2

Sequences, Continuity, and Limits

In this chapter, we introduce the fundamental notions of continuity and limit of a real-valued function of two variables. As in ACICARA, the definitions as well as proofs of basic results will be given using sequences. There are, actually, two possible generalizations of real sequences that seem natural in the setting of two variables. First, functions defined on $\mathbb{N}$ with values in $\mathbb{R}^2$, and second, functions defined on $\mathbb{N}^2$ with values in $\mathbb{R}$. As we shall see, for developing the notions of continuity and limit of a function of two variables, only the former is relevant, and it is studied in this chapter. The study of the latter will be taken up in Chapter 7.

This chapter is organized as follows. Sequences in $\mathbb{R}^2$ are introduced in Section 2.1 below and their fundamental properties, including the Bolzano–Weierstrass Theorem and the Cauchy Criterion, are derived from the corresponding results for sequences in $\mathbb{R}$. We also use the notion of sequence to introduce basic topological notions of closed and open sets, boundary points, and interior points, and also the closure and the interior of subsets of $\mathbb{R}^2$. Section 2.2 deals with the notion of continuity, and it is shown here that continuous functions on path-connected subsets of $\mathbb{R}^2$ or on closed and bounded subsets of $\mathbb{R}^2$ possess several nice properties. An important result known as the Implicit Function Theorem is also proved in this section. Finally, in Section 2.3 we introduce limits of functions of two variables. The definition is given using sequences, while most of the basic properties are proved using a simple observation that the existence of limit of a function at a point is equivalent to the continuity of an associated function at that point.

2.1 Sequences in $\mathbb{R}^2$

A **sequence** in $\mathbb{R}^2$ is a function from $\mathbb{N}$ to $\mathbb{R}^2$. Typically, a sequence in $\mathbb{R}^2$ is denoted by $\big((x_n, y_n)\big)$, $\big((u_n, v_n)\big)$, etc. The value of a sequence $\big((x_n, y_n)\big)$ at $n \in \mathbb{N}$ is given by the element (x_n, y_n) of $\mathbb{R}^2$, and this element is called the

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nth **term** of that sequence. In case the terms of a sequence $((x_n, y_n))$ lie in a subset D of $\mathbb{R}^2$, then we say that $((x_n, y_n))$ is a sequence in D.

The notions of boundedness and convergence extend readily from the setting of sequences in $\mathbb{R}$ to sequences in $\mathbb{R}^2$. Let $((x_n, y_n))$ be a sequence in $\mathbb{R}^2$. We say that $((x_n, y_n))$ is **bounded** if there is $\alpha \in \mathbb{R}$ such that $|(x_n, y_n)| \le \alpha$ for all $n \in \mathbb{N}$. The sequence $((x_n, y_n))$ is said to be **convergent** if there is $(x_0, y_0) \in \mathbb{R}^2$ that satisfies the following condition: For every $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $(x_n, y_n) \in \mathbb{S}_\epsilon(x_0, y_0)$ for all $n \ge n_0$, that is,

$$|x_n - x_0| < \epsilon \text{ and } |y_n - y_0| < \epsilon \quad \text{for all } n \ge n_0.$$

In this case, we say that $((x_n, y_n))$ **converges** to (x_0, y_0) or that (x_0, y_0) is a **limit** of $((x_n, y_n))$, and write $(x_n, y_n) \to (x_0, y_0)$. If $((x_n, y_n))$ does not converge to (x_0, y_0), then we write $(x_n, y_n) \not\to (x_0, y_0)$; if $((x_n, y_n))$ is not convergent, then it is said to be **divergent**.

A sequence $((x_n, y_n))$ in $\mathbb{R}^2$ gives rise to two sequences (x_n) and (y_n) in $\mathbb{R}$, and vice versa. It turns out that the properties of $((x_n, y_n))$ can be completely understood in terms of the properties of the sequences (x_n) and (y_n) in $\mathbb{R}$.

Proposition 2.1. *Given a sequence $((x_n, y_n))$ in $\mathbb{R}^2$, we have the following.*

(i) *If $((x_n, y_n))$ is convergent, then it has a unique limit.*
(ii) *$((x_n, y_n))$ is bounded $\iff$ both (x_n) and (y_n) are bounded.*
(iii) *$((x_n, y_n))$ is convergent $\iff$ both (x_n) and (y_n) are convergent. In fact, for $(x_0, y_0) \in \mathbb{R}^2$, we have $(x_n, y_n) \to (x_0, y_0) \iff x_n \to x_0$ and $y_n \to y_0$.*

Proof. Each of (i), (ii), and (iii) is immediate from the definitions. □

As noted in part (i) of Proposition 2.1, if $((x_n, y_n))$ is a convergent sequence in $\mathbb{R}^2$, then it has a unique limit in $\mathbb{R}^2$. The limit of $((x_n, y_n))$ is sometimes written as $\lim_{n\to\infty}(x_n, y_n)$ or as $\displaystyle\lim_{n\to\infty}(x_n, y_n)$.

Examples 2.2. (i) If $((x_n, y_n))$ is a **constant sequence** in $\mathbb{R}^2$, that is, if there is $(x_0, y_0) \in \mathbb{R}^2$ such that $(x_n, y_n) = (x_0, y_0)$ for all $n \in \mathbb{N}$, then clearly, $((x_n, y_n))$ is convergent and $(x_n, y_n) \to (x_0, y_0)$.

(ii) If $((x_n, y_n))$ is the sequence in $\mathbb{R}^2$ defined by $(x_n, y_n) := (1/n, -1/n)$ for all $n \in \mathbb{N}$, then clearly, $((x_n, y_n))$ is convergent and $(x_n, y_n) \to (0, 0)$.

(iii) The sequence $((x_n, y_n))$ in $\mathbb{R}^2$ defined by $(x_n, y_n) := (1/n, (-1)^n)$ for all $n \in \mathbb{N}$ is divergent, since the sequence $((-1)^n)$ in $\mathbb{R}$ is divergent. ◇

Basic properties of sequences in $\mathbb{R}^2$ readily follow from the corresponding properties of sequences in $\mathbb{R}$. For ease of reference, we recall the relevant results for sequences in $\mathbb{R}$. For proofs, one may refer to pages 45–47 of ACICARA.

Fact 2.3. *Let (a_n) and (b_n) be sequences in $\mathbb{R}$, and let $a, b, \alpha, \beta \in \mathbb{R}$.*

(i) *If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$ and $a_n b_n \to ab$.*

(ii) *If* $a_n \to a$, *then for any* $r \in \mathbb{R}$, *we have* $ra_n \to ra$.
(iii) *If* $a \neq 0$ *and* $a_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $(1/a_n) \to (1/a)$.
(iv) *Let* $a_n \to a$. *If there is* $\ell \in \mathbb{N}$ *such that* $a_n \geq \alpha$ *for all* $n \geq \ell$, *then* $a \geq \alpha$. *Likewise, if there is* $m \in \mathbb{N}$ *such that* $a_n \leq \beta$ *for all* $n \geq m$, *then* $a \leq \beta$.
(v) *If* $a_n \to a$ *and* $a_n \geq 0$ *for all* $n \in \mathbb{N}$, *then* $a_n^{1/k} \to a^{1/k}$ *for any* $k \in \mathbb{N}$.
(vi) **(Sandwich Theorem in** $\mathbb{R}$**)** *If* (b_n) *and* (c_n) *are sequences such that* $b_n \to a$ *and* $c_n \to a$, *and if there is* $m \in \mathbb{N}$ *such that* $b_n \leq a_n \leq c_n$ *for all* $n \geq m$, *then* $a_n \to a$.

A few of these facts yield the result that sums, dot products, and scalar multiples of sequences in $\mathbb{R}^2$ converge, respectively, to the sums, dot products, and scalar multiples of the corresponding limits.

Proposition 2.4. *Let* $\big((x_n, y_n)\big)$ *and* $\big((u_n, v_n)\big)$ *be sequences in* $\mathbb{R}^2$, *and let* $(x_0, y_0), (u_0, v_0) \in \mathbb{R}^2$.

(i) *If* $(x_n, y_n) \to (x_0, y_0)$ *and* $(u_n, v_n) \to (u_0, v_0)$, *then* $(x_n, y_n) + (u_n, v_n) \to (x_0, y_0) + (u_0, v_0)$ *and* $(x_n, y_n) \cdot (u_n, v_n) \to (x_0, y_0) \cdot (u_0, v_0)$.
(ii) *If* $(x_n, y_n) \to (x_0, y_0)$, *then for any* $r \in \mathbb{R}$, $r(x_n, y_n) \to r(x_0, y_0)$.

Proof. Immediate consequence of part (iii) of Proposition 2.1 together with parts (i) and (ii) of Fact 2.3. □

Analogues of properties of sequences in $\mathbb{R}$ that depend on order relations, are considered in Exercise 2.

Subsequences and Cauchy Sequences

Let $\big((x_n, y_n)\big)$ be a sequence in $\mathbb{R}^2$. If $n_1, n_2, \ldots$ are positive integers such that $n_k < n_{k+1}$ for each $k \in \mathbb{N}$, then the sequence $\big((x_{n_k}, y_{n_k})\big)$, whose terms are (x_{n_1}, y_{n_1}), $(x_{n_2}, y_{n_2}), \ldots$, is called a **subsequence** of $\big((x_n, y_n)\big)$. The sequence $\big((x_n, y_n)\big)$ is said to be **Cauchy** if for every $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ and $|y_n - y_m| < \epsilon$ for all $n, m \geq n_0$. It is clear that $\big((x_n, y_n)\big)$ is Cauchy if and only if both (x_n) and (y_n) are Cauchy sequences in $\mathbb{R}$.

Let us recall the following basic facts about sequences in $\mathbb{R}$. For proofs, one may refer to pages 45, 56, and 58 of ACICARA.

Fact 2.5. *Let* (a_n) *be a sequence in* $\mathbb{R}$. *Then we have the following.*

(i) (a_n) *is convergent* $\Longrightarrow$ (a_n) *is bounded.*
(ii) **(Bolzano–Weierstrass Theorem in** $\mathbb{R}$**)** *If* (a_n) *is bounded, then* (a_n) *has a convergent subsequence.*
(iii) (a_n) *is convergent* $\Longleftrightarrow$ (a_n) *is bounded and every convergent subsequence of* (a_n) *has the same limit.*
(iv) **(Cauchy Criterion in** $\mathbb{R}$**)** (a_n) *is Cauchy* $\Longleftrightarrow$ (a_n) *is convergent.*

These facts, in turn, lead to the following results.

Proposition 2.6. *Given a sequence* $\big((x_n, y_n)\big)$ *in* $\mathbb{R}^2$*, we have the following.*

(i) $\big((x_n, y_n)\big)$ *is convergent* $\Longrightarrow$ $\big((x_n, y_n)\big)$ *is bounded.*
(ii) **(Bolzano–Weierstrass Theorem)** *If* $\big((x_n, y_n)\big)$ *is a bounded sequence, then* $\big((x_n, y_n)\big)$ *has a convergent subsequence.*
(iii) $\big((x_n, y_n)\big)$ *is convergent* $\Longleftrightarrow$ $\big((x_n, y_n)\big)$ *is bounded and every convergent subsequence of* $\big((x_n, y_n)\big)$ *has the same limit.*
(iv) **(Cauchy Criterion)** $\big((x_n, y_n)\big)$ *is Cauchy* $\Longleftrightarrow$ $\big((x_n, y_n)\big)$ *is convergent.*

Proof. Clearly, (i) is an immediate consequence of parts (ii) and (iii) of Proposition 2.1 and part (i) of Fact 2.5. To prove (ii), suppose $\big((x_n, y_n)\big)$ is bounded. Then (x_n) is a bounded sequence in $\mathbb{R}$ and hence by part (ii) of Fact 2.5, (x_n) has a convergent subsequence, say (x_{n_k}). Now, (y_n) is a bounded sequence in $\mathbb{R}$ and hence so is (y_{n_k}). So, by part (ii) of Fact 2.5, (y_{n_k}) has a convergent subsequence, say $\big(y_{n_{k_j}}\big)$. Clearly, $\big((x_{n_{k_j}}, y_{n_{k_j}})\big)$ is a convergent subsequence of $\big((x_n, y_n)\big)$. This proves (ii). Next, if $\big((x_n, y_n)\big)$ is convergent, then it is clear that it is bounded and every convergent subsequence of $\big((x_n, y_n)\big)$ has the same limit. To prove the converse, suppose $\big((x_n, y_n)\big)$ is bounded. By (ii), $\big((x_n, y_n)\big)$ has a convergent subsequence. Suppose (x_0, y_0) is the (same) limit for every convergent subsequence of $\big((x_n, y_n)\big)$. If $(x_n, y_n) \not\to (x_0, y_0)$, then there are $\epsilon > 0$ and positive integers $n_1 < n_2 < \cdots$ such that $\max\{|x_{n_k} - x_0|, |y_{n_k} - y_0|\} \geq \epsilon$ for all $k \in \mathbb{N}$. Now, $\big((x_{n_k}, y_{n_k})\big)$ is bounded and hence by (ii), it has a convergent subsequence. Moreover, this subsequence must converge to (x_0, y_0). This is a contradiction. Thus (iii) is proved. Finally, (iv) follows from part (iii) of Proposition 2.1, part (iv) of Fact 2.5, and our earlier observation that $\big((x_n, y_n)\big)$ is Cauchy if and only if both (x_n) and (y_n) are Cauchy sequences in $\mathbb{R}$. □

The result in part (iv) of Proposition 2.6 is sometimes referred to as the **Cauchy completeness** of $\mathbb{R}^2$. A similar result holds for $\mathbb{R}^n$.

Closure, Boundary, and Interior

Let $D \subseteq \mathbb{R}^2$. We say that D is **closed** if every convergent sequence in D converges to a point of D. The set of all points in $\mathbb{R}^2$ that are limits of convergent sequences in D is called the **closure** of D and is denoted by $\overline{D}$. It is clear that D is closed if and only if $\overline{D} = D$. A point of $\mathbb{R}^2$ is said to be a **boundary point** of D if there is a sequence in D that converges to it and also a sequence in $\mathbb{R}^2 \setminus D$ that converges to it. The set of all boundary points of D in $\mathbb{R}^2$ is called the **boundary** of D (in $\mathbb{R}^2$), and is denoted by ∂D. It is easy to see that $\partial D = \partial(\mathbb{R}^2 \setminus D)$, that is, the boundary of a set coincides with the boundary of its complement. A relation between the closure and the boundary is described by the following.

Proposition 2.7. *Given any* $D \subseteq \mathbb{R}^2$*, we have* $\overline{D} = D \cup \partial D$*.*

Proof. Let $(x, y) \in \overline{D}$. Then there is a sequence in D converging to (x, y). Further, if $(x, y) \notin D$, then the constant sequence $\big((x_n, y_n)\big)$ defined by $(x_n, y_n) = (x, y)$ for all $n \in \mathbb{N}$ gives a sequence in $\mathbb{R}^2 \setminus D$ converging to (x, y), and so in this case, $(x, y) \in \partial D$. It follows that $\overline{D} \subseteq D \cup \partial D$. On the other hand, if $(x, y) \in D \cup \partial D$, then it is clear, using either a constant sequence or the definition of ∂D, that $(x, y) \in \overline{D}$, and so $D \cup \partial D \subseteq \overline{D}$. □

Proposition 2.8. *Let D be a nonempty subset of $\mathbb{R}^2$ such that $D \neq \mathbb{R}^2$. Then ∂D is nonempty.*

Proof. Since D is nonempty, there is some $(x_0, y_0) \in D$, and since $D \neq \mathbb{R}^2$, there is some $(x_1, y_1) \in \mathbb{R}^2 \setminus D$. Consider the line segment joining these two points, that is, consider $L := \{t \in [0, 1] : (1 - t)(x_0, y_0) + t(x_1, y_1) \in D\}$. Then L is a nonempty subset of $\mathbb{R}$ bounded above by 1. Let $t^* := \sup L$ and $(x^*, y^*) := (1 - t^*)(x_0, y_0) + t^*(x_1, y_1)$. We claim that (x^*, y^*) is a boundary point of D. To see this, let (t_n) be a sequence in L such that $t_n \to t^*$. Let $(x_n, y_n) := (1 - t_n)(x_0, y_0) + t_n(x_1, y_1)$ for $n \in \mathbb{N}$. Clearly $\big((x_n, y_n)\big)$ is a sequence in D that converges to (x^*, y^*). Further, if $t^* < 1$, then we can find $s_n \in \mathbb{R}$ for $n \in \mathbb{N}$ such that $s_n \to t^*$ and $t^* < s_n \leq 1$, and we let $(u_n, v_n) := (1 - s_n)(x_0, y_0) + s_n(x_1, y_1)$ for $n \in \mathbb{N}$, whereas if $t^* = 1$, then we let $(u_n, v_n) := (x_1, y_1)$ for $n \in \mathbb{N}$. In any case, we see that $\big((u_n, v_n)\big)$ is a sequence in $\mathbb{R}^2 \setminus D$ that converges to (x^*, y^*). This proves the claim. □

Let D be a subset of $\mathbb{R}^2$ and let (x_0, y_0) be any point of $\mathbb{R}^2$. We say that (x_0, y_0) is an **interior point** of D if $(x_0, y_0) \in D$ and (x_0, y_0) is not a boundary point of D. It is easy to see that (x_0, y_0) is an interior point of D if and only if there is $r > 0$ such that $\mathbb{S}_r(x_0, y_0) \subseteq D$. The **interior** of D is defined to be the set of all interior points of D. Clearly, the interior of D is a subset of D. We say that D is **open** if every point of D is an interior point of D. The following proposition shows the connection between the notions of an open set and a closed set.

Proposition 2.9. *Let $D \subseteq \mathbb{R}^2$. Then D is closed if and only if $\mathbb{R}^2 \setminus D$ is open.*

Proof. First, suppose D is a closed set. Let $(x_0, y_0) \in \mathbb{R}^2 \setminus D$. If (x_0, y_0) is not an interior point of $\mathbb{R}^2 \setminus D$, then there is a sequence $\big((x_n, y_n)\big)$ in the complement of $\mathbb{R}^2 \setminus D$, that is, in D, such that $(x_n, y_n) \to (x_0, y_0)$, and so $(x_0, y_0) \in \overline{D} = D$, which is a contradiction. This proves that $\mathbb{R}^2 \setminus D$ is an open set. Conversely, suppose $\mathbb{R}^2 \setminus D$ is open. Let $\big((x_n, y_n)\big)$ be any sequence in D such that $(x_n, y_n) \to (x_0, y_0)$ for some $(x_0, y_0) \in \mathbb{R}^2$. Then (x_0, y_0) cannot be an interior point of $\mathbb{R}^2 \setminus D$. But since $\mathbb{R}^2 \setminus D$ is open, it follows that $(x_0, y_0) \notin \mathbb{R}^2 \setminus D$, that is, $(x_0, y_0) \in D$. This proves that D is closed. □

Example 2.10. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ and $\beta > 0$. Consider the sets $D_1 := \{(x, y) \in \mathbb{R}^2 : |x| \leq \alpha \text{ and } |y| \leq \beta\}$, $D_2 := \{(x, y) \in \mathbb{R}^2 : |x| < \alpha \text{ and } |y| \leq \beta\}$, $D_3 := \{(x, y) \in \mathbb{R}^2 : |x| \leq \alpha \text{ and } |y| < \beta\}$, and $D_4 := \{(x, y) \in \mathbb{R}^2 : |x| < \alpha \text{ and } |y| < \beta\}$. In view of part (iv) of Fact 2.3, we readily see that D_1 is closed, D_4 is open, whereas D_2 and D_3 are neither closed nor open. Further, for each $i = 1, 2, 3, 4$, the closure of D_i is D_1, the interior of D_i is D_4, and the boundary of D_i is the set $\{(x, y) \in \mathbb{R}^2 : |x| = \alpha \text{ and } |y| = \beta\}$. ◇

Remark 2.11. The notions discussed in this section concerning sequences in $\mathbb{R}^2$, closed sets, closure, boundary points, boundary, interior points, interior, and open sets admit a straightforward extension to $\mathbb{R}^3$ and more generally, to $\mathbb{R}^n$ for any $n \in \mathbb{N}$. To avoid a notational conflict, one may denote a sequence in $\mathbb{R}^n$ by $(\mathbf{x}_k)$, where the parameter k runs through $\mathbb{N}$ and $\mathbf{x}_k \in \mathbb{R}^n$ for each $k \in \mathbb{N}$. It may be instructive to formulate precise analogues of the notions and results in this section for $\mathbb{R}^n$ and write down proofs of analogous results in the general case. This may also be a good opportunity to review the results in this section. ◇

2.2 Continuity

Let D be a subset of $\mathbb{R}^2$ and let (x_0, y_0) be any point in D. A function $f : D \to \mathbb{R}$ is said to be **continuous** at (x_0, y_0) if for every sequence $\big((x_n, y_n)\big)$ in D such that $(x_n, y_n) \to (x_0, y_0)$, we have $f(x_n, y_n) \to f(x_0, y_0)$. If f is not continuous at (x_0, y_0), then we say that f is **discontinuous** at (x_0, y_0). When f is continuous at every $(x_0, y_0) \in D$, we say that f is **continuous** on D.

Examples 2.12. (i) If D is any subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ is a constant function on D, that is, if there is $c \in \mathbb{R}$ such that $f(x, y) = c$ for all $(x, y) \in D$, then clearly, f is continuous on D.

(ii) If $f : \mathbb{R}^2 \to \mathbb{R}$ is the norm function given by $f(x, y) := \sqrt{x^2 + y^2}$ for $(x, y) \in \mathbb{R}^2$, then f is continuous on $\mathbb{R}^2$. To see this, let $(x_0, y_0) \in \mathbb{R}$ be any point and let $\big((x_n, y_n)\big)$ be a sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x_0, y_0)$. Then by part (iii) of Proposition 2.1, the sequences (x_n) and (y_n) in $\mathbb{R}$ are such that $x_n \to x_0$ and $y_n \to y_0$. Hence, by parts (i) and (v) of Fact 2.3, we see that $\sqrt{x_n^2 + y_n^2} \to \sqrt{x_0^2 + y_0^2}$. Thus f is continuous on $\mathbb{R}^2$.

(iii) Consider the **coordinate functions** $p_1, p_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $p_1(x, y) := x$ and $p_2(x, y) := y$ for $(x, y) \in \mathbb{R}^2$. Then by part (iii) of Proposition 2.1, we immediately see that p_1 and p_2 are continuous on $\mathbb{R}^2$.

(iv) Let $D \subseteq \mathbb{R}^2$ and let us fix $(x_0, y_0) \in D$. Consider

$$D_1 := \{x \in \mathbb{R} : (x, y_0) \in D\} \quad \text{and} \quad D_2 := \{y \in \mathbb{R} : (x_0, y) \in D\}.$$

Notice that the set D_1 depends on y_0, whereas D_2 depends on x_0. Given any $f : D \to \mathbb{R}$, let $\phi : D_1 \to \mathbb{R}$ and $\psi : D_2 \to \mathbb{R}$ be functions of one variable defined by

$$\phi(x) := f(x, y_0) \quad \text{for } x \in D_1 \quad \text{and} \quad \psi(y) := f(x_0, y) \quad \text{for } y \in D_2.$$

These functions will play a useful role in the study of the function f of two variables around the point (x_0, y_0). If f is continuous at (x_0, y_0), then

ϕ is continuous at x_0 and ψ is continuous at y_0. To see this, let (x_n) be a sequence in D_1 such that $x_n \to x_0$. Then $(x_n, y_0) \to (x_0, y_0)$ and hence $f(x_n, y_0) \to f(x_0, y_0)$, that is, $\phi(x_n) \to \phi(x_0)$. Thus ϕ is continuous at x_0. Similarly, ψ is continuous at y_0. $\diamond$

Let us recall that the sign of a continuous function of one variable is preserved in a neighborhood of that point. More precisely, we have the following. For a proof, one may refer to page 68 of ACICARA.

Fact 2.13. *Let $E \subseteq \mathbb{R}$, $c \in E$, and let $\phi : E \to \mathbb{R}$ be continuous at c. If $\phi(c) > 0$, then there is $\delta > 0$ such that $\phi(x) > 0$ for all $x \in E \cap (c - \delta, c + \delta)$. Likewise, if $\phi(c) < 0$, then there is $\delta > 0$ such that $\phi(x) < 0$ for all $x \in E \cap (c - \delta, c + \delta)$.*

A similar result holds for functions of two variables.

Lemma 2.14. *Let $D \subseteq \mathbb{R}^2$, $(x_0, y_0) \in D$, and let $f : D \to \mathbb{R}$ be a function that is continuous at (x_0, y_0). If $f(x_0, y_0) > 0$, then there is $\delta > 0$ such that $f(x, y) > 0$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$. Likewise, if $f(x_0, y_0) < 0$, then there is $\delta > 0$ such that $f(x, y) < 0$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$.*

Proof. First, suppose $f(x_0, y_0) > 0$. If there is no $\delta > 0$ with the desired property, then for each $n \in \mathbb{N}$, we can find $(x_n, y_n) \in D \cap \mathbb{S}_{1/n}(x_0, y_0)$ such that $f(x_n, y_n) \le 0$. Now $(x_n, y_n) \to (x_0, y_0)$, and since f is continuous at (x_0, y_0), we have $f(x_n, y_n) \to f(x_0, y_0)$. Hence, by part (iv) of Fact 2.3, $f(x_0, y_0) \le 0$, which is a contradiction. The proof when $f(x_0, y_0) < 0$ is similar. $\square$

Proposition 2.15. *Let $D \subseteq \mathbb{R}^2$, $(x_0, y_0) \in D$, $r \in \mathbb{R}$, and let $f, g : D \to \mathbb{R}$ be continuous at (x_0, y_0). Then $f + g$, rf, and fg are continuous at (x_0, y_0). In case $f(x_0, y_0) \ne 0$, there is $\delta > 0$ such that $f(x, y) \ne 0$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$, and the function $1/f : D \cap \mathbb{S}_\delta(x_0, y_0) \to \mathbb{R}$ is continuous at (x_0, y_0). In case there is $\delta > 0$ such that $f(x, y) \ge 0$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$, the function $f^{1/k} : D \cap \mathbb{S}_\delta(x_0, y_0) \to \mathbb{R}$ is continuous at (x_0, y_0) for every $k \in \mathbb{N}$.*

Proof. The continuity of $f + g$, rf, and fg at (x_0, y_0) follows readily from parts (i) and (ii) of Fact 2.3. In case $f(x_0, y_0) \ne 0$, we have either $f(x_0, y_0) > 0$ or $f(x_0, y_0) < 0$. Thus, by Lemma 2.14, there is $\delta > 0$ such that $f(x, y) \ne 0$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$. Now, by part (iii) of Fact 2.3, we see that the function $1/f : D \cap \mathbb{S}_\delta(x_0, y_0) \to \mathbb{R}$ is continuous at (x_0, y_0). Finally, the assertion about the continuity of $f^{1/k}$ at (x_0, y_0) is a direct consequence of part (v) of Fact 2.3. $\square$

As in the case of functions of one variable, we can easily deduce from Proposition 2.15 the following. Suppose $D \subseteq \mathbb{R}^2$ and $f, g : D \to \mathbb{R}$ are continuous at $(x_0, y_0) \in D$. Then the difference $f - g$ is continuous at (x_0, y_0). Also, if $g(x_0, y_0) \ne 0$, then the quotient f/g is continuous at (x_0, y_0). Further, if there is $\delta > 0$ such that $f(x) \ge 0$ for all $x \in D \cap \mathbb{S}_\delta(x_0, y_0)$, then for every positive rational number r, the function f^r is continuous at (x_0, y_0). Similarly, if $f(x_0, y_0) > 0$, then for every negative rational number r the function f^r is continuous at (x_0, y_0).

Examples 2.16. (i) Using Proposition 2.15 and the above remarks, we see that every polynomial function on $\mathbb{R}^2$ is continuous and every rational function is continuous wherever it is defined, that is, if $p(x, y)$ and $q(x, y)$ are polynomials in two variables and if $D := \{(x, y) \in \mathbb{R}^2 : q(x, y) \neq 0\}$, then the rational function $f : D \to \mathbb{R}$ defined by $f(x, y) := p(x, y)/q(x, y)$ for $(x, y) \in D$ is continuous on D. Moreover, if $E = \{(x, y) \in \mathbb{R}^2 : p(x, y) \geq 0 \text{ and } q(x, y) > 0\}$, then for any $m, n \in \mathbb{N}$, the algebraic function $g : E \to \mathbb{R}$ defined by $g(x, y) := p(x, y)^{1/m}/q(x, y)^{1/n}$ for $(x, y) \in E$, is continuous on E.

(ii) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows.

$$f(x, y) := \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then f is not continuous at $(0, 0)$. To see this, consider a sequence in $\mathbb{R}^2$ approaching $(0, 0)$ along the line $y = x$; for example, the sequence $\big((1/n, 1/n)\big)$. Then $(1/n, 1/n) \to (0, 0)$, but $f(1/n, 1/n) \to 1/2 \neq f(0, 0)$.

(iii) Consider a variant of the function in (ii), namely, $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) := \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then f is continuous at $(0, 0)$. To see this, note that for any $(x, y) \in \mathbb{R}^2$, we have $x^2 \leq x^2 + y^2$ and consequently, $|f(x, y)| \leq |y|$. Hence if $\big((x_n, y_n)\big)$ is any sequence in $\mathbb{R}^2$ with $(x_n, y_n) \to (0, 0)$, then $y_n \to 0$, and as a result, $f(x_n, y_n) \to 0 = f(0, 0)$.

(iv) Consider a variant of the function in (iii), namely, $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) := \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then $f(x, y)$ approaches 0 along every line passing through the origin [indeed, $f(0, y) = 0$ and $f(x, mx) = mx/(x^2 + m^2) \to 0$ as $x \to 0$]. However, f is not continuous at $(0, 0)$. To see this, consider a sequence in $\mathbb{R}^2$ approaching $(0, 0)$ along the parabola $y = x^2$; for example, the sequence $\big((1/n, 1/n^2)\big)$. Then $(1/n, 1/n^2) \to (0, 0)$, but $f(1/n, 1/n^2) \to 1/2 \neq f(0, 0)$.

(v) Consider a variant of the function in (iv), namely, $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) := \begin{cases} \dfrac{x^3 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then f is continuous at $(0,0)$. To see this, use the A.M.-G.M. Inequality (given, for example, on page 12 of ACICARA) to obtain $2|x^2y| \le x^4 + y^2$, and hence $|f(x,y)| \le |x|/2$ for all $(x,y) \in \mathbb{R}^2$. Thus, if $\big((x_n, y_n)\big)$ is any sequence in $\mathbb{R}^2$ with $(x_n, y_n) \to (0,0)$, then we see that $x_n \to 0$ and as a result, $f(x_n, y_n) \to 0 = f(0,0)$. ◇

Composition of Continuous Functions

We now show that the composition of continuous functions is continuous. It may be noted that for functions of two variables, three types of composites are possible. Thus, the following result is stated in three parts.

Proposition 2.17. *Let $D \subseteq \mathbb{R}^2$, $(x_0, y_0) \in D$, and let $f : D \to \mathbb{R}$ be continuous at (x_0, y_0).*

(i) *Suppose $E \subseteq \mathbb{R}$ is such that $f(D) \subseteq E$. If $g : E \to \mathbb{R}$ is continuous at $f(x_0, y_0)$, then $g \circ f : D \to \mathbb{R}$ is continuous at (x_0, y_0).*
(ii) *Suppose $E \subseteq \mathbb{R}$, $t_0 \in E$, and $x, y : E \to \mathbb{R}$ are such that $(x(t), y(t)) \in D$ for all $t \in E$ and $(x(t_0), y(t_0)) = (x_0, y_0)$. If x, y are continuous at t_0, then $F : E \to \mathbb{R}$ defined by $F(t) := f(x(t), y(t))$ is continuous at t_0.*
(iii) *Suppose $E \subseteq \mathbb{R}^2$, $(u_0, v_0) \in E$, and $x, y : E \to \mathbb{R}$ are such that $(x(u,v), y(u,v)) \in D$ for all $(u,v) \in E$ and $(x(u_0, v_0), y(u_0, v_0)) = (x_0, y_0)$. If x, y are continuous at (u_0, v_0), then $F : E \to \mathbb{R}$ defined by $F(u,v) := f(x(u,v), y(u,v))$ is continuous at (u_0, v_0).*

Proof. (i) Suppose E and g satisfy the hypotheses in (i). Let $\big((x_n, y_n)\big)$ be a sequence in D such that $(x_n, y_n) \to (x_0, y_0)$. By the continuity of f at (x_0, y_0), we obtain $f(x_n, y_n) \to f(x_0, y_0)$. Now $(f(x_n, y_n))$ is a sequence in $f(D)$, and hence by the continuity of g at $f(x_0, y_0)$, we obtain $g\,(f(x_n, y_n)) \to g\,(f(x_0, y_0))$. So $g \circ f : D \to \mathbb{R}$ is continuous at (x_0, y_0).

(ii) Suppose E, t_0, and the functions x, y satisfy the hypotheses in (ii), and F is as defined in (ii). Let (t_n) be a sequence in E such that $t_n \to t_0$. By the continuity of x and y at t_0, we obtain $x(t_n) \to x(t_0)$ and $y(t_n) \to y(t_0)$. Thus, by part (iii) of Proposition 2.1, $\big(x(t_n), y(t_n)\big)$ is a sequence in D that converges to (x_0, y_0). Hence by the continuity of f at (x_0, y_0), we obtain $f(x(t_n), y(t_n)) \to f(x_0, y_0)$, that is, $F(t_n) \to F(t_0)$. So F is continuous at t_0.

(iii) Suppose E, (u_0, v_0), and the functions x, y satisfy the hypotheses in (iii), and F is as defined in (iii). Let (u_n, v_n) be a sequence in E such that $(u_n, v_n) \to (u_0, v_0)$. By the continuity of x and y at (u_0, v_0), we obtain $x(u_n, v_n) \to x(u_0, v_0)$ and $y(u_n, v_n) \to y(u_0, v_0)$. Thus, by part (iii) of Proposition 2.1, $\big(x(u_n, v_n), y(u_n, v_n)\big)$ is a sequence in D that converges to (x_0, y_0). Hence by the continuity of f at (x_0, y_0), we obtain $f(x(u_n, v_n), y(u_n, v_n)) \to f(x_0, y_0)$, that is, $F(u_n, v_n) \to F(u_0, v_0)$. So F is continuous at (u_0, v_0). □

Examples 2.18. (i) By part (i) of Proposition 2.17, $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := \sin(xy)$ is continuous at each $(x_0, y_0) \in \mathbb{R}^2$, and $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x, y) := \cos(x + y)$ is continuous at each $(x_0, y_0) \in \mathbb{R}^2$.

(ii) By part (ii) of Proposition 2.17, if $f(x, y)$ is any polynomial in two variables, then $F : \mathbb{R} \to \mathbb{R}$ defined by $F(t) := f(e^t, \sin t)$ for $t \in \mathbb{R}$ is continuous at every $t_0 \in \mathbb{R}$.

(iii) By part (iii) of Proposition 2.17, if $f(x, y)$ is any polynomial in two variables, then $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(u, v) := f(\sin(uv), \cos(u + v))$ for $(u, v) \in \mathbb{R}^2$ is continuous at every $(u_0, v_0) \in \mathbb{R}^2$.

(iv) Consider the functions that give the polar coordinates of a point in $\mathbb{R}^2$ other than the origin. (See Section 1.3 and, in particular, Fact 1.26.) More precisely, consider $r : \mathbb{R}^2 \to \mathbb{R}$ and $\theta : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ defined by

$$r(x, y) := \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta(x, y) := \begin{cases} \cos^{-1}\left(\dfrac{x}{r(x, y)}\right) & \text{if } y \geq 0, \\ -\cos^{-1}\left(\dfrac{x}{r(x, y)}\right) & \text{if } y < 0. \end{cases}$$

Then, as seen already in Example 2.12 (ii), the function r is continuous on $\mathbb{R}^2$. Also, we know that $\cos^{-1} : [-1, 1] \to \mathbb{R}$ is a continuous function of one variable. (See, for example, page 252 of ACICARA.) Consequently, by Proposition 2.15 and part (i) of Proposition 2.17, we see that the function θ is continuous at every $(x_0, y_0) \in \mathbb{R}^2$ for which $y_0 \neq 0$. Also, θ is continuous on the positive x-axis. To see this, note that if $(x_0, 0) \in \mathbb{R}^2$ with $x_0 > 0$ and if $\big((x_n, y_n)\big)$ is any sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ converging to $(x_0, 0)$, then

$$|\theta(x_n, y_n)| = \left|\cos^{-1}\left(\frac{x_n}{\sqrt{x_n^2 + y_n^2}}\right)\right| \to \left|\cos^{-1}\left(\frac{x_0}{|x_0|}\right)\right| = \left|\cos^{-1}(1)\right| = 0,$$

and hence $\theta(x_n, y_n) \to 0$. However, at points on the negative x-axis, the function θ is discontinuous. To see this, fix $(x_0, 0) \in \mathbb{R}^2$ with $x_0 < 0$. Clearly, we can find sequences $\big((x_n, y_n)\big)$ and $\big((u_n, v_n)\big)$ in $\mathbb{R}^2 \setminus \{(0, 0)\}$ converging to $(x_0, 0)$ such that $y_n \geq 0$ and $v_n < 0$ for all $n \in \mathbb{N}$. Now,

$$\theta(x_n, y_n) = \cos^{-1}\left(\frac{x_n}{\sqrt{x_n^2 + y_n^2}}\right) \to \cos^{-1}\left(\frac{x_0}{|x_0|}\right) = \cos^{-1}(-1) = \pi,$$

whereas

$$\theta(u_n, v_n) = -\cos^{-1}\left(\frac{u_n}{\sqrt{u_n^2 + v_n^2}}\right) \to -\cos^{-1}\left(\frac{x_0}{|x_0|}\right) = -\cos^{-1}(-1) = -\pi.$$

Thus, θ is discontinuous at every point of $\{(x, y) \in \mathbb{R}^2 : x < 0 \text{ and } y = 0\}$. In fact, given any $x_0 < 0$, we can take $x_n = u_n = x_0$ for all $n \in \mathbb{N}$ in the above argument, and this shows that the function from $(-\infty, 0]$ to $\mathbb{R}$ given by $y \mapsto \theta(x_0, y)$ is discontinuous at 0. On the other hand, the functions that give the rectangular coordinates of a point in the (polar) plane are continuous. More precisely, the functions $x, y : \mathbb{R}^2 \to \mathbb{R}$ defined by $x(r, \theta) := r\cos\theta$ and $y(r, \theta) := r\sin\theta$ are continuous on $\mathbb{R}^2$. $\diamond$

Piecing Continuous Functions on Overlapping Subsets

An effective way to construct continuous functions of one variable is to piece together two continuous function defined on overlapping subsets that intersect at a single point, provided their values agree at the common point of intersection. (See, for example, Proposition 3.5 of ACICARA.) We now obtain a similar result for functions of two variables. A precise statement is given below, and the key hypothesis in this result is illustrated in Figure 2.1.

Proposition 2.19. *Let D_1 and D_2 be subsets of $\mathbb{R}^2$ and let $f_1 : D_1 \to \mathbb{R}$ and $f_2 : D_2 \to \mathbb{R}$ be continuous functions such that $f_1(x, y) = f_2(x, y)$ for all $(x, y) \in D_1 \cap D_2$. Let $D := D_1 \cup D_2$ and let $f : D \to \mathbb{R}$ be defined by*

$$f(x, y) := \begin{cases} f_1(x, y) & \text{if } (x, y) \in D_1, \\ f_2(x, y) & \text{if } (x, y) \in D_2. \end{cases}$$

If D_i is closed in D, that is, $\overline{D_i} \cap D = D_i$ for $i = 1, 2$, then f is continuous.

Proof. Since f_1 and f_2 agree on $D_1 \cap D_2$, it is clear that f is well defined. Assume now that each D_i is closed in D for $i = 1, 2$. Fix $(x_0, y_0) \in D$. Let $\big((x_n, y_n)\big)$ be a sequence in D such that $(x_n, y_n) \to (x_0, y_0)$. In case there is $n_1 \in \mathbb{N}$ such that $(x_n, y_n) \in D_1$ for all $n \geq n_1$, then $(x_0, y_0) \in D_1$ since D_1 is closed in D; further, by the continuity of f_1 on D_1, we obtain $f(x_n, y_n) = f_1(x_n, y_n) \to f_1(x_0, y_0) = f(x_0, y_0)$. Similarly, in case there is $n_2 \in \mathbb{N}$ such that $(x_n, y_n) \in D_2$ for all $n \geq n_2$, then $(x_0, y_0) \in D_2$ and $f(x_n, y_n) \to f(x_0, y_0)$. In the remaining case, there are two subsequences $\big((x_{\ell_k}, y_{\ell_k})\big)$ and $\big((x_{m_k}, y_{m_k})\big)$ of $\big((x_n, y_n)\big)$ such that $(x_{\ell_k}, y_{\ell_k}) \in D_1$ and $(x_{m_k}, y_{m_k}) \in D_2$ for all $k \in \mathbb{N}$, and moreover, $\mathbb{N} = \{\ell_1, \ell_2, \ldots\} \cup \{m_1, m_2, \ldots\}$. Clearly, $(x_{\ell_k}, y_{\ell_k}) \to (x_0, y_0)$ and $(x_{m_k}, y_{m_k}) \to (x_0, y_0)$. Now, since each D_i is closed in D, we have $(x_0, y_0) \in D_1 \cap D_2$; further, since each f_i is continuous at (x_0, y_0), we have $f\left(x_{\ell_k}, y_{\ell_k}\right) = f_1\left(x_{\ell_k}, y_{\ell_k}\right) \to f_1(x_0, y_0) = f(x_0, y_0)$ and $f\left(x_{m_k}, y_{m_k}\right) = f_2\left(x_{m_k}, y_{m_k}\right) \to f_2(x_0, y_0) = f(x_0, y_0)$. Since $\mathbb{N} = \{\ell_1, \ell_2, \ldots\} \cup \{m_1, m_2, \ldots\}$, it follows that $f(x_n, y_n) \to f(x_0, y_0)$. This proves that f is continuous at (x_0, y_0). $\square$

Examples 2.20. (i) Consider the semiopen rectangles $D_1 := (0, 1] \times [-1, 1]$ and $D_2 := [1, 2) \times [-1, 1]$. (See Figure 2.1.) Note that neither D_1 nor D_2 is closed in $\mathbb{R}^2$, but each D_i is closed in $D := D_1 \cup D_2$ for $i = 1, 2$. Thus the hypothesis of Proposition 2.19 is satisfied, and continuous functions on D_1 and D_2 that agree on $D_1 \cap D_2 = \{1\} \times [-1, 1]$ extend to a continuous function on D.

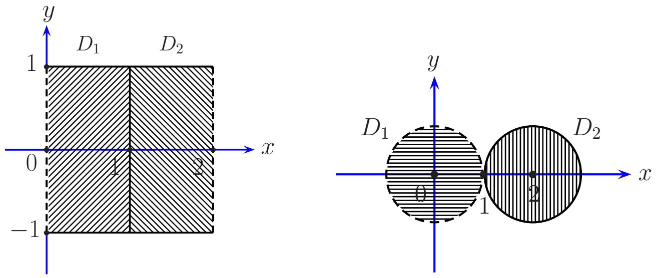

Fig. 2.1. Illustration of the conditions $\overline{D_1} \cap D = D_1$ and $\overline{D_2} \cap D = D_2$ in Proposition 2.19 that are satisfied in Example 2.20(i) and violated in Example 2.20(ii).

(ii) Let D_1 be the open disk $\mathbb{B}_1(0,0)$ and let D_2 the closure of the disk $\mathbb{B}_1(2,0)$, that is, $D_1 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $D_2 := \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 \le 1\}$. (See Figure 2.1 (ii).) Consider $f_1 : D_1 \to \mathbb{R}$ and $f_2 : D_2 \to \mathbb{R}$ defined by $f_1(x,y) := 0$ for all $(x,y) \in D_1$ and $f_2(x,y) := 1$ for all $(x,y) \in D_2$. Clearly, f_1 and f_2 are continuous. Moreover, $D_1 \cap D_2 = \emptyset$ and hence $f : D_1 \cup D_2 \to \mathbb{R}$ as given in Proposition 2.19 is well defined. But f is not continuous at $(1,0)$, since $(x_n, y_n) := \left(1 - \frac{1}{n}, 0\right) \to (1,0)$, whereas $f(x_n, y_n) = f_1(x_n, y_n) = 0$ for all $n \in \mathbb{N}$, and thus $f(x_n, y_n) \not\to 1 = f(1,0)$. This shows that the hypothesis $\overline{D_i} \cap D = D_i$ for $i = 1, 2$ in Proposition 2.19 cannot be dropped. ◇

An easy inductive argument shows that the result in Proposition 2.19 can be extended to piece together continuous functions not just on two overlapping sets, but on any finite number of sets, provided they agree on all pairwise intersections and each of the sets is closed in the union of all the sets. For our purpose, it will suffice to record the following special case of partitioning a set into four quadrants at a given point.

Corollary 2.21. *Let $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be a function. Given any $(x_0, y_0) \in D$, let $D_1 := \{(x,y) \in D : x \ge x_0$ and $y \ge y_0\}$, $D_2 := \{(x,y) \in D : x \le x_0$ and $y \ge y_0\}$, $D_3 := \{(x,y) \in D : x \le x_0$ and $y \le y_0\}$, $D_4 := \{(x,y) \in D : x \ge x_0$ and $y \le y_0\}$, and $f_i = f_{|D_i}$ for $i = 1, \ldots, 4$. Then f is continuous if and only if f_i is continuous for each $i = 1, \ldots, 4$.*

Proof. If f is continuous, then clearly f_i is continuous for each $i = 1, \ldots, 4$. To prove the converse, consider $E_1 := D_1 \cup D_2$ and $E_2 := D_3 \cup D_4$, and also $g_i := f_{|E_i}$ for $i = 1, 2$. Using Proposition 2.19, we see that the continuity of f_1 and f_2 implies the continuity of g_1, while the continuity of f_3 and f_4 implies the continuity of g_2. Further, the continuity of f follows from the continuity of g_1 and g_2 using Proposition 2.19 again. □

Characterizations of Continuity

We have chosen to define continuity of a function at a point using sequences. Alternative definitions are possible, as is shown by the result below.

Proposition 2.22. *Let $D \subseteq \mathbb{R}^2$, $(x_0, y_0) \in D$, and let $f : D \to \mathbb{R}$ be any function. Then the following are equivalent.*

(i) *f is continuous at (x_0, y_0), that is, for every sequence $\big((x_n, y_n)\big)$ in D such that $(x_n, y_n) \to (x_0, y_0)$, we have $f(x_n, y_n) \to f(x_0, y_0)$.*
(ii) *For every $\epsilon > 0$, there is $\delta > 0$ such that $|f(x, y) - f(x_0, y_0)| < \epsilon$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$.*
(iii) *For every open subset V of $\mathbb{R}$ containing $f(x_0, y_0)$, there is an open subset U of $\mathbb{R}^2$ containing (x_0, y_0) such that $f(U \cap D) \subseteq V$, that is, $f(x, y) \in V$ for all $(x, y) \in U \cap D$.*

Proof. Assume that (i) holds. If (ii) does not hold, then there is $\epsilon > 0$ such that for every $\delta > 0$, there is (x, y) in $D \cap \mathbb{S}_\delta(x_0, y_0)$ with the property that $|f(x, y) - f(x_0, y_0)| \geq \epsilon$. Consequently, for each $n \in \mathbb{N}$, there is (x_n, y_n) in $D \cap \mathbb{S}_{1/n}(x_0, y_0)$ such that $|f(x_n, y_n) - f(x_0, y_0)| \geq \epsilon$. But then $(x_n, y_n) \to (x_0, y_0)$ and $f(x_n, y_n) \not\to f(x_0, y_0)$. This contradicts (i). Thus, (i) $\Rightarrow$ (ii).

Next, assume that (ii) holds. Let V be an open subset of $\mathbb{R}$ containing $f(x_0, y_0)$. Then there is $\epsilon > 0$ such that $(f(x_0, y_0) - \epsilon, f(x_0, y_0) + \epsilon) \subseteq V$. By (ii), we can find $\delta > 0$ such that $|f(x, y) - f(x_0, y_0)| < \epsilon$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$. Thus, if we let $U = \mathbb{S}_\delta(x_0, y_0)$, then U is an open subset of $\mathbb{R}^2$ containing (x_0, y_0) such that $f(U \cap D) \subseteq V$. Thus, (ii) $\Rightarrow$ (iii).

Finally, assume that (iii) holds. Let $\big((x_n, y_n)\big)$ be any sequence in D such that $(x_n, y_n) \to (x_0, y_0)$. Given any $\epsilon > 0$, take V to be the open interval $(f(x_0, y_0) - \epsilon, f(x_0, y_0) + \epsilon)$ in $\mathbb{R}$. By (iii), there is an open subset U of $\mathbb{R}^2$ containing (x_0, y_0) such that $f(U \cap D) \subseteq V$. Since U is open, there is $\delta > 0$ such that $\mathbb{S}_\delta(x_0, y_0) \subseteq U$. Further, since $(x_n, y_n) \to (x_0, y_0)$, there is $n_0 \in \mathbb{N}$ such that $(x_n, y_n) \in \mathbb{S}_\delta(x_0, y_0)$ for all $n \geq n_0$. Consequently, $f(x_n, y_n)$ is in $(f(x_0, y_0) - \epsilon, f(x_0, y_0) + \epsilon)$, that is, $|f(x_n, y_n) - f(x_0, y_0)| < \epsilon$ for all $n \geq n_0$. Thus, $f(x_n, y_n) \to f(x_0, y_0)$, and so (iii) $\Rightarrow$ (i).

This proves the equivalence of (i), (ii), and (iii). □

Corollary 2.23. *Let $D \subseteq \mathbb{R}^2$ be open in $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be any function. Then f is continuous on D if and only if for every open subset V of $\mathbb{R}$, the set $f^{-1}(V) := \{(x, y) \in D : f(x, y) \in V\}$ is open in $\mathbb{R}^2$.*

Proof. Follows easily from Proposition 2.22. □

Example 2.24. Clearly, the intervals $(0, \infty)$, $(-\infty, 0)$ and the set $\mathbb{R} \setminus \{0\}$ are open subsets of $\mathbb{R}$. Thus, as a consequence of Corollary 2.23, we see that if $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then each of the sets $\{(x, y) \in \mathbb{R}^2 : f(x, y) > 0\}$, $\{(x, y) \in \mathbb{R}^2 : f(x, y) < 0\}$, and $\{(x, y) \in \mathbb{R}^2 : f(x, y) \neq 0\}$ is open in $\mathbb{R}^2$. ◇

Continuity and Boundedness

A bounded function need not be continuous. Consider, for example, the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) := \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is bounded but f is not continuous at any point of $\mathbb{R}^2$. Also, a continuous function need not be bounded. For example, $g : \mathbb{R}^2 \to \mathbb{R}$ and $h : (0, 1) \times (0, 1) \to \mathbb{R}$ defined by

$$g(x, y) := x + y \quad \text{and} \quad h(x, y) := 1/(x + y)$$

are both continuous, but neither g nor h is a bounded function. It may be noted that the domain of g is closed, but not bounded, whereas the domain of h is bounded, but not closed. The following result shows that the situation is nicer if the domain is closed as well as bounded.

Proposition 2.25. *Let $D \subseteq \mathbb{R}^2$ be closed and bounded, and let $f : D \to \mathbb{R}$ be continuous. Then f is bounded, that is, $f(D) := \{f(x, y) : (x, y) \in D\}$ is a bounded subset of $\mathbb{R}$. Also, $f(D)$ is a closed subset of $\mathbb{R}$. As a consequence, if D is nonempty, then f attains its bounds, that is, there are $(a, b), (c, d) \in D$ such that $f(a, b) = \sup f(D)$ and $f(c, d) = \inf f(D)$.*

Proof. Suppose f is not bounded above. Then for each $n \in \mathbb{N}$, there is $(x_n, y_n) \in D$ such that $f(x_n, y_n) > n$. Since D is bounded, by the Bolzano–Weierstrass Theorem (part (ii) of Proposition 2.6), the sequence $\big((x_n, y_n)\big)$ has a convergent subsequence, say $\big((x_{n_k}, y_{n_k})\big)$. Suppose $(x_{n_k}, y_{n_k}) \to (x_0, y_0)$. Then $(x_0, y_0) \in D$, since D is closed, and $f(x_{n_k}, y_{n_k}) \to f(x_0, y_0)$, since f is continuous. On the other hand, $f(x_{n_k}, y_{n_k}) > n_k$ for each $k \in \mathbb{N}$, and $n_k \to \infty$ as $k \to \infty$, which leads to a contradiction. Hence f must be bounded above. Similarly, it can be seen that f is bounded below. Thus $f(D)$ is bounded. Next, suppose (z_n) is a sequence in $f(D)$ such that $z_n \to r$ for some $r \in \mathbb{R}$. Write $z_n = f(x_n, y_n)$, where $(x_n, y_n) \in D$ for $n \in \mathbb{N}$. As before, $\big((x_n, y_n)\big)$ has a convergent subsequence, say $\big((x_{n_k}, y_{n_k})\big)$, which must converge to a point (x_0, y_0) of D. Since f is continuous at (x_0, y_0), $z_{n_k} = f(x_{n_k}, y_{n_k}) \to f(x_0, y_0)$, and hence $r = f(x_0, y_0)$, which shows that $r \in f(D)$. Thus $f(D)$ is closed. Finally, if D is nonempty, then $f(D)$ is a nonempty bounded subset of $\mathbb{R}$ and thus $M := \sup f(D)$ and $m := \inf f(D)$ are well defined. By the definition of supremum and infimum, for each $n \in \mathbb{N}$, we can find $(a_n, b_n), (c_n, d_n) \in D$ such that $M - \frac{1}{n} < f(a_n, b_n) \leq M$ and $m \leq f(c_n, d_n) < m + \frac{1}{n}$. Consequently, $f(a_n, b_n) \to M$ and $f(c_n, d_n) \to m$. Since $f(D)$ is closed, $M, m \in f(D)$, that is, $f(a, b) = \sup f(D)$ and $f(c, d) = \inf f(D)$ for some $(a, b), (c, d) \in D$. □

Remark 2.26. Subsets of $\mathbb{R}^2$ (and more generally, of $\mathbb{R}^n$) that are both closed and bounded are often referred to as compact sets. Thus, the above proposition says that the continuous image of a compact set is compact. For more on compactness, see Exercise 17. ◇

Continuity and Monotonicity

For functions of one variable, there is no direct relationship between continuity and monotonicity. Indeed, it suffices to consider the integer part function $x \longmapsto [x]$ and the absolute value function $x \longmapsto |x|$ to conclude that a monotonic function need not be continuous and a continuous function need not be monotonic. For functions of two variables, a similar situation prevails. In fact, using the product order on $\mathbb{R}^2$, we have introduced in Chapter 1 two distinct notions: monotonicity and bimonotonicity. We will show below that neither of these implies or is implied by continuity.

Examples 2.27. (i) Consider $f : [-1,1] \times [-1,1] \to \mathbb{R}$ defined by $f(x,y) := xy$. Clearly, f is continuous but not monotonic on $[-1,1] \times [-1,1]$. Note, however, that f is bimonotonically increasing on $[-1,1]\times[-1,1]$, since we have $x_2y_2 + x_1y_1 - x_2y_1 - x_1y_2 = (x_2 - x_1)(y_2 - y_1)$ for all $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$.

(ii) Consider $f : [-1,1] \times [-1,1] \to \mathbb{R}$ defined by $f(x,y) := (x+y)^3$. Clearly, f is continuous. However, f is not bimonotonic on $[-1,1] \times [-1,1]$. To see this, observe that $(x_1,y_1) := (0,0)$ and $(x_2,y_2) := (1,1)$ are points of $[-1,1] \times [-1,1]$ satisfying $(x_1,y_1) \le (x_2,y_2)$ and

$$f(x_1,y_1) + f(x_2,y_2) - f(x_1,y_2) - f(x_2,y_1) = 0 + 8 - 1 - 1 = 6 > 0,$$

whereas $(u_1,v_1) := (-1,-1)$ and $(u_2,v_2) := (0,0)$ are points of $[-1,1]\times[-1,1]$ satisfying $(u_1,v_1) \le (u_2,v_2)$ and

$$f(u_1,v_1) + f(u_2,v_2) - f(u_1,v_2) - f(u_2,v_1) = -8 + 0 + 1 + 1 = -6 < 0.$$

(iii) Consider $f : [-1,1] \times [-1,1] \to \mathbb{R}$ defined by

$$f(x,y) := \begin{cases} 1 & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that f is monotonically as well as bimonotonically increasing, but not continuous on $[-1,1] \times [-1,1]$. ◇

Continuity, Bounded Variation, and Bounded Bivariation

In general, a function of bounded variation need not be continuous. Likewise for a function of bounded bivariation. In fact, Example 2.27 (iii) provides a common counterexample. We have seen earlier that a continuous function need not be monotonic or bimonotonic. The following example shows that it need not even be of bounded variation or of bounded bivariation.

Example 2.28. Consider $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) := \begin{cases} xy\cos(\pi/2x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Clearly, f is continuous on $(0,1] \times [0,1]$. Moreover, since $|f(x,y)| \le |xy|$ for all $(x,y) \in [0,1] \times [0,1]$, it is readily seen that f is continuous at $(0,y)$ for each $y \in [0,1]$. Thus, f is continuous on $[0,1] \times [0,1]$. Next, given any even positive integer n, say $n = 2k$ for some $k \in \mathbb{N}$, if we consider the points

$$x_0 = 0 = y_0 \quad \text{and} \quad x_i := \frac{1}{n+1-i} \quad \text{and} \quad y_i = 1 \text{ for } i = 1, \dots, n,$$

then we have $(0,0) = (x_0, y_0) \le (x_1, y_1) \le \cdots \le (x_n, y_n) = (1,1)$ and moreover, $f(x_i, y_i) = 0$ if i is even and $f(x_i, y_i) = \pm x_i$ if i is odd. Thus

$$\sum_{i=1}^{n} |f(x_i, y_i) - f(x_{i-1}, y_{i-1})| = \frac{1}{n} + \frac{1}{n} + \frac{1}{n-2} + \frac{1}{n-2} + \cdots + \frac{1}{2} + \frac{1}{2} = \sum_{i=1}^{k} \frac{1}{i}.$$

Since the set $\{\sum_{i=1}^{k} (1/i) : k \in \mathbb{N}\}$ is not bounded above (as is shown, for example, on page 51 of ACICARA), it follows that f is not of bounded variation on $[0,1] \times [0,1]$.

Furthermore, if we let $n = 2k$ and $x_0, x_1, \dots, x_n$ be as above, but take $m = 1$, $y_0 = 0$, and $y_1 = 1$, then $0 = x_0 \le x_1 \le \cdots \le x_n = 1$ and $0 = y_0 \le y_1 = 1$, and moreover, for any $i \ge 0$, we have $f(x_i, 0) = 0$, whereas $f(x_i, 1) = 0$ if i is even and $f(x_i, 1) = \pm x_i$ if i is odd, and thus

$$\sum_{i=1}^{n} \sum_{j=1}^{m} |f(x_i, y_j) + f(x_{i-1}, y_{j-1}) - f(x_i, y_{j-1}) - f(x_{i-1}, y_j)| = \sum_{i=1}^{k} \frac{1}{i}.$$

It follows, therefore, that f is not of bounded bivariation on $[0,1] \times [0,1]$. $\diamond$

Remark 2.29. Using Exercise 38, a refined version of the Jordan decomposition (Propositions 1.12 and 1.17) can be obtained for continuous functions. Namely, a continuous function of bounded variation is a difference of continuous monotonic functions, whereas a continuous function of bounded bivariation is a difference of continuous bimonotonic functions. $\diamond$

Continuity and Convexity

In general, a continuous function is neither convex nor concave. For example, consider $D := [-1,1] \times [-1,1]$ and $f : D \to \mathbb{R}$ defined by $f(x,y) := x^3 + y^3$. Clearly, f is continuous. But f is neither convex nor concave. To see this, observe that $\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}(-1,-1) + \frac{1}{2}(0,0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(1,1) + \frac{1}{2}(0,0)$, but $f\left(-\frac{1}{2}, -\frac{1}{2}\right) = -\frac{1}{4} > -1 = \frac{1}{2} f(-1,-1) + \frac{1}{2} f(0,0)$, and $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4} < 1 = \frac{1}{2} f(1,1) + \frac{1}{2} f(0,0)$. Moreover, a convex function need not be continuous. For example, if $D := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ is the closed unit disk and $f : D \to \mathbb{R}$ is a variant of the norm function defined by

$$f(x,y) := \begin{cases} \sqrt{x^2 + y^2} & \text{if } x^2 + y^2 < 1, \\ 2 & \text{if } x^2 + y^2 = 1, \end{cases}$$

then f is convex on D, but not continuous on D. Here, the continuity of f fails precisely at the boundary points of D. In fact, we will show that a convex function is always continuous at the interior points of its domain. First, we prove a couple of auxiliary results, which may also be of independent interest.

Lemma 2.30. *Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Then every real-valued convex function on the closed rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$ is bounded.*

Proof. Let $D := [a, b] \times [c, d]$ and let $f : D \to \mathbb{R}$ be any convex function. Define $M := \max\{f(a, c), f(a, d), f(b, c), f(b, d)\}$. Let $(x, y) \in D$. Then there is s in $[0, 1]$ such that $x = (1-s)a + sb$. Using the convexity of f on D, we see that $f(x, y) \le (1 - s)f(a, y) + sf(b, y)$. Further, there is $t \in [0, 1]$ such that $y = (1 - t)c + td$. Again, using the convexity of f on D, we obtain

$$\begin{aligned} f(x, y) &\le (1 - s)\,[(1 - t)f(a, c) + tf(a, d)] + s\,[(1 - t)f(b, c) + tf(b, d)] \\ &\le (1 - s)\,[(1 - t)M + tM] + s\,[(1 - t)M + tM] \;=\; M. \end{aligned}$$

It follows that M is an upper bound for f. Next, consider the center point $(p, q) := \left(\frac{a+b}{2}, \frac{c+d}{2}\right)$ of D and let $(u, v) := (a + b - x,\, c + d - y)$. Clearly, $(u, v) \in D$ and $(p, q) = \frac{1}{2}(x, y) + \frac{1}{2}(u, v)$. Hence using the convexity of f, we obtain $f(p, q) \le \frac{1}{2}f(x, y) + \frac{1}{2}f(u, v) \le \frac{1}{2}f(x, y) + M$, that is, $f(x, y) \ge m$, where $m := 2\,(f(p, q) - M)$. It follows that m is a lower bound for f. □

Lemma 2.31. *Let D be convex and open in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be convex. Also, let $[a, b] \times [c, d]$ be a closed rectangle contained in D, where $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Then there is $K \in \mathbb{R}$ such that*

$$|f(x, y) - f(u, v)| \le K\,(|x - u| + |y - v|) \quad \textit{for all } (x, y), (u, v) \in [a, b] \times [c, d].$$

Proof. Since D is open, there is $\delta > 0$ such that $[a-\delta, b+\delta] \times [c-\delta, d+\delta] \subseteq D$. By Lemma 2.30, there are $m, M \in \mathbb{R}$ such that $m \le f(z, w) \le M$ for all $(z, w) \in [a - \delta, b + \delta] \times [c - \delta, d + \delta]$. Now, fix any $(x, y), (u, v) \in [a, b] \times [c, d]$. The case $(x, y) = (u, v)$ is trivial, and so we will assume that $(x, y) \ne (u, v)$. Then $\ell := |x - u| + |y - v| > 0$, and we can consider $z := u + \frac{\delta}{\ell}(u - x)$ and $w := v + \frac{\delta}{\ell}(v - y)$. Since $|u - x| \le \ell$, that is, $-\ell \le u - x \le \ell$, we have $u - \delta \le z \le u + \delta$, and hence $z \in [a - \delta, b + \delta]$. Similarly, $w \in [c - \delta, d + \delta]$. In particular, $(z, w) \in D$. Moreover, it can be easily verified that

$$u = \frac{\delta}{\ell + \delta}\, x + \frac{\ell}{\ell + \delta}\, z \quad \text{and} \quad v = \frac{\delta}{\ell + \delta}\, y + \frac{\ell}{\ell + \delta}\, w.$$

Thus $(u, v) = (1 - t)(x, y) + t(z, w)$, where $t := \ell/(\ell + \delta)$. Since $0 < t < 1$, using the convexity of f on D, we obtain $f(u, v) \le (1 - t)f(x, y) + tf(z, w)$. Further, since $0 < t < \ell/\delta$, we see that

$$f(u, v) - f(x, y) \le t\,[f(z, w) - f(x, y)] \le \frac{\ell}{\delta}\,[M - m] = K\,(|x - u| + |y - v|)\,,$$

where $K := (M - m)/\delta$. Similarly, $f(x, y) - f(u, v) \le K\,(|x - u| + |y - v|)$. This proves the desired inequality for $|f(u, v) - f(x, y)|$. □

We are now ready to show that a convex function is continuous at all the interior points of its domain. This is an immediate consequence of the above lemma. (See also Exercise 10.)

Proposition 2.32. *Let D be a convex subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a convex function. Then f is continuous at every interior point of D. In particular, if D is also open in $\mathbb{R}^2$, then f is continuous on D.*

Proof. Let (x_0, y_0) be an interior point of D. Then there is $r > 0$ such that $R := [x_0 - r, x_0 + r] \times [y_0 - r, y_0 + r]$ is contained in D. By Lemma 2.31, there is $K \in \mathbb{R}$ such that $|f(x, y) - f(x_0, y_0)| \leq K (|x - x_0| + |y - y_0|)$ for $(x, y) \in R$. This implies that if $\big((x_n, y_n)\big)$ is a sequence in D such that $(x_n, y_n) \to (x_0, y_0)$, then $f(x_n, y_n) \to f(x_0, y_0)$. Thus, f is continuous at (x_0, y_0). □

Continuity and Intermediate Value Property

A result of fundamental importance in one-variable calculus is that continuous functions possess the intermediate value property (IVP). For ease of reference, we state this result below; see, for example, Proposition 3.13 of ACICARA.

Fact 2.33. (Intermediate Value Theorem) *Let D be a subset of $\mathbb{R}$ and let $\phi : D \to \mathbb{R}$ be a continuous function. Then ϕ has the IVP on every interval $I \subseteq D$, that is, if $a, b \in I$ with $a < b$ and $r \in \mathbb{R}$ is between $\phi(a)$ and $\phi(b)$, then there is $c \in [a, b]$ such that $\phi(c) = r$; in particular, $\phi(I)$ is an interval in $\mathbb{R}$.*

The following result may be viewed as an analogue of Fact 2.33 for real-valued continuous functions of two variables.

Proposition 2.34 (Bivariate Intermediate Value Theorem). *Let D be a subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a continuous function. Then $f(E)$ is an interval in $\mathbb{R}$ for every path-connected subset E of D. In particular, f has the IVP on every 2-interval in D.*

Proof. Suppose $E \subseteq D$ is path-connected. Let $z_1, z_2 \in f(E)$ and let r be any real number between z_1 and z_2. Then $z_1 = f(x_1, y_1)$ and $z_2 = f(x_2, y_2)$ for some $(x_1, y_1), (x_2, y_2) \in E$. Since E is path-connected, there is a path Γ joining (x_1, y_1) to (x_2, y_2) that lies in E. Let $x, y : [\alpha, \beta] \to \mathbb{R}$ be continuous functions such that Γ is given by $(x(t), y(t))$, $t \in [\alpha, \beta]$. Consider $F : [\alpha, \beta] \to \mathbb{R}$ defined by $F(t) := f(x(t), y(t))$. By part (ii) of Proposition 2.17, F is continuous, and by Fact 2.33, F has the IVP on $[\alpha, \beta]$. It follows that $r = F(t_0)$ for some $t_0 \in [\alpha, \beta]$, and hence $r \in f(E)$. This proves that $f(E)$ is an interval in $\mathbb{R}$. Finally, every 2-interval is path-connected (Example 1.5 (iv)), and so in view of Proposition 1.25, we see that f has the IVP on every 2-interval in D. □

The following example shows that the converse of the above result is not true, that is, the IVP does not imply continuity.

Example 2.35. Consider $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) := \begin{cases} \cos(1/y) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Then f is not continuous on $[0,1] \times [0,1]$, because, for example, $(0,\ 1/n\pi) \to (0,0)$, but $f(0,\ 1/n\pi) = (-1)^n \not\to f(0,0) = 0$. Note, however, that f is continuous on $[0,1] \times (0,1]$. We show that f has the IVP on $[0,1] \times [0,1]$. Let $r \in \mathbb{R}$ be an intermediate value of f, that is, r is between $f(x_1,y_1)$ and $f(x_2,y_2)$ for some $(x_1,y_1),(x_2,y_2) \in [0,1] \times [0,1]$. If $y_1 > 0$ and $y_2 > 0$, then by the continuity of f on $[0,1] \times (0,1]$ and Proposition 2.34, we see that $r = f(x,y)$ for some $(x,y) \in I_{(x_1,y_1),(x_2,y_2)}$. If $y_1 = y_2 = 0$, then $f(x_1,y_1) = f(x_2,y_2) = 0$ and there is nothing to prove. Thus we may assume, without loss of generality, that $y_1 = 0$ and $y_2 > 0$. Choose $k \in \mathbb{N}$ such that $(1/k\pi) < y_2$. Now $y_1 < (1/(k+2)\pi) < (1/k\pi) < y_2$, and therefore $\cos(1/y)$ assumes every value between -1 and 1 as y varies from y_1 to y_2. It follows that $r = f(x_1,y)$ for some $y \in [y_1,y_2]$. Thus f has the IVP on $[0,1] \times [0,1]$. ◇

Corollary 2.36. *Let D be a nonempty, path-connected, closed, and bounded subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a continuous function. Then the range $f(D)$ of f is a closed and bounded interval in $\mathbb{R}$.*

Proof. First, note that since D is nonempty, so is $f(D)$. By Proposition 2.25, $f(D)$ is bounded, and moreover, if $m := \inf f(D)$ and $M := \sup f(D)$, then $f(D) \subseteq [m,M]$ and $m, M \in f(D)$. Further, by Proposition 2.34, $f(D)$ is an interval in $\mathbb{R}$. It follows that $f(D) = [m,M]$. □

Uniform Continuity

The notion of uniform continuity for functions of one variable can be easily extended to functions of two variables. Let D be a subset of $\mathbb{R}^2$. A function $f : D \to \mathbb{R}$ is said to be **uniformly continuous** on D if for any sequences $\big((x_n,y_n)\big)$ and $\big((u_n,v_n)\big)$ in D such that $|(x_n,y_n) - (u_n,v_n)| \to 0$, we have $|f(x_n,y_n) - f(u_n,v_n)| \to 0$.

Specializing one of the two sequences to a constant sequence, we readily see that a uniformly continuous function is continuous. As in the case of functions of one variable, the converse is true if the domain is closed and bounded.

Proposition 2.37. *Let $D \subseteq \mathbb{R}^2$ be a closed and bounded set. Then every continuous function on D is uniformly continuous on D.*

Proof. Suppose $f : D \to \mathbb{R}$ is continuous but not uniformly continuous on D. Then there are sequences $\big((x_n,y_n)\big)$ and $\big((u_n,v_n)\big)$ in D such that $|(x_n,y_n) - (u_n,v_n)| \to 0$, but $|f(x_n,y_n) - f(u_n,v_n)| \not\to 0$. The latter implies that there are $\epsilon > 0$ and positive integers $n_1 < n_2 < \cdots$ such that $\big|f(x_{n_k},y_{n_k}) - f(u_{n_k},v_{n_k})\big| \geq \epsilon$ for all $k \in \mathbb{N}$. Now, by the Bolzano–Weierstrass

Theorem (part (ii) of Proposition 2.6), $\big((x_{n_k}, y_{n_k})\big)$ has a convergent subsequence, say $\big((x_{n_{k_j}}, y_{n_{k_j}})\big)$. If $(x_{n_{k_j}}, y_{n_{k_j}}) \to (x_0, y_0)$, then $(u_{n_{k_j}}, v_{n_{k_j}}) \to (x_0, y_0)$, because $|(x_n, y_n) - (u_n, v_n)| \to 0$. Since f is continuous on D, we see that $\left|f(x_{n_{k_j}}, y_{n_{k_j}}) - f(u_{n_{k_j}}, v_{n_{k_j}})\right| \to |f(x_0, y_0) - f(x_0, y_0)| = 0$. But this is a contradiction, since $\left|f(x_{n_{k_j}}, y_{n_{k_j}}) - f(u_{n_{k_j}}, v_{n_{k_j}})\right| \geq \epsilon$ for all $j \in \mathbb{N}$. □

Examples 2.38. (i) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := x + y$. Then it is clear that f is uniformly continuous on $\mathbb{R}^2$.

(ii) If $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ is uniformly continuous, then for every fixed $(x_0, y_0) \in D$, the functions $\phi : D_1 \to \mathbb{R}$ and $\psi : D_2 \to \mathbb{R}$, defined as in Example 2.12(iv), are uniformly continuous. This follows from the definition of uniform continuity by specializing one of the coordinates in the two sequences to a constant sequence.

(iii) Consider $D \subseteq \mathbb{R}^2$ and $f : D \to \mathbb{R}$ given by

$$D := \{(x, y) \in \mathbb{R}^2 : x, y \in [0, 1] \text{ and } (x, y) \neq (0, 0)\} \quad \text{and} \quad f(x, y) := \frac{1}{x + y}.$$

Then f is continuous on D but not uniformly continuous on D. To see the latter, consider the sequences $\big((x_n, y_n)\big)$ and $\big((u_n, v_n)\big)$ in D given by $(x_n, y_n) := (1/n,\, 0)$ and $(u_n, v_n) := (1/(n + 1),\, 0)$ for $n \in \mathbb{N}$. We have $|(x_n, y_n) - (u_n, v_n)| = 1/n(n+1) \to 0$, but $|f(x_n, y_n) - f(u_n, v_n)| = |n - (n + 1)| = 1 \not\to 0$. Alternatively, we could use (ii) above and the fact that $\phi : (0, 1] \to \mathbb{R}$ defined by $\phi(x) = f(x, 0) = 1/x$ is not uniformly continuous on $(0, 1]$. (See Example 3.18 (ii) on page 80 of ACICARA.) It may be noted here that the domain of f is bounded but not closed.

(iv) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := x^2 + y^2$. Then f is continuous on $\mathbb{R}^2$, but not uniformly continuous on $\mathbb{R}^2$. To see the latter, consider the sequences $\big((x_n, y_n)\big)$ and $\big((u_n, v_n)\big)$ in D given by $(x_n, y_n) := (n,\, 0)$ and $(u_n, v_n) := (n-(1/n),\, 0)$ for $n \in \mathbb{N}$. We have $|(x_n, y_n) - (u_n, v_n)| = 1/n \to 0$, but $|f(x_n, y_n) - f(u_n, v_n)| = \left|n^2 - [n^2 - 2 + (1/n^2)]\right| = 2-(1/n^2) \not\to 0$. Alternatively, we could use (ii) above and the fact that $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = f(x, 0) = x^2$ is not uniformly continuous on $\mathbb{R}$. (See Example 3.18 (iii) on page 80 of ACICARA.) It may be noted here that the domain of f is closed but not bounded. On the other hand, the restriction of f to any bounded subset of $\mathbb{R}^2$ is uniformly continuous. ◇

A criterion for the uniform continuity of a function of two variables that does not involve convergence of sequences can be given as follows. The result below may be compared with Proposition 2.22.

Proposition 2.39. *Let $D \subseteq \mathbb{R}^2$. Consider a function $f : D \to \mathbb{R}$. Then f is uniformly continuous on D if and only if it satisfies the following ϵ-δ condition: For every $\epsilon > 0$, there is $\delta > 0$ such that*

$$(x, y), (u, v) \in D \text{ and } |(x, y) - (u, v)| < \delta \Longrightarrow |f(x, y) - f(u, v)| < \epsilon.$$

Proof. Assume that f is uniformly continuous on D. Suppose the ϵ-δ condition does not hold. Then there is $\epsilon > 0$ such that for any $\delta > 0$, we can find $(x, y), (u, v) \in D$ for which $|(x, y) - (u, v)| < \delta$, but $|f(x, y) - f(u, v)| \geq \epsilon$. Considering $\delta := 1/n$ for $n \in \mathbb{N}$, we obtain sequences $\big((x_n, y_n)\big)$ and $\big((u_n, v_n)\big)$ in D such that $|(x_n, y_n) - (u_n, v_n)| < \frac{1}{n}$ and $|f(x_n, y_n) - f(u_n, v_n)| \geq \epsilon$ for all $n \in \mathbb{N}$. Consequently, $|(x_n, y_n) - (u_n, v_n)| \to 0$, but $|f(x_n, y_n) - f(u_n, v_n)| \not\to 0$. This contradicts the assumption that f is uniformly continuous on D.

Conversely, assume that the ϵ-δ condition is satisfied. Suppose $\big((x_n, y_n)\big)$ and $\big((u_n, v_n)\big)$ are any sequences in D such that $|(x_n, y_n) - (u_n, v_n)| \to 0$. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that if $(x, y), (u, v) \in D$ satisfy $|(x, y) - (u, v)| < \delta$, then $|f(x, y) - f(u, v)| < \epsilon$. Now, for this $\delta > 0$, we can find $n_0 \in \mathbb{N}$ such that $|(x_n, y_n) - (u_n, v_n)| < \delta$ for all $n \geq n_0$. Consequently, $|f(x_n, y_n) - f(u_n, v_n)| < \epsilon$ for all $n \geq n_0$. Thus $|f(x_n, y_n) - f(u_n, v_n)| \to 0$. This proves the uniform continuity of f on D. □

Implicit Function Theorem

In the study of functions of one variable, one considers the so-called *implicitly defined curves*, that is, curves given by equations of the form $f(x, y) = 0$, $(x, y) \in D$, where $f : D \to \mathbb{R}$ is a real-valued function of two variables. Heuristically, such an equation defines one of the variables as a function of the other; for example, it may define y as a function of x. In other words, from the equation $f(x, y) = 0$, we may be able to solve for y in terms of x. In fact, this is tacitly assumed when one does implicit differentiation in calculus of functions of one variable. The following result asserts that it is possible to solve the equation $f(x, y) = 0$ locally, around a point (x_0, y_0) satisfying $f(x_0, y_0) = 0$, provided f is continuous in each variable and is either a strictly increasing or a strictly decreasing function of y, for each fixed x. Moreover, the solution $y = \eta(x)$ is unique and it is a continuous function of x.

Proposition 2.40 (Implicit Function Theorem). *Let $D \subseteq \mathbb{R}^2$ and (x_0, y_0) be an interior point of D, and let $f : D \to \mathbb{R}$ satisfy $f(x_0, y_0) = 0$. Assume that there is $r > 0$ with $\mathbb{S}_r(x_0, y_0) \subseteq D$ such that the following conditions hold.*

(a) *Given any $x \in (x_0 - r, x_0 + r)$, the function $\psi : (y_0 - r, y_0 + r) \to \mathbb{R}$ defined by $\psi(y) := f(x, y)$ is continuous. Also, given any $y \in (y_0 - r, y_0 + r)$, the function $\phi : (x_0 - r, x_0 + r) \to \mathbb{R}$ defined by $\phi(x) := f(x, y)$ is continuous.*
(b) *Given any $x \in (x_0 - r, x_0 + r)$, the function $\psi : (y_0 - r, y_0 + r) \to \mathbb{R}$ defined by $\psi(y) := f(x, y)$ is strictly monotonic.*

Then there are $\delta > 0$ and a unique continuous function $\eta : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ with $\eta(x_0) = y_0$ such that $(x, \eta(x)) \in \mathbb{S}_r(x_0, y_0)$ and $f(x, \eta(x)) = 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

Proof. In view of (b), let us first suppose that $\psi_0 : (y_0 - r, y_0 + r) \to \mathbb{R}$ defined by $\psi_0(y) := f(x_0, y)$ is strictly increasing on $(y_0 - r, y_0 + r)$.

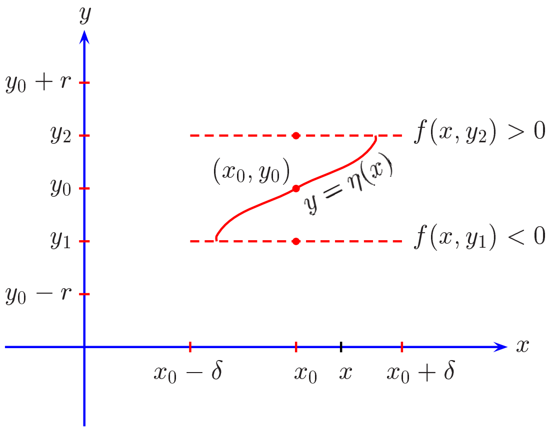

Fig. 2.2. Illustration of the proof of the Implicit Function Theorem.

Choose any $y_1 \in (y_0 - r, y_0)$ and $y_2 \in (y_0, y_0 + r)$. Since $f(x_0, y_0) = 0$ and the function ψ_0 is strictly increasing on $(y_0 - r, y_0 + r)$, we see that $f(x_0, y_1) < 0$ and $f(x_0, y_2) > 0$. By continuity, the sign of f is preserved on small horizontal segments of the lines $y = y_1$ and $y = y_2$. (See Figure 2.2.) More precisely, using (a), we see that the function defined by $x \longmapsto f(x, y_1)$ is continuous on $(x_0 - r, x_0 + r)$, and hence it follows from Fact 2.13 that there is $\delta_1 > 0$ with $\delta_1 \leq r$ such that $f(x, y_1) < 0$ for all $x \in (x_0 - \delta_1, x_0 + \delta_1)$. Similarly, there is $\delta_2 > 0$ with $\delta_2 \leq r$ such that $f(x, y_2) > 0$ for all $x \in (x_0 - \delta_2, x_0 + \delta_2)$. Let $\delta := \min\{\delta_1, \delta_2\}$. Then

$$f(x, y_1) < 0 < f(x, y_2) \quad \text{for all } x \in (x_0 - \delta, x_0 + \delta).$$

Thus, given any $x \in (x_0 - \delta, x_0 + \delta)$, the function $\psi : (y_0 - r, y_0 + r) \to \mathbb{R}$ defined by $\psi(y) := f(x, y)$ satisfies $\psi(y_1) < 0 < \psi(y_2)$. Also by (a), ψ is continuous. Hence by the IVP of ψ, there is $y \in (y_1, y_2)$ such that $\psi(y) = 0$, that is, $f(x, y) = 0$. Moreover, since $\psi(y_1) < \psi(y_2)$, it follows from (b) that ψ is strictly increasing on $(y_0 - r, y_0 + r)$, and hence y is uniquely determined by x. Thus if we write $y = \eta(x)$, then we obtain a unique function $\eta : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ such that $\eta(x) \in (y_1, y_2)$ and $f(x, \eta(x)) = 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. In particular, since $f(x_0, y_0) = 0$ and $y_0 \in (y_1, y_2)$, we have $\eta(x_0) = y_0$.

To prove the continuity of η, fix any $x^\star \in (x_0 - \delta, x_0 + \delta)$ and let (x_n) be a sequence in $(x_0 - \delta, x_0 + \delta)$ such that $x_n \to x^\star$. We have seen above that for any $x \in (x_0 - \delta, x_0 + \delta)$, the function $\psi : (y_0 - r, y_0 + r) \to \mathbb{R}$ defined by $\psi(y) = f(x, y)$ is strictly increasing. Fix $y_1, y_2 \in (y_0 - r, y_0 + r)$ as above, so that $y_1 < \eta(x) < y_2$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Let $\epsilon > 0$ be given and let us suppose ϵ is so small that $y_1 < \eta(x^\star) - \epsilon < \eta(x^\star) + \epsilon < y_2$, that is, $0 < \epsilon < \min\{\eta(x^\star) - y_1,\ y_2 - \eta(x^\star)\}$. Using (a) and (b), we see that

$$f(x_n, \eta(x^\star) - \epsilon) \to f(x^\star, \eta(x^\star) - \epsilon) \quad \text{and} \quad f(x^\star, \eta(x^\star) - \epsilon) < f(x^\star, \eta(x^\star)) = 0.$$

Hence there is $n_1 \in \mathbb{N}$ such that $f(x_n, \eta(x^\star) - \epsilon) < 0$ for all $n \geq n_1$. Similarly, $f(x_n, \eta(x^\star) + \epsilon) \to f(x^\star, \eta(x^\star) + \epsilon) > f(x^\star, \eta(x^\star)) = 0$, and hence there is $n_2 \in \mathbb{N}$ such that $f(x_n, \eta(x^\star) + \epsilon) > 0$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then $f(x_n, \eta(x^\star) - \epsilon) < 0 < f(x_n, \eta(x^\star) + \epsilon)$ for all $n \geq n_0$. But since $f(x_n, \eta(x_n)) = 0$, it follows from (*b*) that $\eta(x^\star) - \epsilon < \eta(x_n) < \eta(x^\star) + \epsilon$, that is, $|\eta(x_n) - \eta(x^\star)| < \epsilon$ for all $n \geq n_0$. Thus, $\eta(x_n) \to \eta(x^\star)$. This proves that η is continuous on $(x_0 - \delta, x_0 + \delta)$.

The case in which $\psi_0 : (y_0 - r, y_0 + r) \to \mathbb{R}$ defined by $\psi_0(y) := f(x_0, y)$ is strictly decreasing on $(y_0 - r, y_0 + r)$ is proved similarly. Alternatively, it follows from applying the result proved above to $-f$. □

Example 2.41. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2 - 1$. Then $C := \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$ is the unit circle in $\mathbb{R}^2$. If $(x_0, y_0) \in C$ and $y_0 \neq 0$, then we can easily see that the hypotheses of the Implicit Function Theorem are satisfied, and the "solution" is given by $\eta(x) := \sqrt{1 - x^2}$ or by $\eta(x) := -\sqrt{1 - x^2}$ according as $y_0 > 0$ or $y_0 < 0$. ◇

Remark 2.42. We have a straightforward analogue of the Implicit Function Theorem for solving $f(x, y) = 0$ for x in terms of y. In this situation, condition (a) in Proposition 2.40 remains the same, while (b) is replaced by the condition that for any $y \in (y_0 - r, y_0 + r)$, the function $\phi : (x_0 - r, x_0 + r) \to \mathbb{R}$ defined by $\phi(x) := f(x, y)$ is strictly monotonic. The conclusion would be that there are $\delta > 0$ and a unique continuous function $\xi : (y_0 - \delta, y_0 + \delta) \to \mathbb{R}$ with $\xi(y_0) = x_0$ such that $(\xi(y), y) \in \mathbb{S}_r(x_0, y_0)$ and $f(\xi(y), y) = 0$ for all $y \in (y_0 - \delta, y_0 + \delta)$. This can be proved in a manner similar to the proof of Proposition 2.40. Alternatively, it follows from applying Proposition 2.40 to the function $(x, y) \mapsto f(y, x)$ and the point (y_0, x_0). ◇

An important consequence of the Implicit Function Theorem is that a continuous real-valued function of one variable that is strictly monotonic in an interval about a point admits a continuous (and strictly monotonic) inverse, locally. A more precise statement appears below. This result may be viewed as a special case of the so-called **Inverse Function Theorem**.

Proposition 2.43. *Let I be an interval in $\mathbb{R}$ and $x_0 \in I$. Suppose $f : I \to \mathbb{R}$ is continuous and strictly monotonic on $I_1 := (x_0 - r, x_0 + r) \cap I$ for some $r > 0$. Let $y_0 := f(x_0)$, $J := f(I)$, and $J_1 := f(I_1)$. Then there are $\delta > 0$ and a unique continuous function $\xi : (y_0 - \delta, y_0 + \delta) \cap J \to \mathbb{R}$ such that $\xi(y_0) = x_0$ and $f(\xi(y)) = y$ for all $y \in (y_0 - \delta, y_0 + \delta) \cap J$. In particular, $f^{-1} : J_1 \to \mathbb{R}$ is continuous at y_0.*

Proof. First, let us consider the case in which x_0 is an interior point of I. Then we may choose $r > 0$ such that $(x_0 - r, x_0 + r) \subseteq I$, and therefore $I_1 = (x_0 - r, x_0 + r)$. Consider $h : \mathbb{S}_r(x_0, y_0) \to \mathbb{R}$ defined by $h(x, y) := f(x) - y$. Then h is continuous, $h(x_0, y_0) = 0$, and given any $y \in (y_0 - r, y_0 + r)$, the function from I_1 to $\mathbb{R}$ given by $x \longmapsto h(x, y)$ is strictly monotonic. Hence by the Implicit Function Theorem (Proposition 2.40 and Remark 2.42), there are $\delta > 0$ and a unique continuous function $\xi : (y_0 - \delta, y_0 + \delta) \to \mathbb{R}$ with $\xi(y_0) = x_0$ such that $(\xi(y), y) \in \mathbb{S}_r(x_0, y_0)$ and $h(\xi(y), y) = 0$ for all $y \in (y_0 - \delta, y_0 + \delta)$. Consequently, $f(\xi(y)) = y$ for all $y \in (y_0 - \delta, y_0 + \delta) \cap J$ and, in particular, $(y_0 - \delta, y_0 + \delta) \subseteq J$. Since f is continuous and strictly monotonic on $I_1 = (x_0 - r, x_0 + r) \subseteq I$, it follows that y_0 is an interior point of $J_1 := f(I_1)$ and $f^{-1} = \xi$ on J_1. Hence $f^{-1} : J_1 \to \mathbb{R}$ is continuous at y_0.

In case x_0 is an endpoint of I, we can extend f to a continuous, strictly monotonic function f^* on a larger interval I^* such that x_0 is an interior point of I^*. For example, if f is strictly increasing and $I = [x_0, b)$, then we may take $I^* := [x_0 - 1, b)$ and $f^*(x) := f(x)$ if $x \in [x_0, b)$ and $f^*(x) := (x - x_0) + y_0$ if $x \in [x_0 - 1, x_0)$. Applying the arguments in the previous paragraph to f^*, we obtain the desired result. □

As an immediate corollary of Proposition 2.43, we obtain an alternative proof of the Continuous Inverse Theorem for functions of one variable (given, for example, on page 78 of ACICARA), which asserts that a continuous one-one function defined on an interval has a continuous inverse. To this end, we shall use the following fact from the theory of functions of one variable, which is completely elementary in the sense that neither the statement nor the proof involves the notions of continuity or limits. For a proof of this fact and also for some related results, one may refer to page 29 of ACICARA.

Fact 2.44. *Let I be an interval in $\mathbb{R}$. If $f : I \to \mathbb{R}$ is one-one and has the IVP on I, then f is strictly monotonic on I.*

Corollary 2.45. *Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a one-one continuous function. Then the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.*

Proof. By part (i) of Fact 2.33, f has the IVP on I. So, by Fact 2.44, f is strictly monotonic on I. Hence by Proposition 2.43, f^{-1} is continuous. □

The notion of continuity can be extended to functions of three or more variables in a completely analogous manner. Most results extend to this case in a straightforward way. A result for which the extension to functions of three variables may not be immediate is the Implicit Function Theorem (Proposition 2.40). Recall that the latter may be roughly stated by saying that if around a point, $f(x, y)$ is continuous in x as well as in y and strictly monotonic in y, then we can solve the equation $f(x, y) = 0$ for y in terms of x around that point. It turns out that for functions of three variables, in order to solve $f(x, y, z) = 0$ for z in terms of x and y around a point, what we need apart from the strict monotonicity in z is not just the continuity in each of the three variables, but the continuity in the variable z and the (bivariate) continuity in x and y. In effect, the statement as well as the proof of Proposition 2.40 generalize easily if the variable x is replaced by two (or more) variables. For ease of reference, we record below a precise statement of this result. Formulation of analogues as in Remark 2.42 and a general result in the case of functions of n variables is left to the reader.

Proposition 2.46 (Trivariate Implicit Function Theorem). *Let* $D \subseteq \mathbb{R}^3$*,* $(x_0, y_0, z_0) \in D$*, and* $f : D \to \mathbb{R}$ *be such that* $f(x_0, y_0, z_0) = 0$*. Assume that there is* $r > 0$ *with* $\mathbb{S}_r(x_0, y_0, z_0) \subseteq D$ *and the following conditions hold:*

(a) *Given any* $(x, y) \in \mathbb{S}_r(x_0, y_0)$*, the function* $\psi : (z_0 - r, z_0 + r) \to \mathbb{R}$ *defined by* $\psi(z) = f(x, y, z)$ *is continuous. Also, given any* $z \in (z_0 - r, z_0 + r)$*, the function* $\phi : \mathbb{S}_r(x_0, y_0) \to \mathbb{R}$ *defined by* $\phi(x, y) = f(x, y, z)$ *is continuous.*
(b) *Given any* $(x, y) \in \mathbb{S}_r(x_0, y_0)$*, the function* $\psi : (z_0 - r, z_0 + r) \to \mathbb{R}$ *defined by* $\psi(z) = f(x, y, z)$ *is strictly monotonic.*

Then there are $\delta > 0$ *and a unique continuous function* $\zeta : \mathbb{S}_\delta(x_0, y_0) \to \mathbb{R}$ *with* $\zeta(x_0, y_0) = z_0$ *such that* $(x, y, \zeta(x, y)) \in \mathbb{S}_r(x_0, y_0, z_0)$ *and* $f(x, y, \zeta(x, y)) = 0$ *for all* $(x, y) \in \mathbb{S}_\delta(x_0, y_0)$*.*

Proof. The proof is similar to that of Proposition 2.40 if we make appropriate notational changes. □

2.3 Limits

Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$. Assume that an open square of positive radius centered at (x_0, y_0), except possibly the center, is contained in D, that is, $\mathbb{S}_r(x_0, y_0) \setminus \{(x_0, y_0)\} \subseteq D$ for some $r > 0$. Let $f : D \to \mathbb{R}$ be any function. We say that a **limit** of f as (x, y) tends to (x_0, y_0) exists if there is a real number ℓ such that whenever a sequence $\big((x_n, y_n)\big)$ in $D \setminus \{(x_0, y_0)\}$ converges to (x_0, y_0), we have $f(x_n, y_n) \to \ell$. We then write $f(x, y) \to \ell$ as $(x, y) \to (x_0, y_0)$. It may be noted that there do exist sequences in $D \setminus \{(x_0, y_0)\}$ that converge to (x_0, y_0). For example,

$$(x_n, y_n) := \left(x_0 - \frac{r}{n+1}, \ y_0 - \frac{r}{n+1} \right) \quad \text{for } n \in \mathbb{N}$$

defines one such sequence. Using this and the fact that the limit of a sequence in $\mathbb{R}^2$ is unique (part (i) of Proposition 2.1), we readily see that if a limit of f as (x, y) tends to (x_0, y_0) exists, then it is unique. With this in view, if $f(x, y) \to \ell$ as $(x, y) \to (x_0, y_0)$, then we may refer to ℓ as *the* limit of $f(x, y)$ as (x, y) tends to (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \ell.$$

Examples 2.47. (i) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(0, 0) := 1$ and $f(x, y) := \sin(xy)$ for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Then the limit of f as (x, y) tends to $(0, 0)$ exists and is equal to 0. Indeed, if $\big((x_n, y_n)\big)$ is a sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(x_n, y_n) \to (0, 0)$, then $x_n y_n \to 0$, and by the continuity of the sine function, $\sin(x_n y_n) \to \sin 0 = 0$, that is, $f(x_n, y_n) \to 0$.

(ii) Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} x+y & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

Then the limit of f as (x,y) tends to $(0,0)$ does not exist. This can be seen by considering two sequences approaching $(0,0)$, one along the line $y = x$ and another staying away from this line. For example, if $(x_n, y_n) := (1/n, 1/n)$ and $(u_n, v_n) := (-1/n, 1/n)$ for $n \in \mathbb{N}$, then $\big((x_n, y_n)\big)$ and $\big((u_n, v_n)\big)$ are sequences in $\mathbb{R}^2 \setminus \{(0,0)\}$ converging to $(0,0)$, but $f(x_n, y_n) \to 1$ and $f(u_n, v_n) \to 0$.

(iii) Consider $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ given by $f(x,y) = xy/(x^2+y^2)$ for $(x,y) \in \mathbb{R}^2$, $(x,y) \neq (0,0)$. Then the limit of f as (x,y) tends to $(0,0)$ does not exist. This can also be seen by considering two sequences approaching $(0,0)$, along different lines through the origin. For example, if $(x_n, y_n) := (1/n, 1/n)$ and $(u_n, v_n) := (1/n, 2/n)$ for $n \in \mathbb{N}$, then $\big((x_n, y_n)\big)$ and $\big((u_n, v_n)\big)$ are sequences in $\mathbb{R}^2 \setminus \{(0,0)\}$ converging to $(0,0)$, but $f(x_n, y_n) \to \frac{1}{2}$ and $f(u_n, v_n) \to \frac{2}{5}$. $\diamond$

Limits and Continuity

The concepts of continuity and limit are related in a similar way as in the case of functions of one variable.

Proposition 2.48. *Let $D \subseteq \mathbb{R}^2$ and let $(x_0, y_0) \in \mathbb{R}^2$ be an interior point of D, that is, $\mathbb{S}_r(x_0, y_0) \subseteq D$ for some $r > 0$. Let $f : D \to \mathbb{R}$ be any function. Then f is continuous at (x_0, y_0) if and only if the limit of f as (x,y) tends to (x_0, y_0) exists and is equal to $f(x_0, y_0)$.*

Proof. Assume that f is continuous at (x_0, y_0). Let $\big((x_n, y_n)\big)$ be any sequence in D such that $(x_n, y_n) \to (x_0, y_0)$. By the continuity of f at (x_0, y_0), we see that $f(x_n, y_n) \to f(x_0, y_0)$. It follows that the limit of f as (x,y) tends to (x_0, y_0) exists and is equal to $f(x_0, y_0)$.

To prove the converse, assume that the limit of f as (x,y) tends to (x_0, y_0) exists and is equal to $f(x_0, y_0)$. Let $\big((x_n, y_n)\big)$ be any sequence in D such that $(x_n, y_n) \to (x_0, y_0)$. If there is $n_0 \in \mathbb{N}$ such that $(x_n, y_n) = (x_0, y_0)$ for all $n \geq n_0$, then it is clear that $f(x_n, y_n) \to f(x_0, y_0)$. Otherwise, there are positive integers $n_1, n_2, \ldots$ such that $n_1 < n_2 < \cdots$ and $\{n \in \mathbb{N} : (x_n, y_n) \neq (x_0, y_0)\} = \{n_k : k \in \mathbb{N}\}$. Now, $\big((x_{n_k}, y_{n_k})\big)$ is a sequence in $D \setminus \{(x_0, y_0)\}$ that converges to (x_0, y_0), and therefore $f(x_{n_k}, y_{n_k}) \to f(x_0, y_0)$. Since $f(x_n, y_n) = f(x_0, y_0)$ for all $n \in \mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}$, it follows that $f(x_n, y_n) \to f(x_0, y_0)$. Hence f is continuous at (x_0, y_0). $\square$

As a consequence, we obtain a useful characterization for the existence of the limit of a function in terms of the continuity of an associated function.

Corollary 2.49. *Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that D contains $\mathbb{S}_r(x_0, y_0) \setminus \{(x_0, y_0)\}$ for some $r > 0$. Given a function $f : D \to \mathbb{R}$ and $\ell \in \mathbb{R}$, let $F : D \cup \{(x_0, y_0)\} \to \mathbb{R}$ be the function defined by*

$$F(x, y) := \begin{cases} f(x, y) & \text{if } (x, y) \in D \setminus \{(x_0, y_0)\}, \\ \ell & \text{if } (x, y) = (x_0, y_0). \end{cases}$$

Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \text{ exists and is equal to } \ell \iff F \text{ is continuous at } (x_0, y_0).$$

Proof. Since $f(x, y) = F(x, y)$ for $(x, y) \in D \setminus \{(x_0, y_0)\}$, it is clear that $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists if and only if $\lim_{(x,y)\to(x_0,y_0)} F(x, y)$ exists, and in this case the two limits are equal. Further, since (x_0, y_0) is an interior point of $D \cup \{(x_0, y_0)\}$ and $F(x_0, y_0) = \ell$, the desired result follows from applying Proposition 2.48 to F. □

Examples 2.50. (i) In view of Proposition 2.48 and Example 2.16 (i), we see that every rational function has a limit wherever it is defined, that is, if $p(x, y)$ and $q(x, y)$ are polynomials in two variables and if $(x_0, y_0) \in \mathbb{R}^2$ is such that $q(x_0, y_0) \neq 0$, then

$$\lim_{(x,y)\to(x_0,y_0)} \frac{p(x, y)}{q(x, y)} = \frac{p(x_0, y_0)}{q(x_0, y_0)}.$$

On the other hand, if $q(x_0, y_0) = 0$, then the limit of $p(x, y)/q(x, y)$ may not exist, in general. For example, for any $m, k \in \mathbb{N}$, the rational function $f(x, y) := x^m/y^k$ does not have a limit as (x, y) tends to $(0, 0)$. To see this, it suffices to approach $(0, 0)$ along the parametric curve given by $(x(t), y(t)) = (\alpha t^k, \beta t^m)$, $t \in [-1, 1]$, where α, β are any nonzero constants. For example, if $(x_n, y_n) := (1/n^k, 1/n^m)$ and $(u_n, v_n) := (2/n^k, 1/n^m)$ for $n \in \mathbb{N}$, then $\big((x_n, y_n)\big)$ and $\big((u_n, v_n)\big)$ are sequences in $\mathbb{R}^2 \setminus \{(0, 0)\}$ converging to $(0, 0)$, but $f(x_n, y_n) \to 1$ and $f(u_n, v_n) \to 2^m$.

(ii) Consider $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ defined by $f(x, y) = x^2y/(x^2 + y^2)$. Then in view of Proposition 2.48 and Example 2.16 (i), we see that the limit of $f(x, y)$ as (x, y) tends to $(0, 0)$ exists and is equal to 0. ◇

Thanks to Corollary 2.49, basic properties of limits of real-valued functions of two variables can be deduced from the corresponding properties of continuous functions.

Proposition 2.51. *Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that D contains $\mathbb{S}_t(x_0, y_0) \setminus \{(x_0, y_0)\}$ for some $t > 0$. Let $f, g : D \to \mathbb{R}$, and let $\ell, m \in \mathbb{R}$ be such that*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \ell \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = m.$$

Then for any $r \in \mathbb{R}$, the limits of $f+g$, rf, and fg as (x, y) tends to (x_0, y_0) exist, and are equal to $\ell+m$, $r\ell$, and ℓm respectively. Moreover, if $\ell \neq 0$, then there is $\delta > 0$ such that $f(x, y) \neq 0$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}$, and the limit of $1/f : D \cap \mathbb{S}_\delta(x_0, y_0) \setminus \{(x_0, y_0)\} \to \mathbb{R}$ as (x, y) tends to (x_0, y_0) exists, and is equal to $1/\ell$.

Proof. Let $F, G : D \cup \{(x_0, y_0)\} \to \mathbb{R}$ be the functions defined by letting $F(x, y) := f(x, y)$ and $G(x, y) := g(x, y)$ for $(x, y) \in D \setminus \{(x_0, y_0)\}$ and setting $F(x_0, y_0) := \ell$ and $G(x_0, y_0) := m$. By Corollary 2.49, F and G are continuous at (x_0, y_0). So the assertion concerning the limits of $f + g$, rf, and fg follow from Proposition 2.15 and Corollary 2.49. If $\ell \neq 0$, then the desired existence of δ and the limit of $1/f$ follow from Lemma 2.14, Proposition 2.15, and Corollary 2.49. □

As in the case of functions of one variable, if there are certain inequalities among the values of real-valued functions of two variables, then the same prevail when we pass to limits, provided the limits exist. But of course, strict inequalities such as $<$ can change to weak inequalities such as $\leq$ when we pass to the limit. (See Exercise 11.) On the other hand, strict inequalities on limits yield strict inequalities on the values of the corresponding function around the point where the limit is taken. (See Exercise 12.) Moreover, for nonnegative functions, extraction of roots is preserved by passing to limits.

Proposition 2.52. *Let $D, (x_0, y_0), r, f, g, \ell$, and m be as in Proposition 2.51.*

(i) *If there is $\delta > 0$ with $\delta \leq r$ such that $f(x, y) \leq g(x, y)$ for all (x, y) in $\mathbb{S}_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}$, then $\ell \leq m$. Conversely, if $\ell < m$, then there is $\delta > 0$ such that $\delta \leq r$ and $f(x, y) < g(x, y)$ for all $(x, y) \in \mathbb{S}_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}$.*

(ii) *If $f(x, y) \geq 0$ for all $(x, y) \in D$, then $\ell \geq 0$ and for each $k \in \mathbb{N}$, the limit of $f^{1/k} : D \to \mathbb{R}$ as (x, y) tends to (x_0, y_0) exists, and is equal to $\ell^{1/k}$.*

(iii) [**Sandwich Theorem**] *If $\ell = m$ and if there is $h : D \to \mathbb{R}$ such that $f(x, y) \leq h(x, y) \leq g(x, y)$ for all $(x, y) \in D$, then the limit of h as (x, y) tends to (x_0, y_0) exists, and is equal to ℓ.*

Proof. Consider $H : D \cup \{(x_0, y_0)\} \to \mathbb{R}$ defined by $H(x, y) := g(x, y) - f(x, y)$ for $(x, y) \in D \setminus \{(x_0, y_0)\}$ and $H(x_0, y_0) := m - \ell$. By Corollary 2.49 and Proposition 2.51, H is continuous at (x_0, y_0). If $\ell > m$, then $H(x_0, y_0) < 0$ and hence by Lemma 2.14, there is $\eta > 0$ such that $H(x, y) < 0$, that is, $f(x, y) > g(x, y)$ for all $(x, y) \in D \cap \mathbb{S}_\eta(x_0, y_0)$. This contradicts the assumption on f and g. Hence $\ell \leq m$. Conversely, suppose $\ell < m$. Then $H(x_0, y_0) > 0$, and hence by Lemma 2.14, there is $\delta > 0$ such that $H(x, y) > 0$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$, and so $f(x, y) < g(x, y)$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$. This proves (i). Next, if $f(x, y) \geq 0$ for all $(x, y) \in D$, then by (i), we obtain $\ell \geq 0$. Further, given any $k \in \mathbb{N}$, the assertion about the limit of $f^{1/k}$ follows from Proposition 2.15 and Corollary 2.49. Finally, (iii) is an immediate consequence of part (vi) of Fact 2.3. □

As in the case of functions of one variable, a criterion for the existence of the limit of a real-valued function of two variables that does not involve convergence of sequences can be given as follows.

Proposition 2.53. *Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that D contains $\mathbb{S}_r(x_0, y_0) \setminus \{(x_0, y_0)\}$ for some $r > 0$ and let $f : D \to \mathbb{R}$ be a function. Then the limit of $f(x, y)$ as (x, y) tends to (x_0, y_0) exists if and only if there is $\ell \in \mathbb{R}$ satisfying the following ϵ-δ condition: For every $\epsilon > 0$, there is $\delta > 0$ such that*

$$(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0) \text{ and } (x, y) \neq (x_0, y_0) \Longrightarrow |f(x, y) - \ell| < \epsilon.$$

Proof. Given $\ell \in \mathbb{R}$, let $F : D \cup \{(x_0, y_0)\} \to \mathbb{R}$ be the function associated with f and ℓ as in Corollary 2.49. Using the equivalence of (i) and (ii) in Proposition 2.22 together with Corollary 2.49, we obtain the desired result. □

The above characterization yields the following analogue of the Cauchy Criterion for sequences in $\mathbb{R}^2$ (part (iv) of Proposition 2.6).

Proposition 2.54 (Cauchy Criterion for Limits of Functions). *Suppose $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ are such that D contains $\mathbb{S}_r(x_0, y_0) \setminus \{(x_0, y_0)\}$ for some $r > 0$. Let $f : D \to \mathbb{R}$ be a function. Then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that*

$$(x, y), (u, v) \in D \cap \mathbb{S}_\delta(x_0, y_0) \setminus \{(x_0, y_0)\} \Longrightarrow |f(x, y) - f(u, v)| < \epsilon.$$

Proof. Assume that $\ell := \lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists. Let $\epsilon > 0$ be given. By Proposition 2.53, there is $\delta > 0$ such that $|f(x, y) - \ell| < \epsilon/2$ for all (x, y) in $D \cap \mathbb{S}_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}$. Hence for $(x, y), (u, v) \in D \cap \mathbb{S}_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}$, we obtain $|f(x, y) - f(u, v)| \leq |f(x, y) - \ell| + |\ell - f(u, v)| < (\epsilon/2) + (\epsilon/2) = \epsilon$, as desired. The converse follows readily from the Cauchy Criterion for limits of sequences in $\mathbb{R}$ (part (iv) of Fact 2.5). □

Limit from a Quadrant

An analogue of the notion of left(-hand) or right(-hand) limits for functions of one variable is given by limits from any one of the four quadrants for functions of two variables. These may be defined as follows.

Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that $(x_0, x_0 + r) \times (y_0, y_0 + r) \subseteq D$ for some $r > 0$. Given a function $f : D \to \mathbb{R}$, we say that a **limit of f from the first quadrant** as (x, y) tends to (x_0, y_0) exists if there is a real number ℓ such that whenever $\big((x_n, y_n)\big)$ is a sequence in $D \setminus \{(x_0, y_0)\}$ satisfying $(x_n, y_n) \geq (x_0, y_0)$ for all $n \in \mathbb{N}$ and $(x_n, y_n) \to (x_0, y_0)$, we have $f(x_n, y_n) \to \ell$. It is easy to see that if such a limit exists, then it is unique. In this case, we write

$$f(x, y) \to \ell \text{ as } (x, y) \to \left(x_0^+, y_0^+\right) \quad \text{ or } \quad \lim_{(x,y)\to\left(x_0^+, y_0^+\right)} f(x, y) = \ell.$$

Similarly, we can define limits of f from the second, the third, and the fourth quadrants. Obvious analogues of the above notation are then used.

Remark 2.55. For limits from a quadrant, Corollary 2.49 admits a straightforward analogue. More precisely, let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that $(x_0, x_0 + r) \times (y_0, y_0 + r) \subseteq D$ for some $r > 0$. Consider $D_1 := \{(x, y) \in D : x \geq x_0 \text{ and } y \geq y_0\}$ and $F_1 : D_1 \cup \{(x_0, y_0)\} \to \mathbb{R}$ defined by

$$F_1(x, y) := \begin{cases} f(x, y) & \text{if } (x, y) \in D_1 \setminus \{(x_0, y_0)\}, \\ \ell & \text{if } (x, y) = (x_0, y_0). \end{cases}$$

Then

$$\lim_{(x,y)\to(x_0^+,y_0^+)} f(x, y) \text{ exists and is equal to } \ell \iff F_1 \text{ is continuous at } (x_0, y_0).$$

This can be proved by a similar argument as in Corollary 2.49. Moreover, analogous results for limits from the second, the third, and the fourth quadrants can be readily obtained. ◇

Proposition 2.56. *Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that D contains $\mathbb{S}_r(x_0, y_0) \setminus \{(x_0, y_0)\}$ for some $r > 0$. Let $f : D \to \mathbb{R}$ be a function and let $\ell \in \mathbb{R}$. Then $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \ell$ if and only if $\lim_{(x,y)\to(x_0^+,y_0^+)} f(x, y)$, $\lim_{(x,y)\to(x_0^-,y_0^+)} f(x, y)$, $\lim_{(x,y)\to(x_0^-,y_0^-)} f(x, y)$, and $\lim_{(x,y)\to(x_0^+,y_0^-)} f(x, y)$ exist and are all equal to ℓ. If, in addition, $(x_0, y_0) \in D$, then f is continuous at (x_0, y_0) if and only if the limit of f from each of the four quadrants as (x, y) tends to (x_0, y_0) exists and they are all equal to $f(x_0, y_0)$.*

Proof. If $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \ell$, then it is clear that the limit of f from each of the four quadrants as (x, y) tends to (x_0, y_0) exists and they are all equal to ℓ. To prove the converse, suppose the limit of f from each of the four quadrants as (x, y) tends to (x_0, y_0) exists and they are all equal to ℓ. Consider $F : D \cup \{(x_0, y_0)\} \to \mathbb{R}$ defined by $F(x_0, y_0) := \ell$ and $F(x, y) := f(x, y)$ for $(x, y) \in D$ with $(x, y) \neq (x_0, y_0)$. Let $D_1 := \{(x, y) \in D : x \geq x_0 \text{ and } y \geq y_0\}$, $D_2 := \{(x, y) \in D : x \leq x_0 \text{ and } y \geq y_0\}$, $D_3 := \{(x, y) \in D : x \leq x_0 \text{ and } y \leq y_0\}$, and $D_4 := \{(x, y) \in D : x \geq x_0 \text{ and } y \leq y_0\}$. Also, let $\widetilde{D}_i := D_i \cup \{(x_0, y_0)\}$ and $F_i := F_{|\widetilde{D}_i}$ for $i = 1, 2, 3, 4$. In view of Remark 2.55, we see that F_i is continuous at (x_0, y_0) for $i = 1, 2, 3, 4$. Hence by Corollary 2.21, F is continuous at (x_0, y_0), and therefore by Corollary 2.49, $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \ell$.

In case $(x_0, y_0) \in D$, the assertion about the continuity of f at (x_0, y_0) follows from what is proved above and Proposition 2.48. □

Approaching Infinity

Let $D \subseteq \mathbb{R}^2$ be such that D contains a product of semi-infinite open intervals of the form $(a, \infty) \times (c, \infty)$, where $a, c \in \mathbb{R}$. Given a function $f : D \to \mathbb{R}$, we say that a **limit** of f as (x,y) tends to (∞,∞) exists if there is a real number ℓ satisfying the following property:

$$\big((x_n,y_n)\big) \text{ any sequence in } D \text{ with } x_n \to \infty \text{ and } y_n \to \infty \implies f(x_n,y_n) \to \ell.$$

In this case the real number ℓ is unique and it is sometimes denoted by $\lim_{(x,y)\to(\infty,\infty)} f(x,y)$. Similarly, we can define a limit of f as $(x,y) \to (-\infty,\infty)$, or as $(x,y) \to (-\infty,-\infty)$, or as $(x,y) \to (\infty,-\infty)$, provided of course the domain D of f contains a product of semi-infinite open intervals of the form $(-\infty,b)\times(c,\infty)$, $(-\infty,b)\times(-\infty,d)$, or $(a,\infty)\times(-\infty,d)$, as the case may be, for some $a,b,c,d \in \mathbb{R}$. An alternative definition that is analogous to the ϵ-δ characterization (Proposition 2.53) can be given for such limits. It should suffice to consider the case of limits as $(x,y) \to (\infty,\infty)$. We leave a formulation of the statement and proofs in the other three cases as an exercise.

Proposition 2.57. *Let $D \subseteq \mathbb{R}^2$ be such that $D \supseteq (a,\infty)\times(c,\infty)$ for some $a,c \in \mathbb{R}$, and let $f : D \to \mathbb{R}$ be a function. Then $\lim_{(x,y)\to(\infty,\infty)} f(x,y)$ exists if and only if there is $\ell \in \mathbb{R}$ satisfying the following ϵ-(α,β) condition: For every $\epsilon > 0$, there are $\alpha,\beta \in \mathbb{R}$ such that*

$$(x,y) \in D \text{ with } (x,y) \geq (\alpha,\beta) \implies |f(x,y)-\ell| < \epsilon.$$

Proof. Assume that $\lim_{(x,y)\to(\infty,\infty)} f(x,y)$ exists and is equal to a real number ℓ. Suppose the ϵ-(α,β) condition is not satisfied. Then there is $\epsilon > 0$ such that for every $\alpha,\beta \in \mathbb{R}$, we can find $(x,y) \in D$ with $(x,y) \geq (\alpha,\beta)$, but $|f(x,y)-\ell| \geq \epsilon$. Taking $(\alpha,\beta) = (n,n)$, as n varies over $\mathbb{N}$, we obtain a sequence $\big((x_n,y_n)\big)$ in D such that $x_n \to \infty$ and $y_n \to \infty$, but $f(x_n,y_n) \not\to \ell$. This contradicts $\lim_{(x,y)\to(\infty,\infty)} f(x,y) = \ell$.

Conversely, assume the ϵ-(α,β) condition. Let $\big((x_n,y_n)\big)$ be a sequence in D such that $x_n \to \infty$ and $y_n \to \infty$. Given any $\epsilon > 0$, find $\alpha,\beta \in \mathbb{R}$ for which $\alpha > a$ and $\beta > c$. Now, there is $n_0 \in \mathbb{N}$ such that $(x_n,y_n) \geq (\alpha,\beta)$ for all $n \geq n_0$, and hence $|f(x_n,y_n)-\ell| < \epsilon$ for all $n \geq n_0$. Thus $f(x_n,y_n) \to \ell$, and so $\lim_{(x,y)\to(\infty,\infty)} f(x,y) = \ell$. □

As in the case of functions of one variable, in some cases ∞ or $-\infty$ can be regarded as a "limit" of a function of two variables. Let $D \subseteq \mathbb{R}^2$ and $(x_0,y_0) \in \mathbb{R}^2$ be such that D contains $\mathbb{S}_r(x_0,y_0) \setminus \{(x_0,y_0)\}$ for some $r > 0$ and let $f : D \to \mathbb{R}$ be any function. We say that $f(x,y)$ tends to ∞ as (x,y) tends to (x_0,y_0) if for every sequence $\big((x_n,y_n)\big)$ in $D\setminus\{(x_0,y_0)\}$ that converges to (x_0,y_0), we have $f(x_n,y_n) \to \infty$. We then write

$$f(x,y) \to \infty \quad \text{as} \quad (x,y) \to (x_0,y_0).$$

Likewise, we say that $f(x,y)$ tends to $-\infty$ as (x,y) tends to (x_0,y_0) if for every sequence $\big((x_n,y_n)\big)$ in $D\setminus\{(x_0,y_0)\}$ that converges to (x_0,y_0), we have $f(x_n,y_n) \to -\infty$. We then write

$$f(x, y) \to -\infty \text{ as } (x, y) \to (x_0, y_0).$$

For example,

$$\frac{1}{x^2 + y^2} \to \infty \text{ as } (x, y) \to (0, 0) \quad \text{and} \quad -\frac{1}{x^2 + y^2} \to -\infty \text{ as } (x, y) \to (0, 0).$$

We now give an analogue of Proposition 2.53 for a real-valued function of two variables that tends to ∞ or to $-\infty$.

Proposition 2.58. *Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that D contains $\mathbb{S}_r(x_0, y_0) \setminus \{(x_0, y_0)\}$ for some $r > 0$ and let $f : D \to \mathbb{R}$ be any function. Then $f(x, y) \to \infty$ as $(x, y) \to (x_0, y_0)$ if and only if the following α-δ condition holds: For every $\alpha \in \mathbb{R}$, there is $\delta > 0$ such that*

$$(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0) \text{ and } (x, y) \neq (x_0, y_0) \Longrightarrow f(x, y) > \alpha.$$

Likewise, $f(x, y) \to -\infty$ as $(x, y) \to (x_0, y_0)$ if and only if the following β-δ condition holds: For every $\beta \in \mathbb{R}$, there is $\delta > 0$ such that

$$(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0) \text{ and } (x, y) \neq (x_0, y_0) \Longrightarrow f(x, y) < \beta.$$

Proof. Assume that $f(x, y) \to \infty$ as $(x, y) \to (x_0, y_0)$. If the α-δ condition does not hold, then there exists $\alpha \in \mathbb{R}$ such that for every $\delta > 0$, there is $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$ with $(x, y) \neq (x_0, y_0)$ and $f(x, y) \leq \alpha$. Taking $\delta = 1/n$ as n varies over $\mathbb{N}$, we obtain a sequence $\big((x_n, y_n)\big)$ in $D \setminus \{(x_0, y_0)\}$ such that $(x_n, y_n) \to (x_0, y_0)$, but $f(x_n, y_n) \not\to \infty$. This contradicts the assumption.

Conversely, assume the α-δ condition. Let $\big((x_n, y_n)\big)$ be a sequence in $D \setminus \{(x_0, y_0)\}$ such that $(x_n, y_n) \to (x_0, y_0)$, and let $\alpha > 0$ be given. Then there is $\delta > 0$ such that $f(x, y) > \alpha$ for all $(x, y) \in D \cap \mathbb{S}_\delta(x_0, y_0)$ with $(x, y) \neq (x_0, y_0)$. Further, there is $n_0 \in \mathbb{N}$ such that $(x_n, y_n) \in \mathbb{S}_\delta(x_0, y_0)$ for $n \geq n_0$. Hence $f(x_n, y_n) > \alpha$ for $n \geq n_0$. Thus $f(x, y) \to \infty$ as $(x, y) \to (x_0, y_0)$.

The equivalence of the condition $f(x, y) \to -\infty$ as $(x, y) \to (x_0, y_0)$ with the β-δ condition is proved similarly. □

Recall that we have defined the notion of a monotonically increasing function of two variables using the product order on $\mathbb{R}^2$. We show below that for such functions, existence of a limit from the first or the third quadrant is equivalent to boundedness properties.

Proposition 2.59. *Let $a, b, c, d \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$ and $c < d$ be such that either $a, c \in \mathbb{R}$ or $a = c = -\infty$, and either $b, d \in \mathbb{R}$ or $b = d = \infty$. Let $f : (a, b) \times (c, d) \to \mathbb{R}$ be a monotonically increasing function. Then*

(i) $\lim_{(x,y)\to(b^-,\, d^-)} f(x, y)$ *exists if and only if f is bounded above; in this case,* $\lim_{(x,y)\to(b^-,\, d^-)} f(x, y) = \sup\{f(x, y) : (x, y) \in (a, b) \times (c, d)\}$. *If f is not bounded above, then $f(x, y) \to \infty$ as $(x, y) \to (b^-, d^-)$.*

(ii) $\lim_{(x,y)\to(a^+,\,c^+)} f(x,y)$ *exists if and only if* f *is bounded below; in this case,* $\lim_{(x,y)\to(a^+,\,c^+)} f(x,y) = \inf\{f(x,y) : (x,y) \in (a,b)\times(c,d)\}$. *If* f *is not bounded below, then* $f(x,y) \to -\infty$ *as* $(x,y) \to (a^+, c^+)$.

Proof. (i) Suppose f is bounded above. Let $M := \sup\{f(x,y) : (x,y) \in (a,b)\times(c,d)\}$. Given any $\epsilon > 0$, there is $(b_0, d_0) \in (a,b)\times(c,d)$ such that $M - \epsilon < f(b_0, d_0) \le M$. Now, if $\big((x_n, y_n)\big)$ is any sequence in $(a,b)\times(c,d)$ such that $(x_n, y_n) \to (b,d)$, then there is $n_0 \in \mathbb{N}$ such that $(b_0, d_0) \le (x_n, y_n)$ for $n \ge n_0$. Since f is monotonically increasing, we obtain $M - \epsilon < f(x_n, y_n) \le M$ for $n \ge n_0$. It follows that $\lim_{(x,y)\to(b^-,\,d^-)} f(x,y)$ exists and is equal to M.

On the other hand, suppose f is not bounded above. Let $\alpha \in \mathbb{R}$. Then there is $(b_0, d_0) \in (a,b)\times(c,d)$ such that $f(b_0, d_0) > \alpha$. Since f is monotonically increasing, we see that $f(x,y) > \alpha$ for all $(x,y) \in (b_0, b)\times(d_0, d)$. Now, if $\big((x_n, y_n)\big)$ is any sequence in $(a,b)\times(c,d)$ such that $(x_n, y_n) \to (b,d)$, then there is $n_0 \in \mathbb{N}$ such that $(b_0, d_0) \le (x_n, y_n)$ for $n \ge n_0$, and hence $f(x_n, y_n) > \alpha$ for $n \ge n_0$. Thus $f(x_n, y_n) \to \infty$ as $(x,y) \to (b^-, d^-)$. It follows that $f(x,y) \to \infty$ as $(x,y) \to (b^-, d^-)$. This proves (i).

(ii) The proof of this part is similar to the proof of part (i) above. □

A result similar to the one above holds for monotonically decreasing functions. (See Exercise 31.) Consequently, we see that if $f : (a,b)\times(c,d) \to \mathbb{R}$ is a monotonic function, then

$$\lim_{(x,y)\to(b^-,\,d^-)} f(x,y) \text{ and } \lim_{(x,y)\to(a^+,\,c^+)} f(x,y) \text{ exist } \iff f \text{ is bounded.}$$

However, for a bounded monotonic function, limits along the other two quadrants may not exist. For example, consider $f : [-1,1]\times[-1,1] \to \mathbb{R}$ defined by

$$f(x,y) := \begin{cases} (x+2)(y+2) & \text{if } x+y \ge 0, \\ (x+1)(y+1) & \text{if } x+y < 0. \end{cases}$$

We have noted in Example 1.8 (i) that f is monotonically increasing. Also, it is clear that f is bounded and (consequently, or otherwise) the limits of f from the first and the third quadrants as (x,y) tends to $(0,0)$ exist. But the limits of f from the second and the fourth quadrants as (x,y) tends to $(0,0)$ do not exist. To see this, consider the sequences in $\mathbb{R}^2$ defined by $(x_n, y_n) := \left(-\frac{1}{n}, \frac{2}{n}\right)$ and $(x'_n, y'_n) := \left(-\frac{2}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Then

$$(x_n, y_n) \to 0 \text{ and } (x'_n, y'_n) \to 0, \quad \text{but} \quad f(x_n, y_n) \to 4 \text{ and } f(x'_n, y'_n) \to 1.$$

Likewise, if $(x_n, y_n) := \left(\frac{2}{n}, -\frac{1}{n}\right)$ and $(x'_n, y'_n) := \left(\frac{1}{n}, -\frac{2}{n}\right)$ for $n \in \mathbb{N}$, then

$$(x_n, y_n) \to 0 \text{ and } (x'_n, y'_n) \to 0, \quad \text{but} \quad f(x_n, y_n) \to 4 \text{ and } f(x'_n, y'_n) \to 1.$$

Thus $\lim_{(x,y)\to(0^-,\,0^+)} f(x,y)$ and $\lim_{(x,y)\to(0^+,\,0^-)} f(x,y)$ do not exist.

In Exercise 40 of Chapter 1, we introduced the notion of an antimonotonic function. It can be seen that if $f : (a,b)\times(c,d) \to \mathbb{R}$ is antimonotonic, then

$\lim_{(x,y)\to(a^+,\,d^-)} f(x,y)$ and $\lim_{(x,y)\to(b^-,\,c^+)} f(x,y)$ exist $\iff$ f is bounded.

(See Exercise 35.)

Remark 2.60. The notion of limit of a real-valued function of two variables admits a straightforward extension to real-valued functions of three or more variables. Moreover, analogues of all the results in Section 2.3 concerning limits can be easily formulated and proved in this case. $\diamond$

Notes and Comments

For the local study around a point in $\mathbb{R}^2$ (and more generally, in $\mathbb{R}^n$), there are at least two natural analogues of the notion of an interval around a point in $\mathbb{R}$: open disks and open squares. These two are essentially equivalent, in the sense that an open disk can be inscribed in an open square with the same center, and vice versa. (See Exercise 3 of Chapter 1). In this book, we have preferred to use open squares instead of open disks. This approach is slightly unusual, but it pays off in several proofs that appear subsequently.

The development of topics discussed in this chapter proceeds along similar lines as in ACICARA. *Sequences in $\mathbb{R}^2$ are introduced first and their basic properties are derived quickly from the corresponding properties of sequences in $\mathbb{R}$. The notion of continuity is defined using convergence of sequences, and basic properties of continuous functions are proved using properties of sequences in $\mathbb{R}^2$. These include a result on piecing together continuous functions on overlapping domains, which does not seem easy to locate in the literature. Standard results about continuous functions on connected domains and on compact domains are included, except that for pedagogical reasons, we have preferred the terminology of path-connected sets and of closed and bounded sets. It may be remarked that the more general notions of connectedness and compactness are of fundamental importance in analysis and topology; for an introduction, we refer to Exercises 17, 18, 19, 20 21, and also the books of Rudin [48] and Munkres [40]. For a convex function of one variable, continuity at an interior point was relegated to an exercise in* ACICARA. *A similar result holds for convex functions of several variables, but proving it is a little more involved, and we have chosen to give a detailed proof for functions of two variables, using arguments similar to those in the book of Roberts and Varberg [47]. For an alternative proof, one may consult the book of Fleming [19].*

Following Hardy [29], we state and prove the Implicit Function Theorem under a weak hypothesis of continuity in each of the two variables and strict monotonicity in one of the variables. That this is possible appears to have been first observed by Besicovitch. (See the footnote on p. 203 of [29].) This version of the Implicit Function Theorem can be used to give an alternative proof of *the Continuous Inverse Theorem. Also, it will pave the way for proving the classical version of the Implicit Function Theorem in Chapter 3.*

Limits of functions of two variables are defined using sequences. We have deduced basic properties of limits from the corresponding properties of continuous functions. Perhaps the only nonstandard notion introduced here is that of a limit from a quadrant. This provides an interesting analogue of the notion in one-variable calculus of left(-hand) and right(-hand) limits. In general, for functions of n variables, the notion will have to deal with 2^n hyperoctants.

Exercises

Part A

1. Consider the sequence in $\mathbb{R}^2$ whose nth term is defined by one of the following. Determine whether it is convergent. If it is, then find its limit.
 (i) $\big(1/n,\, n^2\big)$, (ii) $\big(n,\, 1/n^2\big)$, (iii) $\big(1/n,\, 1/n^2\big)$, (iv) $(1/n,\, (-1)^n/n)$,
 (v) $\big(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!},\, \ln n\big)$, (vi) $\big(\big(1 + \frac{1}{n}\big)^n,\, \big(1 - \frac{1}{n}\big)^n\big)$.
2. A sequence $\big((x_n, y_n)\big)$ in $\mathbb{R}^2$ is said to be
 bounded above if there is $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that $(x_n, y_n) \le (\alpha_1, \alpha_2)$, that is, $x_n \le \alpha_1$ and $y_n \le \alpha_2$ for all $n \in \mathbb{N}$,
 bounded below if there is $(\beta_1, \beta_2) \in \mathbb{R}^2$ such that $(\beta_1, \beta_2) \le (x_n, y_n)$, that is, $\beta_1 \le x_n$ and $\beta_2 \le y_n$ for all $n \in \mathbb{N}$,
 monotonically increasing if $(x_n, y_n) \le (x_{n+1}, y_{n+1})$ for all $n \in \mathbb{N}$,
 monotonically decreasing if $(x_n, y_n) \ge (x_{n+1}, y_{n+1})$ for all $n \in \mathbb{N}$,
 monotonic if it is monotonically increasing or decreasing.

 Prove the following.
 (i) A monotonically increasing sequence in $\mathbb{R}^2$ is bounded above if and only if it is convergent. Also, if $\big((x_n, y_n)\big)$ is monotonically increasing and bounded above, then $\lim_{n\to\infty}(x_n, y_n) = \sup\{(x_n, y_n) : n \in \mathbb{N}\}$.
 (ii) A monotonically decreasing sequence in $\mathbb{R}^2$ is bounded below if and only if it is convergent. Also, if $\big((x_n, y_n)\big)$ is monotonically decreasing and bounded below, then $\lim_{n\to\infty}(x_n, y_n) = \inf\{(x_n, y_n) : n \in \mathbb{N}\}$.
 (iii) A monotonic sequence in $\mathbb{R}^2$ is convergent if and only if it is bounded.
3. Is it true that every sequence in $\mathbb{R}^2$ has a monotonic subsequence? Justify your answer. [Note: It may be remarked that every sequence in $\mathbb{R}$ has a monotonic subsequence; see page 55 of ACICARA.]
4. Let $(x_0, y_0) \in \mathbb{R}^2$. We say that (x_0, y_0) is a **cluster point** of a sequence $\big((x_n, y_n)\big)$ in $\mathbb{R}^2$ if there is a subsequence $\big((x_{n_k}, y_{n_k})\big)$ of $\big((x_n, y_n)\big)$ such that $(x_{n_k}, y_{n_k}) \to (x_0, y_0)$. Show that if $(x_n, y_n) \to (x_0, y_0)$, then (x_0, y_0) is the only cluster point of $\big((x_n, y_n)\big)$. Also, show that the converse is not true, that is, there is a sequence $\big((x_n, y_n)\big)$ in $\mathbb{R}^2$ that has a unique cluster point but is not convergent.
5. If a subset D of $\mathbb{R}^2$ is bounded, then show that its closure $\overline{D}$ is also a bounded subset of $\mathbb{R}^2$.

6. Find the closure, the boundary, and the interior of the following subsets of $\mathbb{R}^2$. Also, determine whether these subsets are closed or open.
(i) $\{(x,y) \in \mathbb{R}^2 : 0 \le x < 1 \text{ and } 0 < y \le 2\}$, (ii) $\{(x,x^2) : x \in \mathbb{R}\}$,
(iii) any finite subset of $\mathbb{R}^2$, (iv) $\{(m,n) : m,n \in \mathbb{N}\}$,
(v) $\{(1/m, 1/n) : m,n \in \mathbb{N}\}$, (vi) $\{(r,s) : r,s \in \mathbb{Q}\}$.
7. Let $D \subseteq \mathbb{R}^2$. Show that the closure of D is the smallest closed subset of $\mathbb{R}^2$ containing D and the interior of D is the largest open subset of D.
8. Let $f, g : [-1,1] \times [-1,1] \to \mathbb{R}$ be the functions defined by
$$f(x,y) := (x+y)^2 \quad \text{and} \quad g(x,y) := \begin{cases} (x+y)^2 & \text{if } x+y \ge 0, \\ -(x+y)^2 & \text{if } x+y < 0. \end{cases}$$
Show that both f and g are continuous on $[-1,1] \times [-1,1]$. Further show that f is bimonotonic but g is not bimonotonic on $[-1,1] \times [-1,1]$.
9. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(0,0) := 0$ and for $(x,y) \ne (0,0)$, by one of the following. In each case, determine whether f is continuous.
(i) $\dfrac{xy^2}{x^2+y^2}$, (ii) $\dfrac{xy^2}{x^2+y^4}$, (iii) $\dfrac{x^3y}{x^6+y^2}$, (iv) $\dfrac{x^2}{x^2+y^2}$,
(v) $xy \ln(x^2+y^2)$, (vi) $\dfrac{x^3}{x^2+y^2}$, (vii) $\dfrac{x^4y}{x^2+y^2}$,
(viii) $\dfrac{x^3y - xy^3}{x^2+y^2}$, (ix) $\dfrac{\sin(x+y)}{|x|+|y|}$, (x) $\dfrac{\sin^2(x+y)}{|x|+|y|}$.
10. Let D be convex and open in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a convex function. If $[a,b] \times [c,d]$ is a closed rectangle contained in D, where $a,b,c,d \in \mathbb{R}$ with $a < b$ and $c < d$, then show that f satisfies a **Lipschitz condition** on $[a,b] \times [c,d]$, that is, there is $L \in \mathbb{R}$ such that
$$|f(x,y) - f(u,v)| \le L\,|(x,y) - (u,v)| \quad \text{for all } (x,y),(u,v) \in [a,b]\times[c,d].$$
(Hint: Use Lemma 2.31, or give a proof similar to that of Lemma 2.31.)
11. Let $D := \mathbb{S}_1(0,0) \setminus \{(0,0)\}$ and let $f, g : D \to \mathbb{R}$ be defined by $f(x,y) := |x| + |y|$ and $g(x,y) := \frac{1}{2}(|x|+|y|)$. Show that $f(x,y) < g(x,y)$ for all $(x,y) \in D$, but $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} g(x,y)$.
12. Show that there is $\delta > 0$ such that $\sin(xy) < \cos(xy)$ for all $(x,y) \in \mathbb{S}_\delta(0,0)$. (Hint: Proposition 2.52.)
13. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by one of (i)–(iv) below. Determine whether the two-variable limit $\lim_{(x,y)\to(0,0)} f(x,y)$ and the two iterated limits $\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right]$ and $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$ exist. If they do, then find them.
(i) $f(x,y) := x + y$, (ii) $f(x,y) := \begin{cases} \dfrac{x^2y^2}{x^2y^2 + (x-y)^2} & \text{if } (x,y) \ne (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$
(iii) $f(x,y) := \begin{cases} \dfrac{x+y}{x-y} & \text{if } x \ne y, \\ 0 & \text{if } x = y, \end{cases}$ (iv) $f(x,y) := \begin{cases} x \sin \dfrac{1}{y} & \text{if } y \ne 0, \\ 0 & \text{if } y = 0. \end{cases}$

Part B

14. Show that a sequence in $\mathbb{R}^2$ is convergent if and only if it is bounded and all its convergent subsequences have the same limit. (Hint: Bolzano–Weierstrass Theorem.)
15. Let m, n be nonnegative integers and let $i, j \in \mathbb{N}$ be even. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0,0) := 0$ and $f(x,y) := x^m y^n/(x^i + y^j)$ for $(x,y) \neq (0,0)$. Show that f is continuous at $(0,0)$ if and only if $mj + ni > ij$.
16. Let $E \subseteq \mathbb{R}$ be open in $\mathbb{R}$ and let $\Phi = (x,y)$ be a pair of real-valued functions $x, y : E \to \mathbb{R}$. Show that both x and y are continuous on E if and only if the set $\Phi^{-1}(V) := \{t \in E : (x(t), y(t)) \in V\}$ is open in $\mathbb{R}$ for every open subset V of $\mathbb{R}^2$.
17. Let $D \subseteq \mathbb{R}^2$. A family $\{U_\alpha : \alpha \in A\}$ indexed by an arbitrary set A is called an **open cover** of D if each U_α is open in $\mathbb{R}^2$ and D is contained in the union of U_α as α varies over A. Such an open cover is said to have a **finite subcover** if there are finitely many indices $\alpha_1, \ldots, \alpha_n \in A$ such that $D \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. The set D is said to be **compact** if every open cover of D has a finite subcover. Prove the following.
    (i) If D is finite, then D is compact.
    (ii) If D is compact and $E \subseteq D$ is closed, then E is compact. (Hint: If $\{U_\alpha : \alpha \in A\}$ is an open cover of D, then consider $\{U_\alpha : \alpha \in A\} \cup \{D \setminus E\}$.)
    (iii) If D is compact, then D is closed. (Hint: If $(x_0, y_0) \in D \setminus \partial D$, then the set of open squares centered at (x,y) and of radius $|(x,y) - (x_0, y_0)|/2$, as (x,y) varies over D, is an open cover of D.)
    (iv) If D is compact, then D is bounded.
    (v) If $D = [a,b] \times [c,d]$ is a closed rectangle, then D is compact. (Hint: Use the midpoints $(a+b)/2$ and $(c+d)/2$ to subdivide D into four smaller rectangles. If an open cover of D has no finite subcover, then the same holds for one of the smaller rectangles. Continue this process and look at the limiting situation.)
    (vi) **(Heine–Borel Theorem)** D is compact $\iff$ D is closed and bounded.

    Generalize the definition and the properties above to subsets of $\mathbb{R}^n$.
18. Let $D \subseteq \mathbb{R}^2$ and $E \subseteq \mathbb{R}$. Prove the following.
    (i) If D is compact and $f : D \to \mathbb{R}$ is continuous, then the range $f(D)$ is closed and bounded.
    (ii) If E is closed and bounded and $x, y : E \to \mathbb{R}$ are continuous, then the subset $\{(x(t), y(t)) : t \in E\}$ of $\mathbb{R}^2$ is compact.
19. If $D \subseteq \mathbb{R}^2$ is path-connected and $f : D \to \mathbb{R}$ is a continuous function such that the image $f(D)$ is a finite set, then show that f is a constant function. Is the conclusion valid if D is not path-connected? Justify your answer. (Hint: If D has two points, take a path $(x(t), y(t))$ joining them. Consider $t \mapsto f(x(t), y(t))$ and use Fact 2.33.)
20. If $D \subseteq \mathbb{R}^2$ is path-connected, then show that D cannot be written as a union of two disjoint, nonempty open subsets of D. (Hint: If it could, then there would be a continuous function $f : D \to \{0, 1\}$. Use Exercise 19.)

21. Let D be an open subset of $\mathbb{R}^2$. If D cannot be written as a union of two disjoint, nonempty open subsets of D, then show that D is path-connected.
22. Let D be a bounded subset of $\mathbb{R}^2$ and let $\overline{D}$ denote its closure. Suppose $f : D \to \mathbb{R}$ be a continuous function. Prove that f is uniformly continuous on D if and only if there is a continuous function $\bar{f} : \overline{D} \to \mathbb{R}$ such that $\bar{f}|_D = f$.
23. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be the **bivariate Thomae function** defined by

$$f(x,y) := \begin{cases} 1 & \text{if } x = 0 \text{ and } y \in \mathbb{Q} \cap [0,1], \\ 1/q & \text{if } x, y \in \mathbb{Q} \cap [0,1] \text{ and } x = p/q \text{ for some} \\ & \text{relatively prime positive integers } p \text{ and } q, \\ 0 & \text{otherwise.} \end{cases}$$

Show that the set of discontinuities of f is $\{(x,y) \in [0,1] \times [0,1] : x, y \in \mathbb{Q}\}$.
24. **(Duhamel's Theorem)** Let $a, b \in \mathbb{R}$ with $a < b$ and $D := [a,b] \times [a,b]$. If $f : D \to \mathbb{R}$ is continuous and $\phi : [a,b] \to \mathbb{R}$ is defined by $\phi(x) := f(x,x)$ for $x \in [a,b]$, then show that ϕ is Riemann integrable on $[a,b]$. Further, show that given any $\epsilon > 0$, there is $\delta > 0$ such that for every partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ with $\mu(P) < \delta$, and every c_i, $\tilde{c}_i \in [x_{i-1}, x_i]$, for $i = 1, \ldots, n$, we have

$$\left| \int_a^b \phi(x)\,dx - \sum_{i=1}^{n} f(c_i, \tilde{c}_i)\ (x_i - x_{i-1}) \right| < \epsilon.$$

25. **(Bliss's Theorem)** If $\phi, \psi : [a,b] \to \mathbb{R}$ are continuous, then show that given any $\epsilon > 0$, there is $\delta > 0$ such that for every partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ with $\mu(P) < \delta$, and every c_i, $\tilde{c}_i \in [x_{i-1}, x_i]$, for $i = 1, \ldots, n$, we have

$$\left| \int_a^b \phi(x)\psi(x)\,dx - \sum_{i=1}^{n} \phi(c_i)\psi(\tilde{c}_i)\ (x_i - x_{i-1}) \right| < \epsilon.$$

26. Let $D \subseteq \mathbb{R}$ and $t_0 \in \mathbb{R}$ be such that D contains $(t_0 - r, t_0) \cup (t_0, t_0 + r)$ for some $r > 0$. For each $t \in D$, let $f_t : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Suppose $f(x) := \lim_{t \to t_0} f_t(x)$ for $x \in [a,b]$, and $f_t \to f$ uniformly in the sense that for every $\epsilon > 0$, there is $\delta > 0$ such that

$$t \in D,\ 0 < |t - t_0| < \delta,\ x \in [a,b] \Longrightarrow |f_t(x) - f(x)| < \epsilon.$$

Show that $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Further, show that $\lim_{t \to t_0} \int_a^b f_t(x)dx$ exists and is equal to $\int_a^b f(x)dx$. Deduce that if $F : [\alpha, \beta] \times [a,b] \to \mathbb{R}$ is continuous, then for each $t_0 \in [\alpha, \beta]$, we have

$$\lim_{t \to t_0} \int_a^b F(t,x)dx = \int_a^b \lim_{t \to t_0} F(t,x)dx = \int_a^b F(t_0,x)dx.$$

Conclude that $\phi : [\alpha, \beta] \to \mathbb{R}$ defined by $\phi(t) := \int_a^b F(t,x)dx$ is continuous.

27. Let $D \subseteq \mathbb{R}$ be such that D contains $[c, \infty)$ for some $c \in \mathbb{R}$. For each $t \in D$, let $f_t : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Suppose $f(x) := \lim_{t\to\infty} f_t(x)$ for $x \in [a, b]$, and $f_t \to f$ uniformly in the sense that for every $\epsilon > 0$, there is $s \in D$ such that $|f_t(x) - f(x)| < \epsilon$ for all $t \in D$ with $t \geq s$ and all $x \in [a, b]$. Show that $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Further, show that $\lim_{t\to\infty} \int_a^b f_t(x)dx$ exists and is equal to $\int_a^b f(x)dx$.
28. Let $D \subseteq \mathbb{R}^2$ and $(x_0, y_0) \in \mathbb{R}^2$ be such that D contains a punctured square $\mathbb{S}_r(x_0, y_0)\setminus\{(x_0, y_0)\}$ for some $r > 0$. Suppose $f : D \to \mathbb{R}$ is such that $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists and is equal to ℓ. Prove the following.
   (i) If $\lim_{y\to y_0} f(x, y)$ exists for every fixed $x \in (x_0 - r, x_0) \cup (x_0, x_0 + r)$, then the **iterated limit** $\lim_{x\to x_0} \left[\lim_{y\to y_0} f(x, y)\right]$ exists and is equal to ℓ.
   (ii) If $\lim_{x\to x_0} f(x, y)$ exists for every fixed $y \in (y_0 - r, y_0) \cup (y_0, y_0 + r)$, then the **iterated limit** $\lim_{y\to y_0} \left[\lim_{x\to x_0} f(x, y)\right]$ exists and is equal to ℓ.
29. Use Exercise 13 (ii) to show that even when both the iterated limits in (i) and (ii) of Exercise 28 exist, they may not be equal. Also, use Exercise 13 (iv) to show that the existence of the two-variable limit does not imply that the one-variable limits in (i) and (ii) of Exercise 28 exist.
30. Let $D \subseteq \mathbb{R}^2$ be such that D contains $(a, \infty) \times (c, \infty)$ for some $a, c \in \mathbb{R}$. Suppose $f : D \to \mathbb{R}$ is such that $\lim_{(x,y)\to(\infty,\infty)} f(x, y)$ exists and is equal to ℓ.
   (i) If $\lim_{y\to\infty} f(x, y)$ exists for every fixed $x \geq a$, then prove that the **iterated limit** $\lim_{x\to\infty} \left[\lim_{y\to\infty} f(x, y)\right]$ exists and is equal to ℓ.
   (ii) If $\lim_{x\to\infty} f(x, y)$ exists for every fixed $y \geq c$, then prove that the **iterated limit** $\lim_{y\to\infty} \left[\lim_{x\to\infty} f(x, y)\right]$ exists and is equal to ℓ.
31. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f : (a, b) \times (c, d) \to \mathbb{R}$ be a monotonically decreasing function. Prove the following.
   (i) $\lim_{(x,y)\to(b^-,\, d^-)} f(x, y)$ exists if and only if f is bounded below; in this case, $\lim_{(x,y)\to(b^-,\, d^-)} f(x, y) = \inf\{f(x, y) : (x, y) \in (a, b) \times (c, d)\}$. If f is not bounded below, then $f(x, y) \to -\infty$ as $(x, y) \to (b^-, d^-)$.
   (ii) $\lim_{(x,y)\to(a^+,\, c^+)} f(x, y)$ exists if and only if f is bounded above; in this case, $\lim_{(x,y)\to(a^+,\, c^+)} f(x, y) = \sup\{f(x, y) : (x, y) \in (a, b) \times (c, d)\}$. If f is not bounded above, then $f(x, y) \to \infty$ as $(x, y) \to (a^+, c^+)$.
32. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f : (a, b) \times (c, d) \to \mathbb{R}$ be a monotonically increasing function. Show that for every $(x_0, y_0) \in (a, b) \times (c, d)$, both $\lim_{(x,y)\to(x_0^-,\, y_0^-)} f(x, y)$ and $\lim_{(x,y)\to(x_0^+,\, y_0^+)} f(x, y)$ exist, and $\lim_{(x,y)\to(x_0^-,\, y_0^-)} f(x, y) \leq f(x_0, y_0) \leq \lim_{(x,y)\to(x_0^+,\, y_0^+)} f(x, y)$. Also, show that if $(x_1, y_1) \in (a, b) \times (c, d)$ with $x_0 < x_1$ and $y_0 < y_1$, then $\lim_{(x,y)\to(x_0^+,\, y_0^+)} f(x, y) \leq \lim_{(x,y)\to(x_1^-,\, y_1^-)} f(x, y)$. Formulate and prove an analogue of these properties for monotonically decreasing functions.

33. Let $D \subseteq \mathbb{R}^2$ and (x_0, y_0) be any point of $\mathbb{R}^2$. If there is a sequence $\big((x_n, y_n)\big)$ in $D \setminus \{(x_0, y_0)\}$ such that $(x_n, y_n) \to (x_0, y_0)$, then (x_0, y_0) is called a **limit point** (or an **accumulation point**) of D.
   (i) Show that (x_0, y_0) is a limit point of D if and only if for every $r > 0$, there is $(x, y) \in D$ such that $0 < |(x, y) - (x_0, y_0)| < r$.
   (ii) If (x_0, y_0) is a limit point of D, then show that for every $r > 0$, the open disk $\mathbb{B}_r(x_0, y_0)$ as well as the open square $\mathbb{S}_r(x_0, y_0)$ contain infinitely many points of the set D.
   (iii) If D is a finite subset of $\mathbb{R}^2$, show that D has no limit point.
   (iv) Determine all the limit points of D if $D := \mathbb{N} \times \mathbb{N}$, or $D := \mathbb{Q} \times \mathbb{Q}$, or $D := \{\big(\frac{1}{n}, \frac{1}{m}\big) : n, m \in \mathbb{N}\}$, or $D := (a, b) \times (c, d)$, or $D := [a, b) \times (c, d]$, where $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$.
   (v) Let $\big((x_n, y_n)\big)$ be a sequence in $\mathbb{R}^2$ and suppose $D = \{(x_n, y_n) : n \in \mathbb{N}\}$ is the set of all its terms. Show that a limit point of D is a cluster point of the sequence $\big((x_n, y_n)\big)$. Give an example to show that a cluster point of $\big((x_n, y_n)\big)$ need not be a limit point of D.
34. Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be a limit point of D. We say that a **limit** of a function $f : D \to \mathbb{R}$ as (x, y) tends to (x_0, y_0) exists if there is a real number ℓ such that whenever $\big((x_n, y_n)\big)$ is any sequence in $D \setminus \{(x_0, y_0)\}$ that converges to (x_0, y_0), we have $f(x_n, y_n) \to \ell$; in this case ℓ is called a **limit** of f as (x, y) tends to (x_0, y_0). Show that if a limit of f as (x, y) tends to (x_0, y_0) exists, then it must be unique. Also, prove analogues of Propositions 2.48, 2.51, 2.52, 2.53, 2.54 and Corollary 2.49.
35. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f : (a, b) \times (c, d) \to \mathbb{R}$ be an antimonotonic function. Show that both $\lim_{(x,y)\to(a^+,\, d^-)} f(x, y)$ and $\lim_{(x,y)\to(b^-,\, c^+)} f(x, y)$ exist if and only if f is bounded. (Hint: Exercise 40 of Chapter 1)
36. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $D := (a, b] \times (c, d]$ and $f : D \to \mathbb{R}$ be a bimonotonic function.
   (i) Define $F : D \to \mathbb{R}$ by $F(x, y) := f(x, y) - f(x, d) - f(b, y) + f(b, d)$. Show that either F is monotonically increasing and bounded below, or F is monotonically decreasing and bounded above.
   (ii) If the one-variable limits $\lim_{x\to b^-} f(x, d)$ and $\lim_{y\to d^-} f(b, y)$ exist, then show that $\lim_{(x,y)\to(b^-,\, d^-)} f(x, y)$ exists.
37. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. State and prove results analogous to those in Exercise 36 above for functions defined on $[a, b) \times [c, d)$, $[a, b) \times (c, d]$, and $(a, b] \times [c, d)$. (Hint: For $[a, b) \times (c, d]$ and $(a, b] \times [c, d)$, consider the notion of antimonotonicity.)
38. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f : [a, b] \times [c, d] \to \mathbb{R}$ be any function. Show that if f is of bounded variation and v_f is continuous, then f is continuous. On the other hand, give an example to show that if f is of bounded bivariation and w_f is continuous, then f need not be continuous.

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