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Noncommutative Functional Calculus

Theory and Applications of Slice Hyperholomorphic Functions

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Chapter 1

Introduction

1.1 Overview

In this book we propose a novel approach to two important problems in the theory of functional calculus: the construction of a general functional calculus for not necessarily commuting n-tuples of operators, and the construction of a functional calculus for quaternionic operators. The approach we suggest is made possible by a series of recent advances in Clifford analysis, and in the theory of quaternion-valued functions (see, e.g., [26] and [49]).

After the success, and recognized importance, of the classical Riesz–Dunford functional calculus, it became apparent that there was a need for a functional calculus for several operators. The necessity of such a calculus was pointed out by Weyl already in the 1930s, see [103], and this issue was first addressed by Anderson in [4] using the Fourier transform and n-tuples of self-adjoint operators satisfying suitable Paley–Wiener estimates.

In his early and seminal work [99], Taylor introduces a new approach which works successfully for n-tuples of commuting operators, while in [100] he considers the Weyl calculus for noncommuting, self-adjoint operators. These works have set the stage for different possible outgrowth of this research.

A promising and successful idea was to address the noncommutativity by exploiting the setting of Clifford algebra-valued functions. This idea has been fruitfully followed in the works of Jefferies, McIntosh and their coworkers, see, e.g., [60], [61], [65], [77], and the book [62] with the references therein for a complete overview of this setting. Note that, despite the noncommutative setting which is useful in the case of several operators, one may still have restriction on the n-tuples of operators and on their spectrum.

Of course, for the sake of generality, one would like to abandon these restrictions. To this purpose we have come to understand that one could attempt the development of a functional calculus based on the use of slice monogenic functions.

These functions were first introduced by the authors in [26], but their theory is by now very well developed, as made evident by the rich literature which is available (see, e.g., [15], [18], [24], [26], [27], [28], [29], [30] and [53], [55]).

As it is well known, in order to construct a functional calculus associated to a class of functions, one of the crucial results is the existence of a suitable integral formula which, for the case of slice monogenic functions, we state and prove in Chapter 2. Such a formula was originally proved by Colombo and Sabadini in [15] (for more details see [18]). It is worth noticing that this integral formula is computed over a path which lies in a complex plane. Moreover, despite what happens with the classical monogenic functions, [7], in the slice monogenic case the analog of the Cauchy kernel is a function which is left or right slice monogenic in a given variable. For this reason, we will need two different kernels when dealing with left or right slice monogenic functions. The Cauchy formula we obtain in the case of slice monogenic functions turns out to be perfectly suited to the definition of a functional calculus for bounded or unbounded n-tuples of not necessarily commuting operators, see Chapter 3.

In the first part of this book therefore, we will develop the main results of the theory of slice monogenic functions and the associated functional calculus for n-tuples of not necessarily commuting operators. This calculus has been introduced in the paper [25] for a particular class of functions and then extended to the general case in [18].

In the second part of the book we deal with a related, and yet independent, problem which has been of interest for many years and which, so far, has proved to be rather difficult to tackle. Specifically, we are interested in attempting to define a function of a single quaternionic linear operator. It is clear that, at least in some sense, there are similarities with the problems discussed above: the setting is noncommutative, and the space of quaternions is a Clifford algebra. Nevertheless, the actual problem is different from the case analyzed before.

When dealing with the functional calculus for n-tuples of operators, our approach is to embed the n-tuple of linear operators (over the real field) into the Clifford algebra setting; in this second case, however, we are given an operator which is quaternionic linear. Since the setting is noncommutative, the operator is either left or right linear, and we shall see that our approach differentiates these two cases. The study of this type of operators is needed to deal, for example, with quaternionic quantum mechanics, see [1].

The first natural issue, of course, is to define the space of functions for which we can construct such a functional calculus. Traditionally, the best understood space of functions defined on quaternions is the space of regular functions as defined by Fueter in his fundamental works [43], [44]. Those functions are differentiable on the space of quaternions and they satisfy a system of first-order linear partial differential equations known as the Cauchy–Fueter system. Note that the Cauchy–Fueter system deals with functions defined in $\mathbb{R}^4$ and hence in $\mathbb{R}^3$ as well. Historically, this last case was introduced before the former one, see [79], by G. Moisil and N. Theodorescu. One may therefore attempt to define a functional calculus in which the functions are regular in the sense of Fueter (and the authors have outlined how this would work in [11]). It turns out, however, that such a functional calculus does not perform as well as one would hope, for a variety of reasons that are described in [11] but that can be easily surmised by noticing, for example, that even the simple function $f(q) = q^2$ is not regular in the sense of Fueter.

However, in a recent series of papers, see, e.g., [9], [12], [48], [49] the authors and some of their collaborators have introduced a completely different notion of regularity, the so-called slice regularity, which was in fact the inspiration for the notion of slice monogenicity. This notion is different from the original one of Fueter, and therefore the second part of this book will show how a functional calculus for quaternionic linear operators over the quaternions can be obtained through the use of slice regular functions. The quaternionic functional calculus, at least for functions admitting a power series expansion, was first introduced in [10], [13] and [14], however the exposition in Chapter 4 is inspired by the more recent papers [16] and [17] which are based on a new Cauchy formula, which becomes the natural tool to define the quaternionic functional calculus for quaternionic bounded or unbounded operators (with components that do not necessarily commute). As an application of the quaternionic functional calculus we define and we study the properties of the quaternionic evolution operator, limiting ourselves to the case of bounded linear operators. The evolution operator is studied in [21] where it is proved that the Hille–Phillips–Yosida theory can be extended to the quaternionic setting. This, it seems to us, is the first step in demonstrating the importance, in physics, of this new functional calculus.

It is worth pointing out that while the definitions and some of the properties of slice monogenic and of slice regular functions appear to be quite similar, there are in fact several important differences, that force an independent treatment for the two cases. Those differences are mainly due to the different algebraic nature of quaternions and of Clifford numbers in higher dimensions, when the number of imaginary units which generate the Clifford algebra is greater than two.

1.2 Plan of the book

Almost all the material presented in this book comes from the recent research of the authors. The only exceptions are the basic notions on Clifford algebras, the Appendix, in which we provide some basic facts on the classical Riesz–Dunford functional calculus, and a few results appearing in some of the notes. To illustrate the central results of this book we provide a quick description.

Slice monogenic functions. Consider the universal Clifford algebra $\mathbb{R}_n$ generated by n imaginary units $\{e_1, \dots, e_n\}$ satisfying $e_i e_j + e_j e_i = -2\delta_{ij}$ and a function f defined on the Euclidean space $\mathbb{R}^{n+1}$, identified with the set of paravectors in $\mathbb{R}_n$, with values in $\mathbb{R}_n$. The notion of slice monogenic function is based on the requirement that all the restrictions of the function f to suitable complex planes