

Chapter 2

Random Variables and Probability; Normal Distribution

2.1 Probability and Random Variables

We say that a phenomenon is random if, on the basis of our best knowledge, we cannot exactly predict its result. What really happens is only one of many possibilities. In every day life we meet random results of lotteries, a non-predictable dispersion of gun shots at a target or a random travel time through a crowded city.

An intuitively understandable idea of random phenomena can be formalized by the concept of random events and probability. Using formal definitions, we say that the *random event* (a collection of sample points) is a result of some random phenomena, and its *probability* is the chance that this phenomenon will occur, expressed by a number from the interval $[0, 1]$.

Example 2.1 (An unbiased coin flipping) During an experiment of a single flip of an unbiased coin, two results are possible: the occurrence of heads and the occurrence of tails. Both results are random events. The probability of heads and the probability of tails are equal and they are $1/2$.

In this example we can say that the set of all possible results of the experiment has two elements (occurrence of heads and occurrence of tails). We interpret the probabilities of occurrence of these elementary events in the following way: if we repeat flipping the coin a sufficient number of times, then the number of occurrences of heads (or, equivalently, the number of occurrences of tails) divided by the number of flips will tend to $1/2$. This is the so-called frequency interpretation of the probability.

Example 2.2 (Dice casting) During an experiment of a single cast of an unbiased die, six results are possible: the occurrence of a face with $n = 1, 2, 3, 4, 5$, or 6 spots. Then the set of the results (elementary events, sample points) contains six elements. The probability of each event (the occurrence of a face with n spots) equals $1/6$. This means that if the number of casts tends to infinity, then the following ratio:

$$\frac{\text{number of casts, when } n \text{ spots occurred}}{\text{total number of casts}}$$

tends to $1/6$, for $n = 1, 2, 3, 4, 5$, and 6 .

The expected outcome of the experiment described in Example 2.2 may be more complicated than the occurrence of a fixed number of spots. For example, we can ask: What is the probability that the single cast results in a face with an even number of spots? What is the probability that we will see a face which has more than 4 spots? Of course, we can easily deduce that in the first case the probability is $1/2$ and in the second one $1/3$.

The above examples show that it is conceptually easy to define an event and the probability of an event if the number of possible outcomes of the experiment (e.g., coin flipping or dice casting) is finite and the outcome of each result is equally probable. In such a case, the probability of some event is defined as the frequency of occurrences of this event when the number of experiments tends to infinity.

In some situations, we can introduce another definition of probability. If the set of results of an experiment is infinite but it is contained in some set on a plane (alternatively: in 3-dimensional space, on a straight line, etc.), then the probability has a geometrical interpretation. The probability of a certain outcome of an experiment is the ratio of the area of the subset corresponding to these results, to the area of the set corresponding to all possible results of the experiment. The geometrical definition of probability has some limitations: the results of the experiment must be located in a bounded set on the plane and, moreover, they must be evenly distributed over this set.

The definitions of an event and the probability of an event used today have their origin in measure theory. The fundamental object of probability theory is the probability space. A *probability space* is defined by the triad (Ω, F, P) , where Ω is the *sample space* containing all elementary events (sample points), F is the σ -algebra of Borel subsets of the sample space Ω containing all possible *events* (elementary and compound), and P is a (probability) measure defined on F .

We will now comment on the above definitions. Elementary events ω (being elements of the sample space Ω) are results of some experiment, mutually excluding each other; this means that only one elementary event can be the result of the experiment. Generally, (compound) events in an experiment are elements of the σ -algebra F . Occurrence of an event A can be the result of several elementary events; knowing the outcome of an experiment we are able to decide if the event A occurred. The probability measure P or, simply, the probability, has the property that it is equal to 1 for the certain event (the whole sample space Ω , that is, the event that the experiment had some outcome). Certainly, the probability of the impossible event (the empty set $\emptyset$) is zero.

Example 2.3 (An unbiased coin flipping, continuation) The probability space for the experiment of a single fair coin flip is (Ω, F, P) , where: the sample space Ω is the following 2-element set:

$$\Omega = (\{\text{heads}\}, \{\text{tails}\});$$

the σ -algebra F consists of four elements: the empty set $\emptyset$, two 1-element sets, and the whole sample space Ω :

$$F = (\emptyset, \{\text{heads}\}, \{\text{tails}\}, \Omega);$$

the probability P is defined as:

$$P(\{\text{heads}\}) = \frac{1}{2}, \quad P(\{\text{tails}\}) = \frac{1}{2}.$$

Example 2.4 (Dice casting, continuation) In the experiment of a single balanced die cast, the probability space is the following: the sample space Ω has 6 elements:

$$\Omega = (\{1 \text{ spot}\}, \{2 \text{ spots}\}, \{3 \text{ spots}\}, \{4 \text{ spots}\}, \{5 \text{ spots}\}, \{6 \text{ spots}\});$$

the σ -algebra F consists of the following elements: the empty set $\emptyset$, all subsets of Ω containing 1, 2, 3, 4, and 5 elements and the whole sample space Ω :

$$F = \left(\begin{array}{l} \emptyset, 6 \text{ one-element subsets, } 15 \text{ two-element subsets,} \\ 20 \text{ three-element subsets, } 15 \text{ four-element subsets,} \\ 6 \text{ five-element subsets, } \Omega \end{array} \right);$$

the probability P is defined as:

$$\begin{aligned} P(\{1 \text{ spot}\}) &= P(\{2 \text{ spots}\}) = P(\{3 \text{ spots}\}) \\ &= P(\{4 \text{ spots}\}) = P(\{5 \text{ spots}\}) = P(\{6 \text{ spots}\}) = \frac{1}{6}. \end{aligned}$$

The concept of randomness and probability presented here identifies events with subsets of the sample space Ω , which are elements of the σ -algebra F . Therefore, we are able to perform on these events the operations analogous to the operations of set theory. For two events $A, B \in F$, we can define the union $A \cup B$ (A or B happens), intersection $A \cap B$ (A and B occur simultaneously), difference $A \setminus B$ (A occurs but B does not), etc. Probability, as we mentioned, is a measure; it has the following properties:

$$0 \leq P(A) \leq 1, \tag{2.1}$$

$$P(\emptyset) = 0, \quad P(\Omega) = 1, \tag{2.2}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B), \tag{2.3}$$

and for a countable number of disjoint events A_j :

$$P\left(\bigcup_j A_j\right) = \sum_j P(A_j). \tag{2.4}$$

In probability theory, it is very important to know the relationship between events: their dependence or independence. We say that two events A and B are *independent* if their probabilities satisfy the following condition:

$$P(A \cap B) = P(A)P(B), \quad (2.5)$$

which means that the probability of the simultaneous occurrence of both events is equal to the product of probabilities of their separate occurrence. If condition (2.5) is not satisfied, the events A and B are *dependent*.

To know to what extent the events A and B are dependent, we can use the *conditional probability* $P(A|B)$, which is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \quad (2.6)$$

The quantity $P(A|B)$, which is the probability of A conditioned on B , we understand to be the probability of occurrence of A under the condition that B has occurred.

Using formula (2.5) in (2.6), we see that if events A and B are independent then

$$P(A|B) = P(A). \quad (2.7)$$

The concept of the conditional probability is strongly related to the definition of *complete probability*. If we have some sequence of mutually excluding events B_j , $j = 1, 2, \dots, n$, $B_k \cap B_l = \emptyset$ for $k \neq l$, satisfying additionally $\bigcup_j B_j = \Omega$, then the probability of any event A can be represented as

$$P(A) = \sum_j P(A|B_j)P(B_j). \quad (2.8)$$

The last equation enables us to calculate the probability of some event A if we know its probability under some additional conditions, that is, if we know that some event B_j has occurred.

Example 2.5 (Dice casting, continuation) Consider the experiment of the die single cast and define two events: A , the outcome is a face with an even number of spots, and B , the face with a number of spots greater than 4. We can verify whether these two events are independent. Using the elementary events defined in Example 2.4 we find that the events are: $A = (\{2 \text{ spots}\}, \{4 \text{ spots}\}, \{6 \text{ spots}\})$, $B = (\{5 \text{ spots}\}, \{6 \text{ spots}\})$, and their probabilities are: $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$.

The intersection of the events is: $A \cap B = (\{6 \text{ spots}\})$, and the probability of intersection, $P(A \cap B) = \frac{1}{6}$. It is seen that the events A and B satisfy condition (2.5), that is, they are independent.

If we replace the event B with a new one: B_1 —the number of spots is greater than 5 (that is, $B_1 = (\{6 \text{ spots}\})$ and $P(B_1) = \frac{1}{6}$), then the intersection of the events is $A \cap B_1 = (\{6 \text{ spots}\})$ and it is seen that the events A and B_1 are dependent, because $P(A)P(B_1) = \frac{1}{12}$ and $P(A \cap B_1) = \frac{1}{6}$, so the condition (2.5) is not satisfied.

The description of results of experiments or observations of random phenomena in terms really existing in these processes is very complicated. To make the modeling of the processes more convenient we can introduce the concept of a random variable.

The real-valued function $X(\omega)$ defined on the sample space Ω of random events ω is called a *random variable* if a pre-image¹ A of every interval of real numbers of the form $I = (-\infty, x)$ is a random event (an element of the σ -algebra F).

Probability P describing properties of random events can also describe random variables. It is transferred from the σ -algebra of events to the space of real-valued random variables by pre-images of the intervals I :

$$P(I) = P(\omega \text{ such that } X(\omega) < x). \quad (2.9)$$

For a given sample space we can consider various random variables. Our choice depends on the purpose of the modeling.

Example 2.6 (An unbiased coin flipping, continuation)

- (a) Consider the experiment of a symmetric coin single flip. Assign number 1 to the outcome of heads and number -1 to the outcome of tails. Such a random variable may be used for description of a random walk on a straight line. We start from $x = 0$ and repeat the coin flipping. If the outcome is heads then we add 1 to x , if tails, we subtract 1. After every trial the value of x is greater by 1 or smaller by 1 than the value in the previous step. We repeat the trial many times obtaining the x -coordinate of the walking particle in every step (see, e.g., [11]).
- (b) Consider the same experiment. We assign number 1 to heads and number 0 to tails. Repeating the trials many times and writing down the obtained numbers we generate random numbers in binary notation.

Analogously to the events, we can define independence of random variables. We will say that two random variables X and Y (defined on the probability space (Ω, F, P)) are *independent* if for all $x_1 \leq x_2$ and $y_1 \leq y_2$ the events of the form $\{\omega : x_1 \leq X(\omega) < x_2\}$ and $\{\omega : y_1 \leq Y(\omega) < y_2\}$ are independent.

The theorem concerning the complete probability (2.8) makes it possible to apply in many technical problems the so-called *conditioning technique*. This method is based on the procedure of decomposition of the initial complicated problem into a number of tasks easy to solve when we assume certain conditions to be satisfied with a certain probability. Then the simplified problems are solved and, finally, the general non-conditioned solution is obtained by averaging of the set of solutions with respect to the assumed probability distribution. Such a technique lets us calculate the parameters (e.g., moments) or distributions of random variables in various

¹Assume, we have a function $X : \Omega \rightarrow R$ and let A be a subset of the set of real numbers R . The pre-image (or inverse image) A for the function X is a set $B \subset \Omega$, containing all the elements $\omega \in \Omega$ such that their image belongs to A , which means $X(\omega) \in A$. In such a case we write: $B = X^{-1}(A)$.

engineering problems. The reader can find more about this technique in the papers [12, 13] or the textbook [20].

2.2 The Cumulative Distribution Function; the Probability Density Function

Most problems of the error calculus arising in technological applications concern the analysis of random variables with continuous distributions. Random variables of such a nature may assume any value from a certain range. The *cumulative distribution function* (or: *probability distribution function*) $F(x)$ of any one-dimensional random variable X is defined by the expression²:

$$F(x) = P(X < x), \quad (2.10)$$

which means that the cumulative distribution function is defined as a function, the value of which for a given x is equal to the probability of an event that the random variable X is smaller than the number x .

The cumulative distribution function is defined for all real numbers and it is a non-decreasing, continuous on the left, function. Moreover, for x tending to minus infinity and plus infinity, it satisfies the following conditions:

$$F(-\infty) = 0, \quad F(\infty) = 1. \quad (2.11)$$

The probability distribution function can be applied to the calculation of probabilities of the events related to the random variable X . For instance, the probability of an event that a random variable X belongs to the interval $[x_1, x_2]$ can be expressed by means of the probability distribution function (see Fig. 2.1):

$$P(x_1 \leq X < x_2) = F(x_2) - F(x_1). \quad (2.12)$$

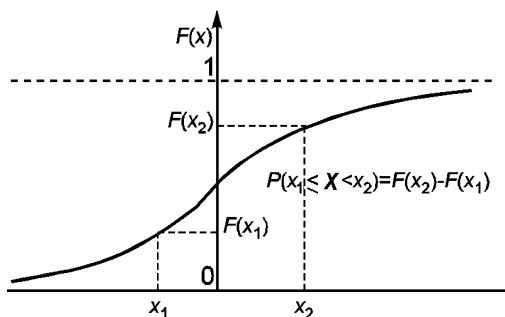


Fig. 2.1 The cumulative distribution function

²We shall denote random variables by capital letters X, Y , while their values, being numbers, will be denoted by small letters x, y . This does not refer to cases when a random variable in a particular formula has a physical meaning and is usually denoted by a small letter. $P(A)$ is the probability of an event A .

If the random variable X is *discrete*, that is, if it takes values from a finite (or countable) set: $\{x_j, j = 1, 2, \dots, N\}$ (or $\{x_j, j = 1, 2, \dots\}$), then the cumulative distribution function is discontinuous at these points and its jumps are equal to p_j . Moreover, the following equality holds:

$$P(X = x_j) = p_j. \quad (2.13)$$

Over the intervals of continuity, $x \in [x_j, x_{j+1})$, the cumulative distribution function $F(x)$ of the discrete random variable X is constant and equal to $F(x) = \sum_{k=1}^j p_k = F_j$. An example of the cumulative distribution function of some discrete random variable is presented in Fig. 1.3.

The cumulative distribution function of a random variable with a continuous distribution (the *continuous random variable*) may be expressed in the form of the integral

$$F(x) = \int_{-\infty}^x f(\xi) d\xi. \quad (2.14)$$

Function $f(x)$ in (2.14) is referred to as the *probability density function* (or simply *probability density*) of a random variable X . If the cumulative frequency distribution $F(x)$ has a derivative at any point x , then such a derivative represents the density

$$f(x) = F'(x). \quad (2.15)$$

Since the cumulative distribution function describes the normalized probability measure (the probability of the certain event equals 1, which means that $P(-\infty < X < \infty) = 1$) and is a non-decreasing function, the probability density function $f(x)$ has the following two properties:

$$A = \int_{-\infty}^{\infty} f(x) dx = 1 \quad (2.16)$$

and

$$f(x) \geq 0. \quad (2.17)$$

Thus, the area A between the graph of function $f(x)$ and the horizontal axis x of the random variable is equal to unity.

The probability of any event that the variable X lies in the interval $[x_1, x_2)$, which is, that it will have the value $P(x_1 \leq X < x_2)$, is defined by the following:³

$$P(x_1 \leq X < x_2) = \int_{x_1}^{x_2} f(x) dx. \quad (2.18)$$

The relation (2.18) is presented graphically in Fig. 2.2.

³For continuous distributions the probability that a random variable is located in a closed interval is the same as in an interval closed on one side or as in an open interval. In (2.18) we decided to choose an option of the interval closed on the left-hand side.

Fig. 2.2 The probability density function

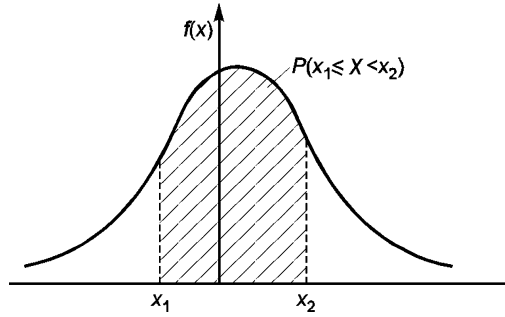
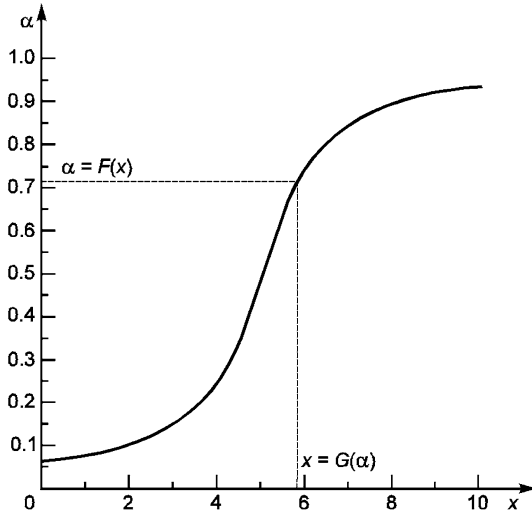


Fig. 2.3 The quantile function, see (2.19)



The cumulative distribution function and the probability density function are not the only functions characterizing a random variable. In some situations the inverse distribution function $G(\alpha)$, sometimes called the quantile function, is more convenient. For a given cumulative distribution function $F(x)$, the *quantile function* is defined as a function satisfying the following conditions:

$$\begin{aligned} x &= G(\alpha) = G(F(x)), \\ P(X \leq G(\alpha)) &= \alpha. \end{aligned} \tag{2.19}$$

This mutual relation between $F(x)$ and $G(\alpha)$ is shown graphically in Fig. 2.3.

In some applications of inspection theory and reliability theory, and also in some problems of mathematical statistics, the *survival function* $S(x)$ is useful. It is defined as the probability that the random variable X is greater than or equal to x :

$$S(x) = P(X \geq x) = 1 - F(x). \tag{2.20}$$

More definitions of functions describing the properties of distributions of random variables can be found in handbooks dealing with probability theory or mathematical statistics (see, e.g., [6, 7]).

2.3 Moments

Moments play an important role in the error calculus, particularly when multidimensional problems are considered. For the one-dimensional random distributions discussed in this chapter, the expressions for moments take simple forms.

The *first-order moment* with respect to the line perpendicular to the x -axis and crossing it at $x = 0$ is defined by the formula

$$m = \int_{-\infty}^{\infty} x f(x) dx. \quad (2.21)$$

Assuming such a value $\bar{x}$ that the equality

$$A\bar{x} = m$$

holds true, we obtain, remembering that $A = 1$ (comp. (2.16)), the formula

$$\bar{x} = \int_{-\infty}^{\infty} x f(x) dx. \quad (2.22)$$

Using (2.22) one can calculate the average value $\bar{x}$. In other words, $\bar{x}$ represents the abscissa of the gravity center of the area between the graph of the probability density function and the x -axis. The moment m may be interpreted as the statical moment of that field with respect to the $x = 0$ straight line.

The *second-order moment* is the quantity J defined as

$$J = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x) dx. \quad (2.23)$$

Such a moment calculated with respect to the straight line $x = \bar{x}$ is called the *central second order moment*.

Assuming now a quantity σ^2 such that the equality

$$A\sigma^2 = J$$

holds true, we get, still remembering that $A = 1$, the relation

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x) dx, \quad (2.24)$$

where σ^2 is the *variance* of the distribution $f(x)$. The square root of the variance, denoted by σ , represents the *standard deviation of the distribution*.

Note that the quantity J given by formula (2.23) is, in terms used in engineering applications, the central inertia moment of the area between the graph of the function $f(x)$ and the x -axis. Using such an interpretation it is seen that the standard deviation represents the so-called inertia radius of that field.

Of practical significance is also the *average deviation* d defined as

$$d = \int_{-\infty}^{\infty} |x - \bar{x}| f(x) dx. \quad (2.25)$$

The concept of the average value $\bar{x}$ may be generalized; in this way we obtain moments of order n , $n = 0, 1, 2, 3, \dots$ (called the *ordinary moments* of n -th order), defined as:

$$m_n = \overline{x^n} = \int_{-\infty}^{\infty} x^n f(x) dx. \quad (2.26)$$

In the new notation the average value (or: the mean value) is the moment of order 1, namely m_1 .

The generalization of the variance are *central moments* of order n , $n = 2, 3, 4, \dots$, defined as:

$$\mu_n = \int_{-\infty}^{\infty} (x - m_1)^n f(x) dx. \quad (2.27)$$

Using definition (2.27) of the central moment we obtain the following relation between central moments and ordinary moments:

$$\begin{aligned} \mu_n &= \int_{-\infty}^{\infty} (x - m_1)^n f(x) dx \\ &= \int_{-\infty}^{\infty} \left(\sum_{j=0}^n (-1)^j \binom{n}{j} x^{n-j} m_1^j \right) f(x) dx = \sum_{j=0}^n (-1)^j \binom{n}{j} m_{n-j} m_1^j. \end{aligned} \quad (2.28)$$

In particular, the variance σ^2 can be represented as:

$$\sigma^2 = \mu_2 = m_2 - m_1^2. \quad (2.29)$$

Except for the ordinary and central moments defined above, the *absolute moments* (*ordinary* and *central*), that is, average values of powers of the absolute value of x , can be defined by the following formulas:

$$m_n^{abs} = \int_{-\infty}^{\infty} |x|^n f(x) dx, \quad (2.30)$$

$$\mu_n^{abs} = \int_{-\infty}^{\infty} |x - m_1|^n f(x) dx. \quad (2.31)$$

The most often used absolute moment is the *average deviation* d , defined by (2.25).

Let us remark that for even values of n , the absolute moments and moments (ordinary and central) are identical.

Existence of moments is strongly connected with integrability of the probability density function $f(x)$ multiplied by some power of x . The condition of existence of the moment of a given order n is the convergence of the integral $\int_{-\infty}^{\infty} x^n f(x) dx$; from the existence of the moment for a certain given range $n = n_0$ we obtain the moments of lower orders. Therefore, the greatest n for which the moments exist is called the *range of the random variable*. In applications, the most often required assumption is that random variables have finite variances, that is, they are random variables of the second order.

Example 2.7 (The Cauchy distribution) The probability distribution with the probability density function

$$f(x) = \frac{1}{\pi b \{[(x-a)/b]^2 + 1\}} \quad (2.32)$$

and the cumulative distribution function

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x-a}{b}\right), \quad (2.33)$$

is called the Cauchy distribution. It is an example of distribution which has no moments (for each $n = 1, 2, \dots$ the integral

$$m_n = \int_{-\infty}^{\infty} \frac{x^n dx}{\pi b \{[(x-a)/b]^2 + 1\}}$$

is divergent).

Example 2.8 (The normal distribution) The probability distribution with the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left[\frac{-(x-m)^2}{2\sigma^2}\right] \quad (2.34)$$

is called the normal distribution. It is an example of distribution which has moments of any order (for each $n = 1, 2, \dots$ the integral

$$m_n = \int_{-\infty}^{\infty} \frac{x^n}{\sqrt{2\pi}\sigma^2} \exp\left[\frac{-(x-m)^2}{2\sigma^2}\right] dx$$

is finite).

Remark 2.1 Assume that a certain random variable X has a finite mean value m_X and variance σ_X^2 . Then we can consider the new random variable $\tilde{X}$, defined as

$$\tilde{X} = X - m_X$$

and called the centered random variable. This new random variable $\tilde{X}$ (sometimes called the fluctuation of X) has zero average (mean) value and a variance equal to the variance of the original random variable X ,

$$m_{\tilde{X}} = 0, \quad \sigma_{\tilde{X}}^2 = \sigma_X^2.$$

Such decompositions of random variables are often applied in error analysis. In the above procedure we interpret the random variable X as the result of a measurement with some random error, the mean value m_X as the nominal value of the measured quantity, and the fluctuation $\tilde{X}$ as the random measurement error itself.

2.4 The Normal Probability Distribution

The *normal distribution*, called also the *Gaussian distribution*, plays a basic role in error calculus. In most engineering applications random variables, such as small errors of measurements, small errors of positioning accuracy of certain mechanisms, e.g., robot manipulators or small deviations of magnitudes of certain parameters of objects in mass production, may be treated as those having normal probability distribution. They are called normal (Gaussian) random variables.

In the normal distribution, the probability density function takes the form [2]:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - \bar{x})^2}{2\sigma^2}\right], \quad (2.35)$$

where $\bar{x}$ is the average value, comp. (2.22), and σ stands for the standard deviation (comp. (2.24)).

Introducing a new random variable

$$T = \frac{X - \bar{x}}{\sigma}, \quad (2.36)$$

which is called the *normalized random variable corresponding to X* (comp. [19]), we get another form of the probability density function,

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right]. \quad (2.37)$$

Between the two forms of the probability density function, there exists the relation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - \bar{x})^2}{2\sigma^2}\right] = \frac{1}{\sigma} \phi(t), \quad t = \frac{x - \bar{x}}{\sigma}. \quad (2.38)$$

The numerical values of the normalized Gaussian distribution $\phi(t)$ may be calculated with the use of a computer or even a pocket calculator. Moreover, they are

Table 2.1 The probability density function $\phi(t)$ of the normalized Gaussian distribution

t	0	2	4	6	8
0.0	0.3989	0.3989	0.3986	0.3982	0.3977
0.1	0.3970	0.3961	0.3951	0.3939	0.3925
0.2	0.3910	0.3894	0.3876	0.3857	0.3836
0.3	0.3814	0.3790	0.3765	0.3739	0.3712
0.4	0.3683	0.3653	0.3621	0.3589	0.3555
0.5	0.3521	0.3485	0.3443	0.3410	0.3372
0.6	0.3332	0.3292	0.3251	0.3209	0.3166
0.7	0.3123	0.3079	0.3034	0.2989	0.2943
0.8	0.2897	0.2850	0.2803	0.2756	0.2709
0.9	0.2661	0.2613	0.2565	0.2516	0.2468
1.0	0.2420	0.2371	0.2323	0.2275	0.2227
1.1	0.2179	0.2131	0.2033	0.2036	0.1989
1.2	0.1942	0.1895	0.1849	0.1804	0.1758
1.3	0.1714	0.1669	0.1626	0.1582	0.1539
1.4	0.1497	0.1456	0.1415	0.1374	0.1334
1.5	0.1295	0.1257	0.1219	0.1182	0.1145
1.6	0.1109	0.1074	0.1040	0.1006	0.0973
1.7	0.0940	0.0909	0.0878	0.0848	0.0818
1.8	0.0790	0.0761	0.0734	0.0707	0.0681
1.9	0.0656	0.0632	0.0608	0.0584	0.0562
2.0	0.0540	0.0519	0.0498	0.0478	0.0459
2.1	0.0440	0.0422	0.0404	0.0387	0.0371
2.2	0.0355	0.0339	0.0325	0.0310	0.0297
2.3	0.0283	0.0270	0.0258	0.0246	0.0235
2.4	0.0224	0.0213	0.0203	0.0194	0.0184
2.5	0.0175	0.0167	0.0158	0.0151	0.0143
2.6	0.0136	0.0129	0.0122	0.0116	0.0110
2.7	0.0104	0.0099	0.0093	0.0089	0.0084
2.8	0.0079	0.0075	0.0071	0.0063	0.0063
2.9	0.0060	0.0056	0.0053	0.0050	0.0047
3.0	0.0044	0.0042	0.0039	0.0037	0.0035

tabulated in numerous books (comp. [9, 10]). To make this book sufficiently self-contained, the values are given in Table 2.1.⁴ Knowing the function $\phi(t)$ and the standard deviation σ of a particular non-normalized normal distribution, we may

⁴The numbers 0, 2, 4, 6 and 8 in the heading of the table are values of the second fractional digit of the number t .

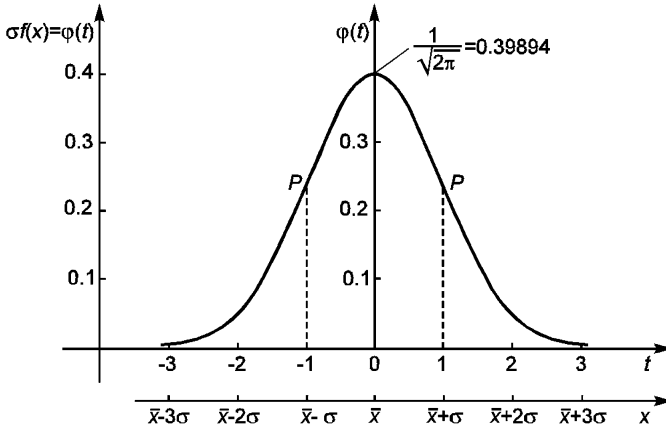


Fig. 2.4 The normalized probability density function of the normal (Gaussian) distribution

calculate by means of formula (2.38) the values of $f(x)$ for any value of the independent variable x . In practical calculations one can use the graph of the function $\phi(t)$ shown in Fig. 2.4. The graph has two inflexion points P , for $t = +1$ and for $t = -1$.

In Fig. 2.4 is also shown a simple graphical procedure allowing us to find the graph of the function $f(x)$ if the graph of the normalized density function $\phi(t)$ is given. The smaller is the standard deviation σ of the normal distribution, the smaller will be the dispersion of the random variable X around the average value $\bar{x}$. This property of normal distribution is illustrated in Fig. 2.5, in which three various normal distributions are presented. Their average value is of the same magnitude $\bar{x} = 0$, while standard deviations are different having the values $\sigma = 0.5$, $\sigma = 1.0$, and $\sigma = 2.0$, respectively.

The diagrams of normal probability densities are symmetrical with respect to the average value $\bar{x}$, at which they have a maximum. This maximum value of the density is given by the formula

$$f(\bar{x}) = \frac{1}{\sigma\sqrt{2\pi}}. \quad (2.39)$$

The relation between the half-width t_α of any arbitrarily chosen range $(-t_\alpha, t_\alpha)$ and the probability α that the random variable T takes the value located inside this range is of great practical significance. Some selected values of the pairs t_α , $(1 - \alpha)$ are collated in Table 2.2 (comp. Fig. 2.6). The quantity $(1 - \alpha)$ is called the *residual probability*.

In practice, certain specific ranges are often used, bounded by the multiplicities of the standard deviation σ , namely:

$$(-\sigma, \sigma), \quad \text{the probability is } \alpha = 0.6826,$$

$$(-2\sigma, 2\sigma), \quad \text{the probability is } \alpha = 0.9544,$$

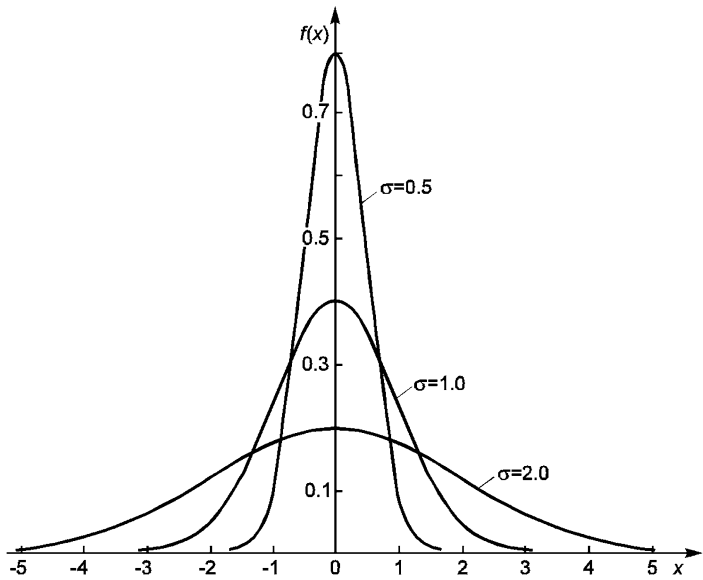


Fig. 2.5 The probability density function of the normal distribution for several values of the standard deviation

Table 2.2 The residual probabilities of the normal distribution

t_α	$1 - \alpha$
0.0	1.0000
0.5	0.6170
1.0	0.3174
1.5	0.1336
2.0	0.0456
2.5	0.0124
3.0	0.0027

$(-3\sigma, 3\sigma)$, the probability is $\alpha = 0.9973$.

These numbers indicate that the normal distribution of a random variable is concentrated in the vicinity of the average value $\bar{x}$. The probability that the value of a random variable X with the normal distribution differs from its average value by more than 3σ equals 0.0027. Such a significant property justifies to a certain degree the so-called *three-sigma rule*, that is, often used also in cases when other distributions are involved, not only when the normal distribution is considered. This rule should not, however, be used uncritically for any arbitrary probability distribution (comp. [6]).

Fig. 2.6 The residual probability $(1 - \alpha)$

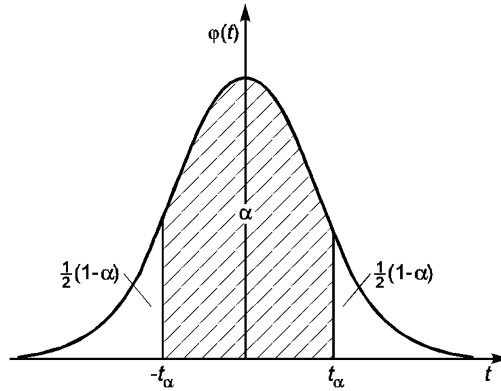
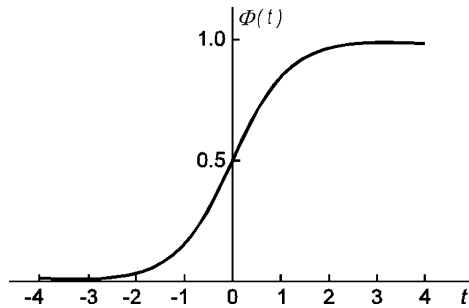


Fig. 2.7 The cumulative distribution function of the standard normal distribution



Between the standard deviation σ of the normal probability distribution and its average deviation d we have the following, sometimes useful, relation:

$$d = \sqrt{\frac{2}{\pi}} \sigma \approx 0.798\sigma. \quad (2.40)$$

The cumulative distribution function of the normal random variable is given by the formula

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x \exp\left[-\frac{(\xi - \bar{x})^2}{2\sigma^2}\right] d\xi \equiv \Phi(t), \quad t = \frac{x - \bar{x}}{\sigma}, \quad (2.41)$$

where $\Phi(t)$ is the cumulative distribution function of the normalized Gaussian random variable:

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp\left[-\frac{1}{2}\tau^2\right] d\tau. \quad (2.42)$$

The function $\Phi(t)$ is tabulated, see Table 2.3; its graph is presented in Fig. 2.7. This distribution is sometimes called the *standard normal distribution* $N(0, 1)$.

Table 2.3 The cumulative distribution function of the normalized Gaussian distribution

t	$\Phi(t)$	t	$\Phi(t)$	t	$\Phi(t)$	t	$\Phi(t)$	t	$\Phi(t)$
-3.00	0.0013	-1.75	0.0401	-0.50	0.3085	0.75	0.7734	2.00	0.9773
-2.95	0.0016	-1.70	0.0446	-0.45	0.3264	0.80	0.7881	2.05	0.9798
-2.90	0.0019	-1.65	0.0495	-0.40	0.3446	0.85	0.8023	2.10	0.9821
-2.85	0.0022	-1.60	0.0548	-0.35	0.3632	0.90	0.8159	2.15	0.9842
-2.80	0.0026	-1.55	0.0606	-0.30	0.3821	0.95	0.8289	2.20	0.9861
-2.75	0.0030	-1.50	0.0668	-0.25	0.4013	1.00	0.8413	2.25	0.9878
-2.70	0.0035	-1.45	0.0745	-0.20	0.4207	1.05	0.8531	2.30	0.9893
-2.65	0.0040	-1.40	0.0808	-0.15	0.4404	1.10	0.8643	2.35	0.9906
-2.60	0.0047	-1.35	0.0885	-0.10	0.4602	1.15	0.8749	2.40	0.9918
-2.55	0.0056	-1.30	0.0968	-0.05	0.4801	1.20	0.8849	2.45	0.9929
-2.50	0.0062	-1.25	0.1056	0.00	0.5000	1.25	0.8944	2.50	0.9938
-2.45	0.0071	-1.20	0.1151	0.05	0.5199	1.30	0.9032	2.55	0.9946
-2.40	0.0082	-1.15	0.1251	0.10	0.5398	1.35	0.9115	2.60	0.9953
-2.35	0.0094	-1.10	0.1357	0.15	0.5596	1.40	0.9192	2.65	0.9960
-2.30	0.0107	-1.05	0.1469	0.20	0.5793	1.45	0.9265	2.70	0.9965
-2.25	0.0122	-1.00	0.1587	0.25	0.5987	1.50	0.9332	2.75	0.9979
-2.20	0.0139	-0.95	0.1711	0.30	0.6179	1.55	0.9394	2.80	0.9974
-2.15	0.0158	-0.90	0.1841	0.35	0.6368	1.60	0.9452	2.85	0.9978
-2.10	0.0179	-0.85	0.1977	0.40	0.6554	1.65	0.9505	2.90	0.9981
-2.05	0.0202	-0.80	0.2119	0.45	0.6736	1.70	0.9554	2.95	0.9984
-2.00	0.0227	-0.75	0.2266	0.50	0.6915	1.75	0.9599	3.00	0.9987
-1.95	0.0256	-0.70	0.2420	0.55	0.7088	1.80	0.9641	—	—
-1.90	0.0287	-0.65	0.2578	0.60	0.7257	1.85	0.9678	—	—
-1.85	0.0322	-0.60	0.2743	0.65	0.7422	1.90	0.9713	—	—
-1.80	0.0359	-0.55	0.2912	0.70	0.7580	1.95	0.9744	—	—

In practical calculations often the so-called *error function*,

$$\operatorname{erf}(t) = \frac{1}{\sqrt{2\pi}} \int_0^t \exp\left[-\frac{1}{2}\tau^2\right] d\tau, \quad (2.43)$$

is used instead of the cumulative distribution function of the normalized Gaussian random variable. The error function is tabulated and given in various books (comp., e.g., [1, 10]). Its values are also given in Table 2.4.

The error function is directly connected with the cumulative distribution function $\Phi(t)$ by the simple formulas:

$$\begin{aligned} \Phi(t) &= \frac{1}{2} - \operatorname{erf}(-t) \quad \text{for } t \leq 0, \\ \Phi(t) &= \frac{1}{2} + \operatorname{erf}(t) \quad \text{for } t > 0. \end{aligned} \quad (2.44)$$

Table 2.4 The error function

t	$\operatorname{erf}(t)$	t	$\operatorname{erf}(t)$	t	$\operatorname{erf}(t)$
0.00	0.0000	1.00	0.3413	2.00	0.4773
0.05	0.0199	1.05	0.3531	2.05	0.4798
0.10	0.0398	1.10	0.3643	2.10	0.4821
0.15	0.0596	1.15	0.3749	2.15	0.4842
0.20	0.0793	1.20	0.3849	2.20	0.4861
0.25	0.0987	1.25	0.3944	2.25	0.4878
0.30	0.1179	1.30	0.4032	2.30	0.4893
0.35	0.1368	1.35	0.4115	2.35	0.4906
0.40	0.1554	1.40	0.4192	2.40	0.4918
0.45	0.1736	1.45	0.4265	2.45	0.4929
0.50	0.1915	1.50	0.4332	2.50	0.4938
0.55	0.2088	1.55	0.4394	2.55	0.4946
0.60	0.2257	1.60	0.4452	2.60	0.4953
0.65	0.2422	1.65	0.4505	2.65	0.4960
0.70	0.2580	1.70	0.4554	2.70	0.4965
0.75	0.2734	1.75	0.4599	2.75	0.4979
0.80	0.2881	1.80	0.4641	2.80	0.4974
0.85	0.3023	1.85	0.4678	2.85	0.4978
0.90	0.3159	1.90	0.4713	2.90	0.4981
0.95	0.3289	1.95	0.4744	2.95	0.4984
—	—	—	—	3.00	0.4987

The error function $\operatorname{erf}(t)$ is a special function and has no representation in the form of a combination of elementary functions. However, in certain books one can find approximate expressions allowing one to calculate the values of that function by means of elementary functions. Two such practical methods are presented below. They are based on the asymptotic expansions (comp. [1, 7]).

Method 2.1

$$\operatorname{erf}(t) = 1 - \left(a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 \right) \exp[-t^2] + \varepsilon(t), \tag{2.45}$$

where

$$z = \frac{1}{1 + pt}, \qquad |\varepsilon(t)| \leq 1.5 \times 10^{-7}$$

and

$$\begin{aligned} p &= 0.3275911, & a_1 &= 0.254829592, \\ a_2 &= -0.284496736, & a_3 &= 1.421413741, \end{aligned}$$

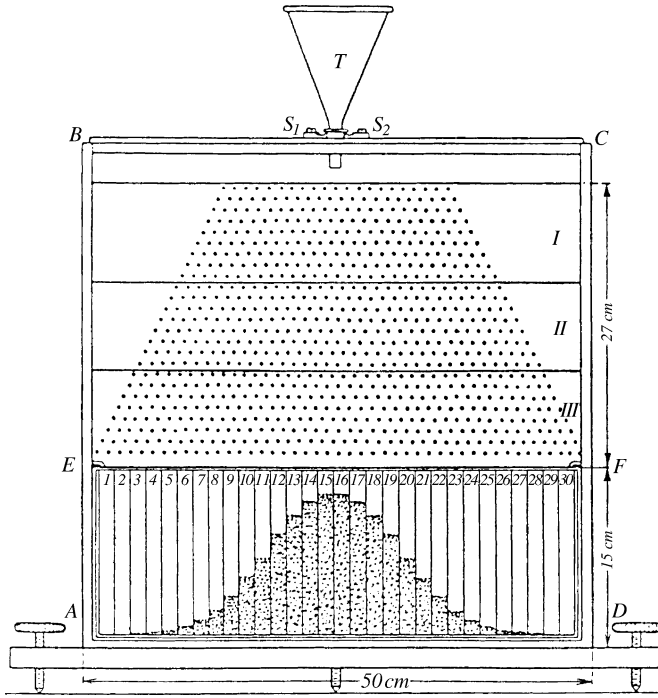


Fig. 2.8 Experimental generation of the normal distribution, the Galton box

$$a_4 = -1.453152027, \quad a_5 = 1.061405429.$$

Method 2.2

$$\operatorname{erf}(t) = 1 - \frac{1}{(a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6)^{16}} + \varepsilon(t), \quad (2.46)$$

where

$$|\varepsilon(t)| \leq 3 \times 10^{-7}$$

and

$$\begin{aligned} a_1 &= 0.0705230784, & a_2 &= 0.0422820123, \\ a_3 &= 0.0092705272, & a_4 &= 0.0001520143, \\ a_5 &= 0.0002765672, & a_6 &= 0.0000430638. \end{aligned}$$

The numerical values of the error function calculated according to each of these approximate formulas are often more accurate (the accuracy order 10^{-7} for all $x \in [0, \infty)$) than those given in the popular textbooks.

For clarity, the generation of the normal distribution can be demonstrated by using simple devices, such as that shown in Fig. 2.8 (cf. [5]). Small metal balls

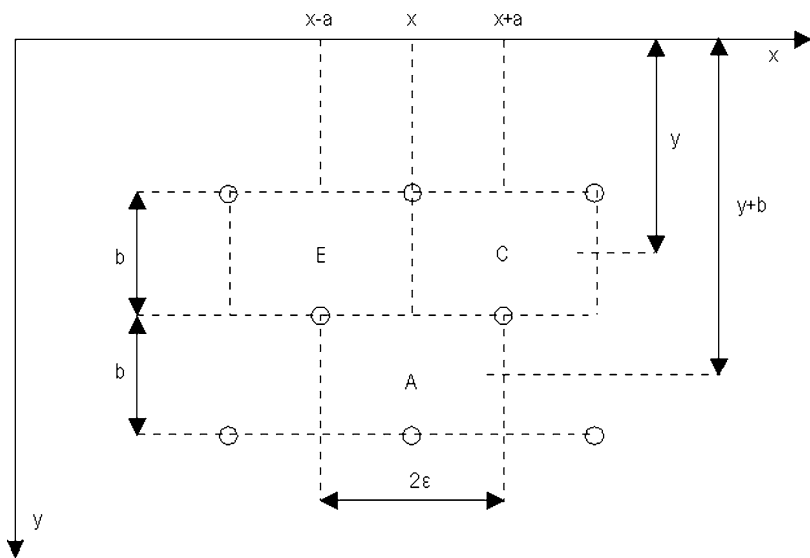


Fig. 2.9 The scheme of cells in the Galton box

falling down from a container T and striking numerous metal pins are randomly directed to the right or to the left. Finally, they fall at random into one of the separate small containers at the bottom of the device. The distribution of the number of balls in consecutive containers is close to the normal distribution. Similar examples may be found in [21].

Such a result of this educational experiment may be interpreted in two ways. From the mathematical point of view we can say that the normal distribution is formed as the consequence of the so-called central limit theorem, comp. [6]. Each ball falling down is randomly directed to the right or to the left, suffering a unit displacement with the same probability. Its final location in a specific container at the bottom is the sum of such displacements, which is the sum of independent random variables.

The fact that the distribution obtained in such an experimental device tends to the normal distribution may be proved by simple calculus, see [17]. Let us consider an arbitrary set of three cells A , B , C separated from the device in Fig. 2.8, and, moreover, let us assume that the probability distribution in the model may be described by a continuous function $P(x, y)$, if the distances between the pins are tending to zero ($a \rightarrow 0$ and $b \rightarrow 0$). The configuration of these separated cells is shown in Fig. 2.9.

Let the probabilities that a moving downwards ball falls to the cell B or C are:

$$P(x - a, y) \quad \text{and} \quad P(x + a, y),$$

respectively. Hence, we may express the momentary probabilities of migration of balls to the cell A located in the lower layer (marked as $y + b$), using the formula

for the complete probability, see (2.8). We obtain:

$$P(x, y + b) = \frac{1}{2}P(x - a, y) + \frac{1}{2}P(x + a, y). \quad (2.47)$$

Then, using Taylor's expansion of all terms of (2.47) around the point (x, y) , we can write:

$$\begin{aligned} P(x, y + b) - P(x, y) &= b \frac{\partial P(x, y)}{\partial y} + \frac{1}{2}b^2 \frac{\partial^2 P(x, y)}{\partial y^2} + \dots, \\ P(x - a, y) - P(x, y) &= -a \frac{\partial P(x, y)}{\partial x} + \frac{1}{2}a^2 \frac{\partial^2 P(x, y)}{\partial x^2} + \dots, \\ P(x + a, y) - P(x, y) &= a \frac{\partial P(x, y)}{\partial x} + \frac{1}{2}a^2 \frac{\partial^2 P(x, y)}{\partial x^2} + \dots. \end{aligned}$$

Substituting the above equations in (2.47) and decreasing the dimensions of the cells to a zero limit in such a way that simultaneously two conditions are satisfied:

$$a \rightarrow 0, \quad b \rightarrow 0, \quad \text{and} \quad \frac{a^2}{2b} = D = \text{const.}, \quad (2.48)$$

we obtain the following partial differential equation for the probability density function $P(x, y)$:

$$\frac{\partial P(x, y)}{\partial y} - D \frac{\partial^2 P(x, y)}{\partial x^2} = 0. \quad (2.49)$$

Equation (2.49), obtained in [17], is of the same type as the equation of conduction of heat in solids, cf., e.g., [4]. Its solution can be written in the form

$$P(x, y) = \frac{\beta}{\sqrt{y}} \exp \left[-\frac{x^2}{4Dy} \right]. \quad (2.50)$$

To make the solution $P(x, y)$ of (2.50) to be a probability density function we take the parameter β such that the integral with respect to x of the right-hand of (2.50), for each $y = \text{const.}$ is equal to 1,

$$\int_{-\infty}^{\infty} P(x, y) dx = 2\beta\sqrt{\pi D} = 1,$$

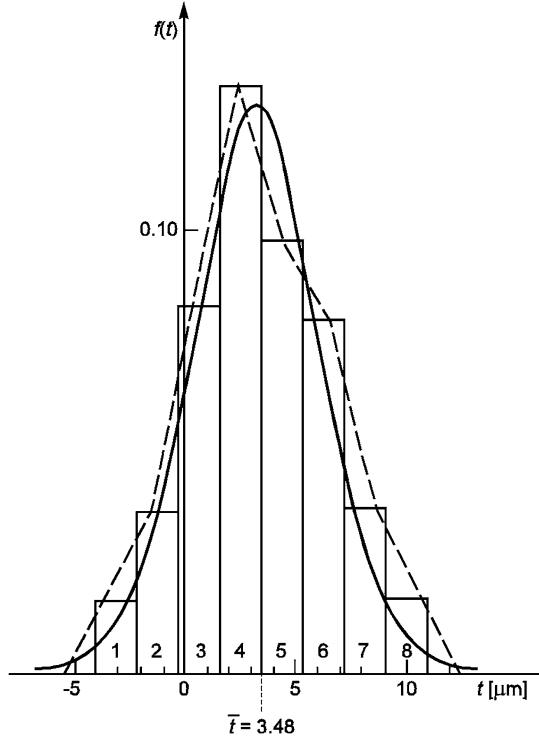
from which it follows that

$$\beta = \frac{1}{2\sqrt{\pi D}}. \quad (2.51)$$

Thus, the solution to (2.49) can be written as

$$P(x, y) = \frac{1}{2\sqrt{\pi D y}} \exp \left[-\frac{x^2}{4Dy} \right], \quad (2.52)$$

Fig. 2.10 Approximation of a histogram presented in Fig. 1.2 by the normal probability density function



and after substitution

$$\sigma = \sqrt{2Dy} \quad (2.53)$$

as

$$P(x, y) \equiv f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma^2}\right]. \quad (2.54)$$

Comparing the obtained expression (2.54) with the known probability density function of the normal distribution (2.35) of a zero mean value ($\bar{x} = 0$) we obtain an argument that the probability distribution, which is a result of random symmetric (that is with probability $\frac{1}{2}$ in every side) reflections of balls on pins of the Galton box presented in Fig. 2.8, is really the normal distribution.

This result can be also interpreted more generally: we deal with the normal probability distribution of a random variable when this variable is influenced by numerous independent factors. Such an interpretation explains why the normal distribution corresponds so well to the distribution of errors of measurements, which usually arise as a result of numerous unknown external factors.

Let us now consider an example of application of the continuous normal distribution to the description of the quasi-stepwise distribution shown in the form of the histogram presented in Fig. 1.2. For the quasi-stepwise distribution, the average

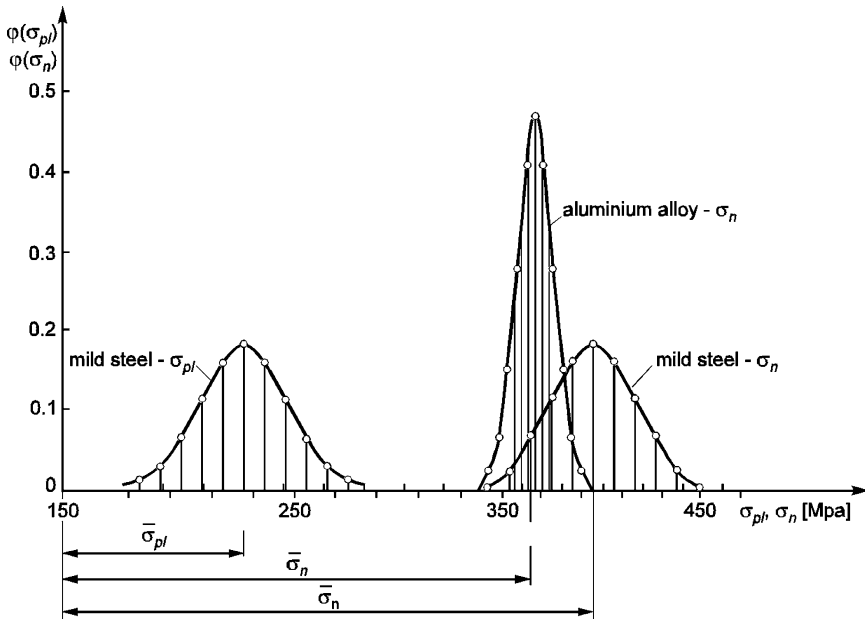


Fig. 2.11 Examples of distributions of certain mechanical properties of metal alloys, see [8]

value and the standard deviation are equal, respectively, to

$$\bar{t} = 3.48 \mu\text{m}, \quad \sigma = 3.11 \mu\text{m}.$$

The diagram of the normal distribution calculated for such values of $\bar{t}$ and σ is presented in Fig. 2.10 along with the transformed original histogram. The area below the upper stepwise boundary of the transformed histogram equals unity. It is seen that this stepwise boundary corresponds fairly well to the graph of the normal probability distribution.

As another example, Fig. 2.11 shows distributions of the yield locus σ_{pl} and limit stress σ_n under uniaxial tension of a steel sheet 2 mm thick, measured during a tension test on 330 specimens cut out at various places of the same sheet, comp. [8]. For an aluminum alloy sheet, results of similar tests also shown in the figure display much smaller dispersion of the limit stress.

Random distributions of mechanical properties observed even in one large piece of a material contribute to the so-called scale effect: large specimens display smaller limit stress and yield locus than small specimens made of the same material.

Another example of a practical application of the normal distribution to the description of the cohesion c of soils is presented in Fig. 2.12. The figure was prepared on the basis of the experimental results given in [18].

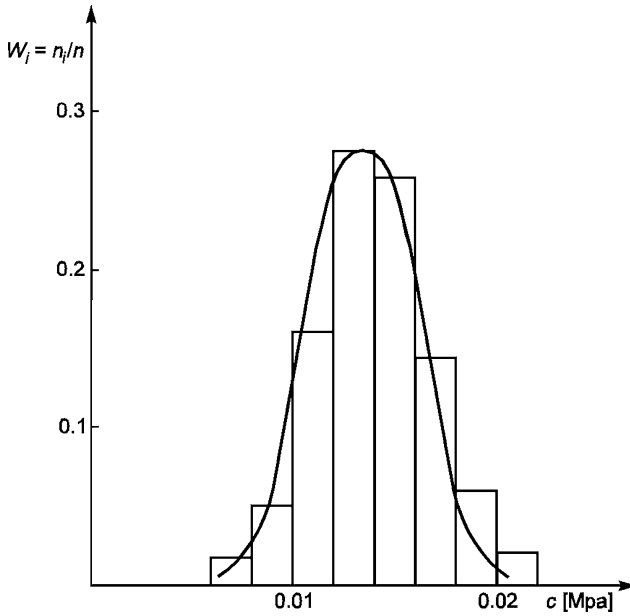


Fig. 2.12 Application of the normal distribution to the description of the cohesion c of soils, see [18]

2.5 Two-dimensional Gravity Flow of Granular Material

Before giving more information concerning probability distributions, let us analyze an example showing that using even the elementary theory of probability one can solve numerous problems of real practical significance. In the papers [15–17] J. Litwiniszyn ingeniously analyzed the inverse problem in which the cavities in a bulk of a loose material move randomly upwards from the bottom. To illustrate his idea, let us consider a two-dimensional problem of a relatively wide container with an outlet at the middle of the bottom. Fig. 2.13 shows the assumed initial system of finite cells analogous to that shown previously in Fig. 2.9. The width to height ratio of cells connected with the parameter (2.48) should be determined experimentally for the granular medium in question; for details see [23].

A portion of the loose medium has just now left cell A leaving an empty space in it. The cavity in A formed in such a manner migrates upwards. We assume, as in the inverse problem shown in Fig. 2.8 that each time a portion of that cavity moves upwards, the probability of migrating into the right-hand or into the left-hand cell lying just above is equal to $\frac{1}{2}$. It means that at the beginning of the migration process, one half of the initial cavity A moves to the cell B and the other half is shifted to the cell C . If the volume of each cell is assumed to be a unit volume, the numbers in consecutive cells indicate how large a portion of the initial unit volume A passed through the cell during the migration process. Since after migration each portion of empty space must be filled by the granular medium falling downwards,

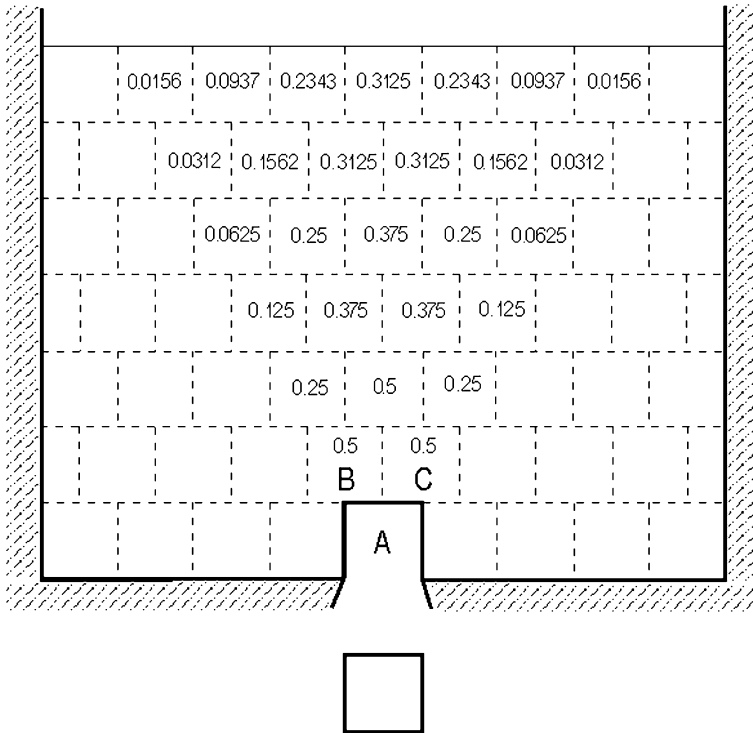


Fig. 2.13 Assumed system of cells for the problem of gravity flow from a bin

these numbers correspond to the average vertical displacement of the medium in particular cells. These vertical displacements are represented in Fig. 2.14. However, each particle of the medium is displaced also horizontally.

Below is presented a simple approximate method of determining total displacements [22]. Let us analyze an arbitrary set of three adjacent cells taken from the system of cells shown in Fig. 2.13. They are represented in Fig. 2.15a. The numbers in them correspond to the fraction of the initial volume of the cavity A, which passed through the cell during the migration towards the free surface of the bulk of the medium. According to the finite cells methodology, only one half of these fractions migrates from each cell A and B to the cell C. It is assumed that this migration takes place along the respective lines $A - C$ or $B - C$ joining central points of the cells. Directions and magnitudes of these migrating portions of the cavity may be represented by vectors W_{BC} and W_{AC} as shown in Fig. 2.15b. They may be treated as components of the resulting vector W_{cav} representing the direction and the magnitude of the averaged momentary flux of the cavity into cell C during the migration process. The opposite vector W_{mat} may be treated as a representation of the flux of the mass of granular medium filling the space left by cavities moving upwards. In order to calculate the magnitude of the averaged displacement vector u of the particles of the medium, it is assumed that its direction coincides with the direction

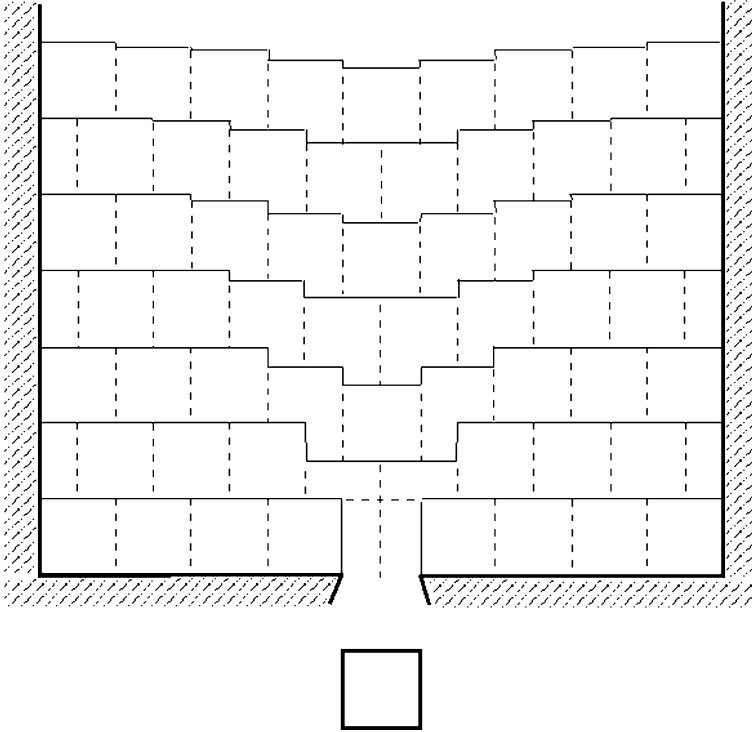


Fig. 2.14 Vertical displacement of granular material in cells

of the vector W_{mat} . To make this procedure consistent with that described before, it is assumed that the vertical component of the displacement vector u is equal to the vertical displacement of the respective sector of the stepwise deformed boundary between the rows of cells (cf. Fig. 2.14). Using this approximate procedure, the vectors of displacements have been calculated for the problem shown in Fig. 2.13 and Fig. 2.14. Results are shown in Fig. 2.16.

In Fig. 2.17 is presented an analogous solution for prediction of the movements of a crowd in a relatively narrow exit [14].

Figure 2.18 shows the theoretical field of displacements vectors calculated in the manner described above.

In order to verify experimentally such a theoretical motion pattern, a preliminary simple experimental simulation model composed of an assembly of coins of three different diameters has been used. The initial configuration of the assembly corresponding to the theoretical problem shown in Fig. 2.17 is presented in Fig. 2.19. The coins are located on a glass plate in the initial horizontal position. Then the plate is inclined with respect to the horizontal plane and the coins begin to slide downwards due to the gravity forces. This movement is disturbed by random mutual contacts between neighbors. The final configuration of displaced coins is shown in Fig. 2.20.

Fig. 2.15 Calculation of displacements of the granular material in cells, after [22]

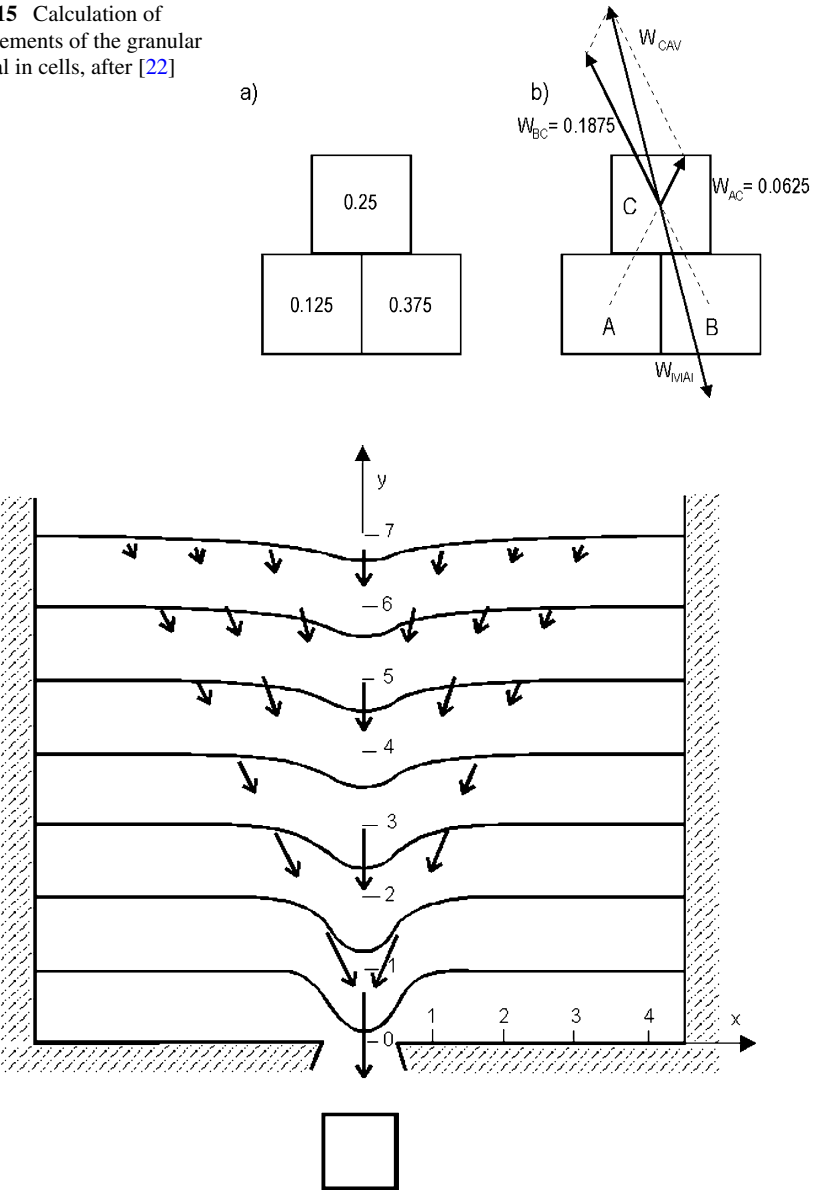


Fig. 2.16 Calculated displacements of granular medium in a bin, after [22]

The experiment was performed in three stages. In each stage one of the blocking strips at the bottom was removed. For each stage displacements of particular coins were measured. They are shown in Fig. 2.21. The stochastic nature of the movements of coins is visible. Let us notice, however, that their general layout is close to that shown previously in Fig. 2.18.

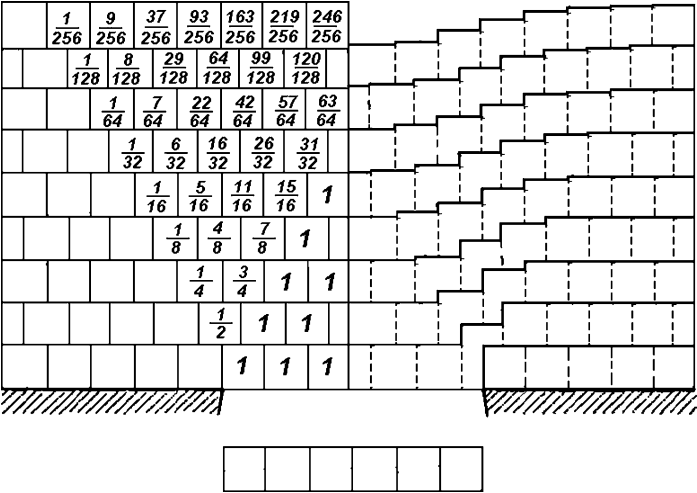


Fig. 2.17 Assumed system of finite cells and vertical displacements in a crowd in narrow exits, see [14]

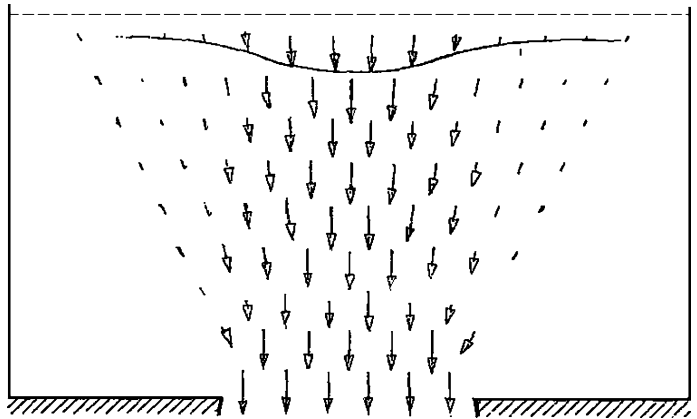


Fig. 2.18 Calculated displacements of a crowd in a narrow exit, see [14]

The next example concerns the problem of terrain subsidence caused by subterranean exploitation. The solution is shown in Fig. 2.22 (cf. [24]).

In the lower part of the soil resting on a bedrock, the empty space $A - B - C - D$ has been left by underground exploitation. In the following process of subsidence this empty space will be filled by the soil migrating downwards. Let us divide this empty space into a number of cells, each of them being of unit volume. These unit cavities migrate upwards through the system of cells shown in the figure. It is assumed that each time a cavity in the particular cell migrates upwards, the probability that it moves to the left or to the right cell, just above it, is equal to $1/2$. Numbers shown in particular cells indicate how large was the portion of a unit cavity which

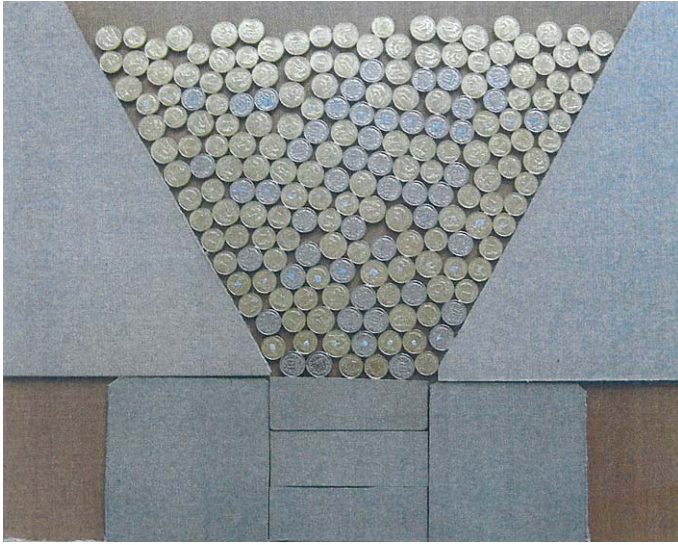


Fig. 2.19 Initial configuration of coins located on a glass plate, see [14]

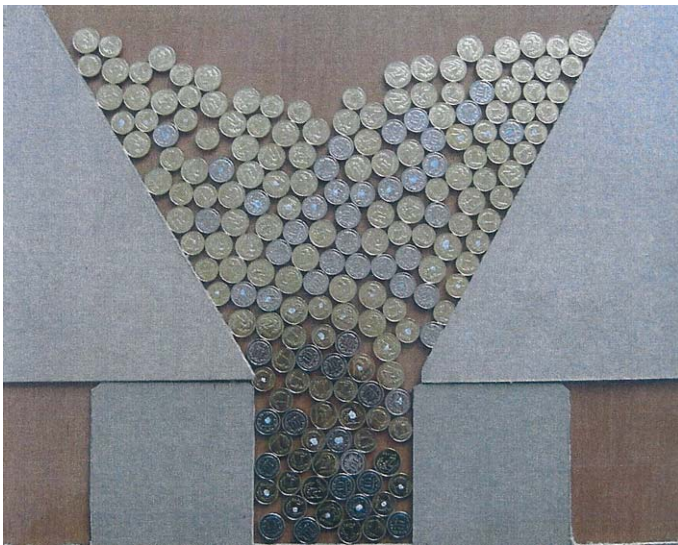


Fig. 2.20 Final configuration of coins in an experimental simulation of movements of a crowd, see [14]

has passed through the cell during the migration process. On the basis of these numbers, the diagram representing a stepwise approximation of the final subsidence shown in Fig. 2.23 has been prepared. The procedure described above allows us to calculate the vectors of displacements in the entire deformation zone Fig. 2.24.

Fig. 2.21 Experimentally determined displacements of coins in the test shown in Figs. 2.19 and 2.20

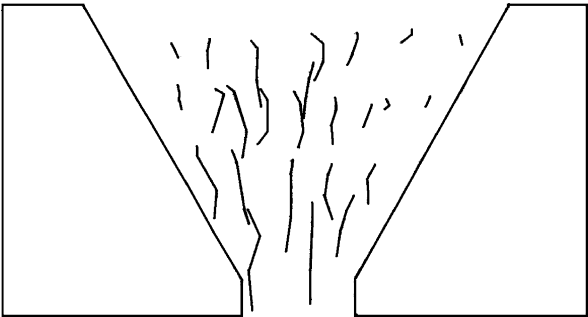


Fig. 2.22 Assumed system of finite cells for the analysis of terrain subsidence, see [24]

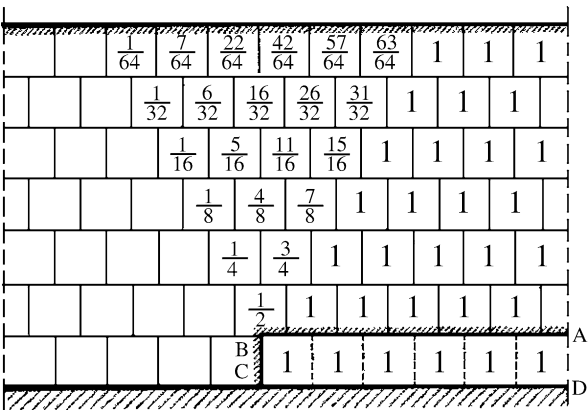
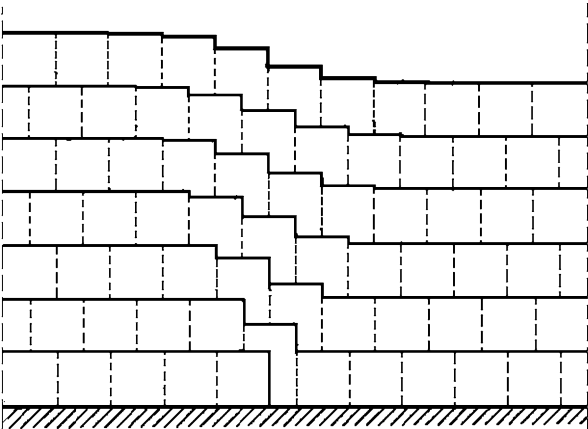


Fig. 2.23 Vertical displacement of granular medium in cells of the assumed system, see [24]



In Fig. 2.25 is presented a simple experimental simulation of such a subsidence process. The coins of different diameters are located on a glass plate as shown in the photograph. To simulate the initial configuration corresponding to that shown in Fig. 2.22, two bottom rows on the right side have been left without coins. Then

Fig. 2.24 Calculated displacements of a granular medium in the process of terrain subsidence shown in Figs. 2.22 and 2.23, after [24]

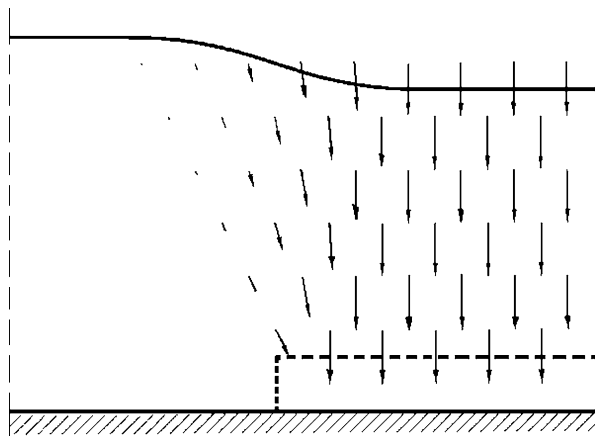


Fig. 2.25 Initial configuration of coins located on a glass plate, see [24]

the blocking strip at the bottom has been removed and the plate was inclined with respect to its initial horizontal position. The coins slid downwards due to gravity force. The final configuration of coins is shown in Fig. 2.26.

The displacements of central points of several coins resulting from this experimental simulation are shown in Fig. 2.27.

Let us note that this experimental result is similar to that resulting from theoretical solution shown in Fig. 2.24.

Summarizing the considerations of this subsection we see that the calculation methods proposed by J. Litwiniszyn were, both, very effective in solving quite involved geotechnics problems and very illustrative. They were also an inspiration for mathematically more advanced models, e.g., description of the random walk of voids by means of diffusive Markov processes, cf. [3].

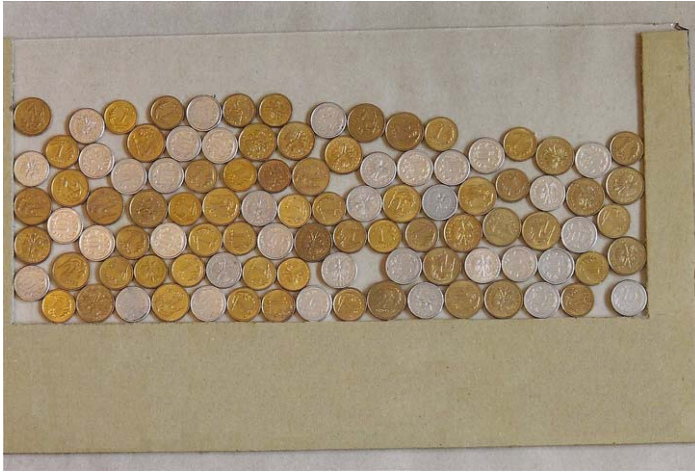


Fig. 2.26 Final configuration of coins in an experimental simulation of terrain subsidence

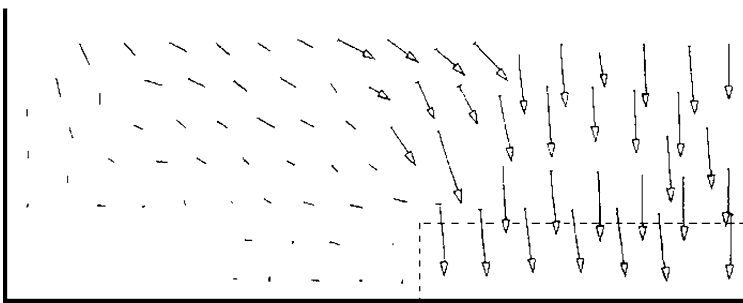


Fig. 2.27 Experimentally determined displacements of coins in the test shown in Figs. 2.25 and 2.26, after [23]

Problem 2.1 Consider a system of four electric elements connected in series. The probability of defective operation of these elements after one year of work is, respectively, 0.6, 0.5, 0.4, and 0.3, and is independent one from the other. Calculate the probability of defective operation of the system of elements. Calculate the probability that the system works correctly.

Problem 2.2 A sample of 200 mass-produced elements is tested by random choice of 10 elements. It is rejected if at least one of the elements is defective. Calculate the probability of the rejection of the sample of elements if 5% of the elements in the sample are defective.

Problem 2.3 Calculate the probability that a sample of 100 mass-produced elements will be accepted if it contains 5 defective elements and we test 50 elements allowing at the most two defective elements among them.

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