_{\partial M = L} \mathrm{cone}(L),$$

where L, a closed manifold of dimension $n-1$, is the link of the singularity, and M, a compact manifold with boundary $\partial M = L$, is the complement of a small open cone-neighborhood of the singularity. Assume that L gives rise to an object L in $\mathbf{C}$. The intersection space $I^{\bar{p}}X$ is defined to be the homotopy cofiber of the composition

$$t_{<k}(L) \overset{\mathrm{emb}_k(L)}{\longrightarrow} pi(L) = L = \partial M \hookrightarrow M,$$

where $k = n - 1 - \bar{p}(n)$, see Definition 2.10. In other words: we attach the cone on a suitable spatial homology truncation of the link to the exterior of the singularity along the boundary of the exterior. The two extreme cases of this construction arise when $k = 1$ and when k is larger than the dimension of the link. In the former case, $t_{<1}(L)$ is a point (at least when L is path connected) and thus $I^{\bar{p}}X$ is homotopy equivalent to the nonsingular part $X - \mathrm{Sing}$ of X. In the latter case no actual truncation has to be performed, $t_{<k}(L) = L$, $\mathrm{emb}_k(L)$ is the identity map and thus $I^{\bar{p}}X = X$. If there are several isolated singularities, then we perform spatial homology truncation on each of the links. If the singularities are not isolated, a process of fiberwise spatial homology truncation applied to the link bundle has to be used, see Section 2.9. If there are more than two nested strata, then more elaborate homotopy colimit constructions involving iterated truncation techniques can be used.

Theorem 2.12 establishes generalized Poincaré duality for the rational homology of intersection spaces and simultaneously analyzes the relation to intersection homology, both in the isolated singularity case. This relation is of a "reflective" nature (which is also responsible for both theories being mirrors of each other in the context of singular Calabi–Yau threefolds). The requisite abstract language of reflective diagrams is introduced in Section 2.1. Of particular interest here is to understand what happens at the cut-off degree k, which is the middle dimension for the middle perversity. The reflective diagram shows that while $IH_k^{\bar{p}}(X)$ is generally smaller than both $H_k(X - \mathrm{Sing})$ and $H_k(X)$, being a quotient of the former and a subgroup of the latter, $H_k(I^{\bar{p}}X)$, on the other hand, is generally bigger than both $H_k(X - \mathrm{Sing})$ and $H_k(X)$, containing the former as a subgroup and mapping to the latter surjectively. Section 3.9 contains an example of a singular quintic S (a conifold) in $\mathbb{P}^4$ such that $H_3(IS)$ has rank 204, but $IH_3(S)$ has only rank 2. Corollary 2.14 computes the difference of the Euler characteristics of the two theories. As far as Witt groups are concerned, both theories lead to equivalent intersection forms: We prove in Theorem 2.28 that for a pseudomanifold X of dimension $n = 4m$, the symmetric intersection form on $IH_{2m}^{\bar{m}}(X)$ and the symmetric intersection form on $H_{2m}(I^{\bar{m}}X)$ determine the same element in the Witt group of the rationals. In particular, the signature of the two forms are equal. Definition 2.41 contains the construction of $I^{\bar{p}}X$ for a space X with a positive dimensional singular stratum with untwisted link bundle. Theorem 2.47 establishes generalized Poincaré duality in this context.

As our approach relies on the ability to perform spatial homology truncation, Chapter 1 is devoted to a systematic investigation of this problem. The investigation and results are of a general nature and can be read and used independently of any interest in intersection spaces. Throughout the development, we strive to remain firmly on the plane of elementary homotopy theory, using only classical instruments, working unstably, avoiding simplicial or model theoretic language, as such language does not seem to yield any particular advantage here. Our spaces in this chapter will be simply connected CW-complexes because, just as Hilton [Hil65] does, we wish to avail ourselves of the Hurewicz and the Whitehead theorem. Spatial homology truncation *on the object level* has been studied by several researchers: the Eckmann–Hilton dual of the Postnikov decomposition is the homology decomposition (or Moore space decomposition) of a space, see [Hil65, BJCJ59, Moo]. It seems that the problem has not received much attention on the morphism level; see, however, [Bau88] for a tower of categories. Consequently, we focus on aspects of functoriality, and this is where homology truncation turns out to be harder than Postnikov truncation because obstructions surface that do not arise in the Postnikov picture. Given a space X, let $p_n(X) : X \to P_n(X)$ denote a stage-n Postnikov approximation for X. If $f : X \to Y$ is any map, then there exists, uniquely up to homotopy, a map $p_n(f) : P_n(X) \to P_n(Y)$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p_n(X)"'] & Y \arrow[d, "p_n(Y)"] \\ P_n(X) \arrow[r, "p_n(f)"] & P_n(Y) \end{tikzcd}$$

homotopy commutes. In the introductory Section 1.1.1 we give an example that shows that this property does not Eckmann–Hilton dualize to spatial homology truncation. Thus a homology truncation functor in this naive sense cannot exist. Our solution proposes to consider spaces endowed with an extra structure. Morphisms should preserve this extra structure; one obtains a category $\mathbf{CW}_{n\supset\partial}$. What is this extra structure? Hilton's homology decomposition really depends on a choice of complement to the group of n-cycles inside of the nth chain group. Such a complement always exists and pairs (space, choice of complement) are the objects of $\mathbf{CW}_{n\supset\partial}$; morphisms are cellular maps that map the complement chosen for the domain to the complement chosen for the codomain. The Compression Theorem 1.32 shows that such morphisms can always be compressed into spatial homology truncations. The upshot at this stage is that we obtain a covariant assignment

$$t_{<n} : \mathbf{CW}_{n\supset\partial} \longrightarrow \mathbf{HoCW}_{n-1}$$

of objects and morphisms into the rel $(n-1)$-skeleton homotopy category of CW-complexes together with a natural transformation emb_n from $t_{<n}$ to the identity, such that for every object (K,Y) of $\mathbf{CW}_{n\supset\partial}$, where K is a simply connected CW-complex and Y a complement as discussed above,

$$\mathrm{emb}_n(K,Y)_* : H_r(t_{<n}(K,Y)) \longrightarrow H_r(K)$$

is an isomorphism for $r < n$ and $H_r(t_{<n}(K,Y)) = 0$ for $r \geq n$, see the first part of Theorem 1.41. (Note that we do not at this stage claim that $t_{<n}$ is a functor on all of $\mathbf{CW}_{n\supset\partial}$.) This solves the first order problem of the existence of compressions of maps. Immediately, the second order problem of the uniqueness of compressions presents itself. Example 1.9 shows that even when domain and codomain of a map f have unique homological n-truncations and f does have a homological n-truncation, the homotopy class of that truncation may not be uniquely determined by f. The obvious idea of imposing the above requirement of complement-preservation also on homotopies and then just applying the Compression Theorem 1.32 to compress the homotopy into spatial homology truncations does not work. We call a map *n-compression rigid*, if its compression into n-truncations

agrees with f on the $(n-1)$-skeleton and is unique up to rel $(n-1)$-skeleton homotopy, see Definition 1.33 and Proposition 1.34. Example 1.35 exposes a map that is not compression rigid, even though its domain and codomain have unique n-truncations. As an instrument for understanding compression rigidity, we introduce virtual cell groups VC_n of a space, so named because they are homotopy groups which are not themselves cellular chain groups, but they sit naturally between two actual cellular chain groups of certain cylinders. The virtual cell groups come equipped with an endomorphism so that we can formulate the concept of a 1-eigenclass (or eigenclass for short) for elements of VC_n. We show that a map is compression rigid if and only if the homotopies coming from the homotopy commutativity of the transformation square associated to emb_n can be chosen to be eigenclasses in VC_n. For 2-connected spaces, virtual cell groups are computed in Proposition 1.18. An obstruction theory for compression rigidity is set up in Section 1.2. Case studies of compression rigid categories are presented in Section 1.3. The second part of Theorem 1.41 asserts that the covariant assignment $t_{<n}$ is a *functor* on n-compression rigid subcategories of $\mathbf{CW}_{n\supset\partial}$. The dependence of the spatial homology truncation $t_{<n}(K,Y)$ on Y is discussed by Proposition 1.25, Scholium 1.26, Proposition 1.27 and Corollaries 1.30, 1.31. Proposition 1.25 gives a necessary and sufficient condition for $t_{<n}(K,Y)$ and $t_{<n}(K,\overline{Y})$ to be homotopy equivalent rel $(n-1)$-skeleton, where $Y,\overline{Y}$ are two choices of complements. Section 1.4 deals with the truncation of homotopy equivalences, Section 1.5 with the truncation of inclusions, and Section 1.6 with iterated truncation. In Section 1.7, we investigate spatial homology truncation followed by localization at odd primes. Theorem 1.61 establishes that this composite assignment $t_{<n}^{(\mathrm{odd})}$ is a functor on 2-connected spaces. The key ingredients here are the compression rigidity obstruction theory together with Proposition 1.50, which calculates a pertinent homotopy group and shows that it is all 2-torsion.

There are important classes of spaces where no complement Y has to be chosen and the compression rigidity obstructions vanish. We study one such class in detail, namely spaces with vanishing odd-dimensional homology. We refer to this class as the *interleaf category*, $\mathbf{ICW}$. It includes for instance simply connected 4-manifolds, smooth compact toric varieties, homogeneous spaces arising as the quotient of a complex simply connected semisimple Lie group by a parabolic subgroup (e.g. flag manifolds, Grassmannians), and smooth Schubert varieties. A truncation functor $t_{<n}: \mathbf{ICW} \to \mathbf{HoCW}$ and cotruncation functor $t_{\geq n}: \mathbf{ICW} \to \mathbf{HoCW}$ are defined. Mostly, but not exclusively, in the context of the interleaf category, we investigate continuity properties of the homology truncation of homeomorphisms. We show in Theorem 1.78 that truncation of cellular self-homeomorphisms of an interleaf space is a continuous H-map into the grouplike topological monoid of self-homotopy equivalences of the homology truncation of the space. In Section 1.11, we discuss fiberwise homology truncation for mapping tori (general simply connected fiber), flat bundles over spaces whose fundamental group G has a $K(G,1)$ of dimension at most 2 (for example flat bundles over closed surfaces other than $\mathbb{R}P^2$; again for general simply connected fiber), and fiber bundles over a sphere of dimension greater than 1, with interleaf fiber.

Since spatial homology truncation of a space L in general requires making a choice of a certain type of subgroup Y in the nth chain group of L in order to obtain an object (L,Y) in $\mathbf{CW}_{n\supset\partial}$, and since the construction of intersection spaces uses this truncation on the links L of singularities, the homotopy type of the intersection space $I^{\bar{p}}X$ may well depend, to some extent, on choices. We show (Theorem 2.18) that the rational homology of $I^{\bar{p}}X$ is well-defined and independent of choices. Furthermore, we give sufficient conditions, in terms of the homology of the links in X and the homology of $X - \mathrm{Sing}$, for the integral homology of $I^{\bar{p}}X$ in the cut-off degree to be independent of choices. Away from the cut-off degree, the integral homology is always independent of choices. The conditions are often satisfied in algebraic geometry for the middle perversity, for instance when X is a complex projective algebraic threefold with isolated hypersurface singularities that are weighted homogeneous and "well-formed," see Theorem 2.24. This class of varieties includes in particular conifolds, to be discussed below. Theorem 2.26 asserts that the homotopy type of $I^{\bar{p}}X$ is well-defined independent of choices when all the links are interleaf spaces.

It was mentioned before that the homology of intersection spaces addresses certain questions in type II string theory – let us expand on this. Our viewpoint is informed by [GSW87, Str95, Hüb97]. In addition to the four dimensions that model space–time, string theory requires six dimensions for a string to vibrate. Due to supersymmetry considerations, these six dimensions must be a Calabi–Yau space, but this still leaves a lot of freedom. It is thus important to have mechanisms to move from one Calabi–Yau space to another. A topologist's take on this might be as follows, disregarding the Calabi–Yau property for a moment. Since any two 6-manifolds are bordant ($\Omega_6^{\mathrm{SO}} = 0$) and since, by Morse theory, any bordism is obtained by performing a finite sequence of surgeries, surgery is not an unreasonable vessel to travel between 6-manifolds. Note also that every three-dimensional homology class in a simply connected smooth 6-manifold can be represented, by the Whitney embedding theorem, by an embedded 3-sphere with trivial normal bundle. Physicists' *conifold transition* starts out with a nonsingular Calabi–Yau threefold, passes to a singular variety (the conifold) by a deformation of complex structure, and arrives at a different nonsingular Calabi–Yau threefold by a small resolution of singularities. The deformation collapses embedded 3-spheres to isolated singular points, whose link is $S^3 \times S^2$. The resolution resolves the singular points by replacing each one with a $\mathbb{C}P^1$. As we review in Section 3.6, massless particles in four dimensions should be recorded as classes by good cohomology theories for Calabi–Yau varieties. In type IIA string theory, there are charged twobranes present that wrap around the $\mathbb{C}P^1$ 2-cycles and that become massless when those 2-cycles are collapsed to points by the resolution map, see Section 3.5. We show that intersection homology accounts for all of these massless twobranes and thus is the physically correct homology theory for type IIA string theory. However, in type IIB string theory, there are charged threebranes present that wrap around the 3-spheres and that become massless when those 3-spheres are collapsed to points by the deformation of complex structure. Neither the ordinary homology of the conifold, nor its intersection homology (or L^2-cohomology) accounts for these massless threebranes. In

Proposition 3.6 we prove that the homology of the intersection space of the conifold yields the correct count of these threebranes. From this point of view, the homology of intersection spaces appears to be a physically suitable homology theory in the IIB regime. The theory in particular answers a question posed by [Hüb97] in this regard. Given a Calabi–Yau threefold M, the mirror map associates to it another Calabi–Yau threefold W such that type IIB string theory on $\mathbb{R}^4 \times M$ corresponds to type IIA string theory on $\mathbb{R}^4 \times W$. If M and W are nonsingular, then $b_3(W) = (b_2 + b_4)(M) + 2$ and $b_3(M) = (b_2 + b_4)(W) + 2$ for the Betti numbers of ordinary homology. The preceding discussion suggests that if M and W are singular, $\mathcal{H}_*^{\mathrm{IIA}}$ is a type IIA D-brane-complete homology theory with Poincaré duality, and $\mathcal{H}_*^{\mathrm{IIB}}$ is a type IIB D-brane-complete homology theory with Poincaré duality, then one should expect that

$$\begin{aligned}
\operatorname{rk}\mathcal{H}_3^{\mathrm{IIA}}(M) &= \operatorname{rk}\mathcal{H}_2^{\mathrm{IIB}}(W) + \operatorname{rk}\mathcal{H}_4^{\mathrm{IIB}}(W) + 2,\\
\operatorname{rk}\mathcal{H}_3^{\mathrm{IIA}}(W) &= \operatorname{rk}\mathcal{H}_2^{\mathrm{IIB}}(M) + \operatorname{rk}\mathcal{H}_4^{\mathrm{IIB}}(M) + 2,\\
\operatorname{rk}\mathcal{H}_3^{\mathrm{IIB}}(M) &= \operatorname{rk}\mathcal{H}_2^{\mathrm{IIA}}(W) + \operatorname{rk}\mathcal{H}_4^{\mathrm{IIA}}(W) + 2, \text{ and}\\
\operatorname{rk}\mathcal{H}_3^{\mathrm{IIB}}(W) &= \operatorname{rk}\mathcal{H}_2^{\mathrm{IIA}}(M) + \operatorname{rk}\mathcal{H}_4^{\mathrm{IIA}}(M) + 2.
\end{aligned}$$

Corollary 3.14 establishes that this is indeed the case for $\mathcal{H}_*^{\mathrm{IIA}}(-) = IH_*(-)$ and $\mathcal{H}_*^{\mathrm{IIB}}(-) = H_*(I-)$ when M and W are conifolds. Thus $(IH_*(-), H_*(I-))$ is a mirror-pair in this sense. Intersection homology and the homology of intersection spaces reveal themselves as the two sides of one coin.

Prerequisites. In Chapter 1, we assume that the reader is acquainted with the elementary homotopy theory of CW complexes [Whi78, Hil53, Hat02]. In Chapter 2, a rudimentary knowledge of stratification theory, pseudomanifolds, and intersection homology is useful. In addition to the references already mentioned in the beginning of this preface, the reader may wish to consult [GM88, Wei94, Sch03, Pfl01]. A geometric understanding of intersection homology in terms of PL or singular chains is sufficient. Sheaf-theoretic methods are neither used nor required in this book. Regarding Chapter 3, we have made an attempt to collect in Sections 3.1–3.6 all the background material from string theory that we need for our predominantly mathematical arguments in Sections 3.7–3.9. Specific competence in, say, quantum field theory, is not required to read this chapter.

Notation and Conventions. Our convention for the mapping cylinder $Y \cup_f X \times I$ of a map $f: X \to Y$ is that the attaching is carried out at time 1, that is, the points of $X \times \{1\} \subset X \times I$ are attached to Y using f. For products in cohomology and homology, we will use the conventions of Spanier's book [Spa66]. In particular, for an inclusion $i: A \subset X$ of spaces and elements $\xi \in H^p(X)$, $x \in H_n(X, A)$, the formula $\partial_*(\xi \cap x) = i^*\xi \cap \partial_* x$ holds for the connecting homomorphism ∂_* (no sign). For the compatibility between cap- and cross-product, one has the sign

$$(\xi \times \eta) \cap (x \times y) = (-1)^{p(n-q)} (\xi \cap x) \times (\eta \cap y),$$

where $\xi \in H^p(X)$, $\eta \in H^q(Y)$, $x \in H_m(X)$, and $y \in H_n(Y)$.

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