Lecture Notes in Physics 774

Noncommutative Spacetimes

Symmetries in Noncommutative Geometry and Field Theory

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Chapter 2
Deformed Gauge Theories

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Gauge theories are studied on a space of functions with the Moyal product. The development of these ideas follows the differential geometry of the usual gauge theories, but several changes are forced upon us. The Leibniz rule has to be changed such that the theory is now based on a twisted Hopf algebra. Nevertheless, this twisted symmetry structure leads to conservation laws. The symmetry has to be extended from Lie algebra valued to enveloping algebra valued and new vector potentials have to be introduced. As usual, field equations are subjected to consistency conditions that restrict the possible models. Some examples are studied.

2.1 Introduction

Gauge theories have been formulated and developed on the algebra of functions with a pointwise product:

$$\mu\{f \otimes g\} = f \cdot g. \tag{2.1}$$

This product is associative and commutative.

Recently, algebras of functions with a deformed product have been studied intensively [1–5]. These deformed (star) products remain associative but not commutative.

The simplest example is the Moyal product,[1] see Chap. 1 for details

$$\mu_\star\{f \otimes g\} = \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho \otimes \partial_\sigma} f \otimes g\}. \tag{2.2}$$

It had its first appearance in quantum mechanics [6, 7].

The star product can be seen as a higher order f-dependent differential operator acting on the function g. For the example of the Moyal product this is

[1] Note that in this and in the following chapters in the first part of the book the deformation parameter h is absorbed in $\theta^{\rho\sigma}$. Therefore, from now on we refer to $\theta^{\rho\sigma}$ as the deformation parameter.

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$$f \star g = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^{\rho_1\sigma_1} \ldots \theta^{\rho_n\sigma_n} \left(\partial_{\rho_1} \ldots \partial_{\rho_n} f\right) \left(\partial_{\sigma_1} \ldots \partial_{\sigma_n} g\right). \tag{2.3}$$

The differential operator maps the function g to the function $f \star g$.

The inverse map also exists [8, 9]. It $\star$-maps the function g to the function obtained by pointwise multiplying it with f

$$X_f^\star \star g = f \cdot g \tag{2.4}$$

For the Moyal product we obtain

$$X_f^\star = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{2}\right)^n \theta^{\rho_1\sigma_1} \ldots \theta^{\rho_n\sigma_n} \left(\partial_{\rho_1} \ldots \partial_{\rho_n} f\right) \star \partial_{\sigma_1}^\star \ldots \partial_{\sigma_n}^\star. \tag{2.5}$$

The star-acting derivatives we denote by $\partial_\rho^\star$. For the Moyal product the $\star$-derivatives and the usual derivatives are the same. Star differentiation and star differential operators have been thoroughly discussed in Chap. 1 and in [9, 10].

In this chapter we are going to study gauge transformations on Moyal or θ-deformed spaces.[2]

2.2 Gauge transformations

Undeformed infinitesimal gauge transformations are Lie algebra valued:

$$\begin{aligned}
&\delta_\alpha \phi(x) = i\alpha(x)\phi(x), \\
&\alpha(x) = \sum_a \alpha^a(x) T^a, \\
&[T^a, T^b] = i f^{abc} T^c, \\
&[\delta_\alpha, \delta_\beta]\phi = [\alpha, \beta]\phi = -i\delta_{[\alpha,\beta]}\phi,
\end{aligned} \tag{2.6}$$

where $\phi(x)$ is a matter field which belongs to an irreducible representation of the gauge group.

In the previous chapter *deformed* gauge transformations were introduced. Here we analyze them in more detail. They are defined as follows [11, 12]:

$$\delta_\alpha^\star \phi = iX_\alpha^\star \star \phi = iX_{\alpha^a}^\star T^a \star \phi = i\alpha \cdot \phi. \tag{2.7}$$

From the fact that $X_f^\star \star X_g^\star = X_{f \cdot g}^\star$, we conclude

$$\begin{aligned}
&[X_\alpha^\star \stackrel{\star}{,} X_\beta^\star] = X_{-i[\alpha,\beta]}^\star, \\
&[\delta_\alpha^\star, \delta_\beta^\star]\phi = -i\delta_{[\alpha,\beta]}^\star \phi.
\end{aligned} \tag{2.8}$$

[2] A comparison between the present approach to noncommutative gauge theories and an earlier one, so-called Seiberg–Witten map approach, is in Chap. 5.

The $\star$-transformations $\delta^\star_\alpha$ represent the algebra via the usual[3] commutator. However, written in terms of the operators $X^\star_\alpha$ the same algebra is represented via the $\star$-commutator.

Before we construct gauge theories we have to learn how products of fields transform.

In the *undeformed* situation we use, without even thinking, the Leibniz rule:

$$\delta_\alpha(\phi \cdot \psi) = (\delta_\alpha \phi) \cdot \psi + \phi \cdot (\delta_\alpha \psi), \tag{2.9}$$

and we can easily verify that this Leibniz rule is consistent with the Lie algebra:

$$[\delta_\alpha, \delta_\beta](\phi \cdot \psi) = -i\delta_{[\alpha,\beta]}(\phi \cdot \psi). \tag{2.10}$$

For the *deformed* transformation law of a $\star$-product of fields we demand a transformation law that is in the class of transformations defined in (2.7) [8, 9, 11, 13, 14]. This amounts to first decomposing the representation $\phi \star \psi$ for x-independent parameters into its irreducible parts and then follow (2.7) for gauging

$$\delta^\star_\alpha(\phi \star \psi) = iX^\star_{\alpha^a} \star \{T^a\phi \star \psi + \phi \star T^a\psi\}. \tag{2.11}$$

Certainly it is consistent with the Lie algebra:

$$[\delta^\star_\alpha, \delta^\star_\beta](\phi \star \psi) = -i\delta^\star_{[\alpha,\beta]}(\phi \star \psi). \tag{2.12}$$

Because $\phi \star \psi$ is a function we can use the definition of $X^\star_f$ given in (2.4) and simplify (2.11)

$$\delta^\star_\alpha(\phi \star \psi) = i\alpha^a \cdot \{T^a\phi \star \psi + \phi \star T^a\psi\}. \tag{2.13}$$

As α^a does not commute with the $\star$-operation this is different from (2.9). To see this difference more clearly we expand (2.13) in θ

$$\begin{aligned}\delta^\star_\alpha(\phi \star \psi) = i\alpha^a \Big\{ & T^a\phi \cdot \psi + \phi \cdot T^a\psi \\ & + \frac{i}{2}\theta^{\rho\sigma}\left(T^a\partial_\rho\phi \cdot \partial_\sigma\psi + \partial_\rho\phi \cdot T^a\partial_\sigma\psi\right) + O(\theta^2)\Big\}.\end{aligned} \tag{2.14}$$

The final version of the Leibniz rule for the $\star$-product should be entirely expressed with $\star$-operations. Thus we express (2.14) with $\star$-products. A short calculation (see Chap. 1, Sect. 1.6 for details) shows

$$\begin{aligned}\delta^\star_\alpha(\phi \star \psi) &= i(\alpha\phi) \star \psi + i\phi \star (\alpha\psi) \\ & - \frac{i}{2}\theta^{\rho\sigma}\left(i\left((\partial_\rho\alpha)\phi\right) \star (\partial_\sigma\psi) + (\partial_\rho\phi) \star i\left((\partial_\sigma\alpha)\psi\right)\right) + O(\theta^2).\end{aligned} \tag{2.15}$$

[3] Here the usual commutator $[A, B] = AB - BA$ stands in contrast to the $\star$-commutator which is defined in the following way $[A \stackrel{\star}{,} B] = A \star B - B \star A$.

With more work we can prove by induction to all orders in θ the following equation:

$$\begin{aligned}\delta^\star_\alpha(\phi\star\psi) = {} & i(\alpha\phi)\star\psi + i\phi\star(\alpha\psi)\\ & + i\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\right)^n \theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\{(\partial_{\rho_1}\dots\partial_{\rho_n}\alpha)\phi\star(\partial_{\sigma_1}\dots\partial_{\sigma_n}\psi)\\ & + (\partial_{\rho_1}\dots\partial_{\rho_n}\phi)\star(\partial_{\sigma_1}\dots\partial_{\sigma_n}\alpha)\psi\}. \end{aligned} \tag{2.16}$$

This is different from what we obtain by putting just stars in the Leibniz rule (2.9). But this difference has a well-defined meaning if we use the Hopf algebra language to derive the Leibniz rule.

2.3 Hopf algebra techniques

The essential ingredient for a Hopf algebra [15, 16] is the comultiplication $\Delta(\alpha)$: For the *undeformed* situation we define

$$\Delta(\alpha) = \alpha\otimes 1 + 1\otimes\alpha. \tag{2.17}$$

It allows us to write the Leibniz rule (2.9) in the Hopf algebra language:

$$\delta_\alpha(\phi\cdot\psi) = \mu\{\Delta(\alpha)\phi\otimes\psi\}. \tag{2.18}$$

In the *deformed* situation we use a twisted coproduct:

$$\begin{aligned}\Delta_{\mathscr{F}}(\alpha) &= \mathscr{F}(\alpha\otimes 1 + 1\otimes\alpha)\mathscr{F}^{-1},\\ \mathscr{F} &= e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}. \end{aligned} \tag{2.19}$$

Here $\mathscr{F}$ is a twist that has all the properties to define a Hopf algebra with $\Delta_{\mathscr{F}}(\alpha)$ as a comultiplication [17–24]. Details about Hopf algebra methods, twists, and twisted Hopf algebras will be given in Chaps. 7 and 8. We can show that the transformation (2.16) can be written in the form

$$\delta^\star_\alpha(\phi\star\psi) = i\mu_\star\{\Delta_{\mathscr{F}}(\alpha)\phi\otimes\psi\}, \tag{2.20}$$

with the multiplication $\mu_\star$ defined in (2.2). Equation (2.20) defines the Leibniz rule in terms of the twisted comultiplication and the product $\mu_\star$. To show this we start from Eq. (2.13) and write it with the explicit definition of the $\star$-product:

$$\begin{aligned}\delta^\star_\alpha(\phi\star\psi) &= i\alpha^a\mu\left\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}(T^a\phi\otimes\psi + \phi\otimes T^a\psi)\right\}\\ &= i\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n}\left(\alpha^aT^a(\partial_{\rho_1}\dots\partial_{\rho_n}\phi)(\partial_{\sigma_1}\dots\partial_{\sigma_n}\psi)\right.\\ &\quad \left. + (\partial_{\rho_1}\dots\partial_{\rho_n}\phi)\alpha^aT^a(\partial_{\sigma_1}\dots\partial_{\sigma_n}\psi)\right). \end{aligned} \tag{2.21}$$

This we now rewrite as follows:

$$\begin{aligned}\delta_\alpha^\star(\phi\star\psi) &= i\mu(\alpha\otimes 1+1\otimes\alpha)e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\phi\otimes\psi \\ &= i\mu\left\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\cdot e^{-\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}(\alpha\otimes 1+1\otimes\alpha)e^{\frac{i}{2}\theta^{\rho\sigma}\partial_\rho\otimes\partial_\sigma}\phi\otimes\psi\right\} \\ &= i\mu_\star\{\Delta_{\mathscr{F}}(\alpha)\phi\otimes\psi\}. \end{aligned} \tag{2.22}$$

The last line is exactly (2.20).

Gauge fields can be included in this formalism as well. In the *undeformed* situation they are Lie algebra valued, $A_\mu(x)=A_\mu^a(x)T^a$, and under infinitesimal gauge transformations transform as follows:

$$\delta A_\mu = \partial_\mu\alpha + i\alpha^a[T^a, A_\mu]. \tag{2.23}$$

Let us calculate the contribution of the gauge field to the Leibniz rule. As an example we calculate

$$\delta_\alpha^\star(A_\mu\star\phi) = \mu_\star\{\Delta_{\mathscr{F}}(\alpha)A_\mu\otimes\phi\} \tag{2.24}$$

and obtain

$$\begin{aligned}\delta_\alpha^\star(A_\mu\star\psi) &= i\alpha^a\left([T^a,A_\mu]\star\psi\right)+i\alpha^a\left(A_\mu\star T^a\psi\right)+(\partial_\mu\alpha^a)T^a\psi \\ &= i\alpha^a\left((T^aA_\mu)\star\psi-(A_\mu T^a)\star\psi\right) \\ &\quad +i\alpha^a(A_\mu T^a)\star\psi+(\partial_\mu\alpha^a)T^a\psi \\ &= i\alpha^aT^a(A_\mu\star\psi)+(\partial_\mu\alpha)\psi. \end{aligned} \tag{2.25}$$

Now we define a covariant derivative

$$D_\mu^\star\psi = \partial_\mu\psi - iA_\mu\star\psi. \tag{2.26}$$

It will transform covariantly

$$\delta_\alpha^\star(D_\mu^\star\psi) = i\alpha^aT^a(D_\mu^\star\psi) = iX_{\alpha^a}^\star\star T^a(D_\mu^\star\psi), \tag{2.27}$$

if the vector field A_μ transforms as in (2.23)

$$\delta_\alpha^\star A_\mu = \partial_\mu\alpha + i\alpha^a[T^a,A_\mu] = \partial_\mu\alpha + iX_{\alpha^a}^\star\star[T^a,A_\mu]. \tag{2.28}$$

From (2.28) we see that a Lie algebra valued vector field remains Lie algebra valued by the transformation (2.28).

2.4 Field equations

Now we proceed as in the *undeformed* situation. First we define the field strength tensor:

$$F_{\mu\nu} = i[D_\mu^\star \stackrel{\star}{,} D_\nu^\star]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu]. \tag{2.29}$$

Here we see already that $F_{\mu\nu}$ will not be Lie algebra valued even for Lie algebra-valued vector fields. Namely, assuming that the gauge field is Lie algebra valued $A_\mu = A_\mu^a T^a$ the field strength tensor $F_{\mu\nu}$ (2.29) can be decomposed in two parts

$$F_{\mu\nu} = F^a_{1\mu\nu} T^a + F^{ab}_{2\mu\nu} \frac{1}{2}\{T^a, T^b\}. \tag{2.30}$$

Since anticommutator of generators $\{T^a, T^b\}$ is not Lie algebra valued in general, the full $F_{\mu\nu}$ will not be Lie algebra valued in general.

Using the twisted gauge transformations of the gauge field A_μ (2.28) and the deformed Leibniz rule (2.16) we derive the transformation law of the field strength tensor:

$$\delta^\star_\alpha F_{\mu\nu} = i X^\star_{\alpha^a} \star [T^a, F_{\mu\nu}] = i[\alpha, F_{\mu\nu}]. \tag{2.31}$$

The expression $F^{\mu\nu} \star F_{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$ will transform accordingly

$$\delta^\star_\alpha (F^{\mu\nu} \star F_{\mu\nu}) = i X^\star_{\alpha^a} \star [T^a, F^{\mu\nu} \star F_{\mu\nu}] = i[\alpha, F^{\mu\nu} \star F_{\mu\nu}]. \tag{2.32}$$

Hint, use the transformation law (2.31) and the deformed Leibniz rule (2.16).

The Lagrangian that is invariant under the twisted gauge transformations (2.28) we define as in the gauge theory on commutative space:

$$\mathscr{L} = \frac{1}{c} \mathrm{Tr}(F^{\mu\nu} \star F_{\mu\nu}), \tag{2.33}$$

where c is an arbitrary constant. It is invariant and it is a deformation[4] of the *undeformed* Lagrangian of a gauge theory.

To speak about an action we have to define integration. We take the usual integral over x on the commutative space and we can verify that

$$\int \mathrm{d}^4x\, f \star g = \int \mathrm{d}^4x\, g \star f = \int \mathrm{d}^4x\, f \cdot g \tag{2.34}$$

by partial integration. This is called the trace property of the integral or cyclicity .

Equation (2.34) allows a cyclic permutation of the fields under the integral. To derive the field equations we use the usual Leibniz rule for the functional variation, that is, we vary the field where it stands. The trace property is then used to derive the final result. As an example we look at the action for the gauge field

$$S = \frac{1}{c} \int \mathrm{d}^4x\, \mathrm{Tr}(F^{\mu\nu} \star F_{\mu\nu}). \tag{2.35}$$

[4] One can expand the $\star$-products appearing in the Lagrangian (2.33) and check that in the zeroth order in the deformation parameter $\theta^{\rho\sigma}$ the Lagrangian of the undeformed theory is obtained. Higher order terms give new contributions due to the noncommutativity (deformation) of the commutative space.

From the trace property we compute

$$
\begin{aligned}
\frac{\delta S}{\delta A_\rho(z)} &= \frac{1}{c}\frac{\delta}{\delta A_\rho(z)}\int \mathrm{d}^4x\ \mathrm{Tr}(F^{\mu\nu}\star F_{\mu\nu}) \\
&= \frac{1}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\left(\left(\frac{\delta F^{\mu\nu}(x)}{\delta A_\rho(z)}\right)\star F_{\mu\nu} + F^{\mu\nu}\star\left(\frac{\delta F_{\mu\nu}(x)}{\delta A_\rho(z)}\right)\right) \\
&= \frac{2}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\frac{\delta F_{\mu\nu}(x)}{\delta A_\rho(z)}\star F^{\mu\nu}(x) \qquad (2.36)\\
&= \frac{2}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\frac{\delta}{\delta A_\rho(z)}(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu])\star F^{\mu\nu}(x) \\
&= \frac{4}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\frac{\delta}{\delta A_\rho(z)}(\partial_\mu A_\nu - iA_\mu\star A_\nu)\star F^{\mu\nu}(x)
\end{aligned}
$$

because $F^{\mu\nu}$ is antisymmetric. Then we have

$$
\begin{aligned}
\frac{\delta S}{\delta A_\rho(z)} &= \frac{4}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\{-\delta^{(4)}(x-z)\star(\partial_\mu F^{\mu\rho}) \\
&\quad -i\delta^{(4)}(x-z)\star A_\mu\star F^{\rho\mu} - iA_\mu\star\delta^{(4)}(x-z)\star F^{\mu\rho}\} \qquad (2.37)\\
&= -\frac{4}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\delta^{(4)}(x-z)\star\{\partial_\mu F^{\mu\rho} - iA_\mu\star F^{\mu\rho} + iF^{\mu\rho}\star A_\mu\}.
\end{aligned}
$$

The field equations follow after using (2.34)

$$
\frac{\delta S}{\delta A_\rho(z)} = -\frac{4}{c}\int \mathrm{d}^4x\ \mathrm{Tr}\delta^{(4)}(x-z)\{\partial_\mu F^{\mu\rho} - iA_\mu\star F^{\mu\rho} + iF^{\mu\rho}\star A_\mu\}. \qquad (2.38)
$$

These are exactly the equations we have expected from covariance:

$$
D^\star_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - i[A_\mu \stackrel{\star}{,} F^{\mu\nu}] = 0. \qquad (2.39)
$$

We have already seen that $F_{\mu\nu}$ cannot be Lie algebra valued. From the field equations (2.39), considered as equations for the vector potential A_μ, we see that A_μ cannot be Lie algebra valued either. We have to consider $F_{\mu\nu}$ and A_μ to be enveloping algebra valued. The additional vector fields (coming from the non-Lie algebra-valued parts) will introduce additional ghosts in the Lagrangian. To eliminate them we have to enlarge the symmetry to be enveloping algebra valued as well. For simplicity we assume α, A_μ, and $F_{\mu\nu}$ to be matrix valued when the matrices act in the representation space of T^a.

From the field equations (2.39) follows a consistency equation because $F^{\mu\nu}$ is antisymmetric in μ and ν:

$$
\partial_\nu[A_\mu \stackrel{\star}{,} F^{\mu\nu}] = 0. \qquad (2.40)
$$

To verify this condition we have to use the field equations (2.39). First we differentiate (2.40)

$$
\partial_\nu[A_\mu \stackrel{\star}{,} F^{\mu\nu}] = [\partial_\nu A_\mu \stackrel{\star}{,} F^{\mu\nu}] + [A_\mu \stackrel{\star}{,} \partial_\nu F^{\mu\nu}]. \qquad (2.41)
$$

In the first term we replace $\partial_\nu A_\mu$ by $\frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu)$ because $F_{\mu\nu}$ is antisymmetric in μ and ν. Then we express this term by $F_{\mu\nu}$ according to (2.29):

$$\frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu) = \frac{i}{2}F_{\nu\mu} + \frac{i}{2}[A_\nu \stackrel{\star}{,} A_\mu]. \tag{2.42}$$

The $\star$-commutator $[F^{\mu\nu} \stackrel{\star}{,} F_{\mu\nu}] = F^{\mu\nu} \star F_{\mu\nu} - F_{\mu\nu} \star F^{\mu\nu}$ vanishes and we are left with $\frac{i}{2}[[A_\nu \stackrel{\star}{,} A_\mu] \stackrel{\star}{,} F^{\mu\nu}]$ for the first term in (2.41). For the second term in (2.41) we use the field equations (2.39). Finally all terms left add up to zero if we use the Jacobi identity. In all these equations A_μ and $F_{\mu\nu}$ are supposed to be matrices. We have suppressed the matrix indices.

A conserved current is found

$$j^\nu = [A_\mu \stackrel{\star}{,} F^{\mu\nu}], \qquad \partial_\nu j^\nu = 0. \tag{2.43}$$

For $\theta^{\rho\sigma} = 0$ this is the current of a non-abelian gauge theory **on commutative space**.

2.5 Matter fields

Matter fields can be coupled covariantly to the gauge fields via a covariant derivative. We start from a multiplet of the gauge group ψ_A not necessarily irreducible. The index A denotes the component of the field ψ in the representation space. The transformation law of ψ is $\delta_\alpha^\star \psi_A = iX^\star_{\alpha_{AB}} \star \psi_B = i\alpha_{AB}\psi_B$. For the usual gauge transformations α_{AB} will be Lie algebra valued. The covariant derivative is

$$(D_\mu^\star \psi)_A = \partial_\mu \psi_A - iA_{\mu AB} \star \psi_B. \tag{2.44}$$

The gauge potential A_μ in now supposed to be matrix valued in the representation space spanned by the matter fields.

For a spinor field

$$\bar{\psi}_{\alpha A} \star \gamma^\mu_{\alpha\beta}(D_\mu^\star \psi)_A \tag{2.45}$$

will be invariant and therefore suitable for a covariant Lagrangian.

We consider the Lagrangian

$$\mathscr{L} = \frac{1}{c}\mathrm{Tr}(F^{\mu\nu} \star F_{\mu\nu}) + \bar{\psi} \star \gamma^\mu (i\partial_\mu + A_\mu \star)\psi - m\bar{\psi} \star \psi. \tag{2.46}$$

We have suppressed the matrix indices.

The field equations are obtained from (2.46) by varying the fields in the same way as in Sect. 2.4:

$$\frac{\delta \mathscr{L}}{\delta A_\rho} = \partial_\mu F^{\mu\rho}_{AB} + i[A_\mu \stackrel{\star}{,} F^{\rho\mu}]_{AB} + \gamma^\rho_{\alpha\beta}\psi_{\beta A} \star \bar{\psi}_{\alpha B} = 0, \tag{2.47}$$

and for the matter fields

$$\begin{aligned}\frac{\delta \mathscr{L}}{\delta \bar{\psi}} &= \gamma^{\mu}(\partial_{\mu}\psi_A - iA_{\mu AB}\star\psi_B) + im\psi_A = 0 \\ \frac{\delta \mathscr{L}}{\delta \psi} &= (\partial_{\mu}\bar{\psi}_A\gamma^{\mu} + i\bar{\psi}_B\gamma^{\mu}\star iA_{\mu AB}) - im\bar{\psi}_A = 0.\end{aligned} \tag{2.48}$$

Again, Eq. (2.47) leads to a consistency relation that can be verified with the help of the field equations. It is, however, important that the representation space for the field ψ and the vector potential $A_{\mu AB}$ are the same. The representation space of the matter fields determines the space for the gauge potentials.

We conclude that there is a conserved current:

$$j^{\rho}_{AB} = i[A_{\mu} \stackrel{\star}{,} F^{\mu\rho}]_{AB} - \gamma^{\rho}_{\alpha\beta}\psi_{\beta A}\star\bar{\psi}_{\alpha B}. \tag{2.49}$$

We were again able to find a conserved current as a consequence of a deformed symmetry. Even if we put the vector potential to zero there remains the part from the matter field. There are conservation laws due to a deformed symmetry. It is remarkable that we have found conserved currents in the twisted theory as well. In the *undeformed* theory we can derive them with the help of the Noether theorem. In the deformed theory this is not possible. Nevertheless the property that a theory has a conserved current is preserved by a deformation. This is an important step to convince ourselves that a deformed gauge theory has properties close to what we need for physics.

2.6 Examples

1) Maxwell equations

We start from the simplest gauge theory based on U(1) and describing gauge fields only. We proceed schematically. The transformation law of the gauge field A_{μ}:

$$\delta^{\star}_{\alpha}A_{\mu} = \partial_{\mu}\alpha. \tag{2.50}$$

The covariant derivative:

$$D^{\star}_{\mu} = \partial_{\mu} - iA_{\mu}\star. \tag{2.51}$$

The field strength tensor:

$$F_{\mu\nu} = [D^{\star}_{\mu} \stackrel{\star}{,} D^{\star}_{\nu}] = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - i[A_{\mu} \stackrel{\star}{,} A_{\nu}]. \tag{2.52}$$

The Lagrangian:

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}\star F_{\mu\nu}. \tag{2.53}$$

The field equations:

$$\partial^\mu F_{\mu\nu} - i[A^\mu \overset{\star}{,} F_{\mu\nu}] = 0. \tag{2.54}$$

Consistency equations:

$$\partial^\nu [A^\mu \overset{\star}{,} F_{\mu\nu}] = 0. \tag{2.55}$$

A schematic proof of the consistency condition:

$$[\partial^\nu A^\mu \overset{\star}{,} F_{\mu\nu}] + [A^\mu \overset{\star}{,} \partial^\nu F_{\mu\nu}] = \tag{2.56}$$

$$= \frac{i}{2}[[A^\nu \overset{\star}{,} A^\mu] \overset{\star}{,} F_{\mu\nu}] + i[A^\mu \overset{\star}{,} [A^\nu \overset{\star}{,} F_{\mu\nu}]]. \tag{2.57}$$

We have used the field equations and the fact that $[F_{\mu\nu} \overset{\star}{,} F^{\mu\nu}] = 0$. The terms left can now be rearranged

$$[[A^\nu \overset{\star}{,} A^\mu] \overset{\star}{,} F_{\mu\nu}] + [[A^\mu \overset{\star}{,} F_{\mu\nu}] \overset{\star}{,} A^\nu] + [[F_{\mu\nu} \overset{\star}{,} A^\nu] \overset{\star}{,} A^\mu] \tag{2.58}$$

and vanish due to the Jacobi identity.

We found a conserved current:

$$j_\nu = [A^\mu \overset{\star}{,} F_{\mu\nu}], \qquad \partial_\nu j^\nu = 0. \tag{2.59}$$

2) Electrodynamics with one charged spinor field

Transformation law of the gauge field and the spinor field:

$$\delta^\star_\alpha \psi = i\alpha\psi, \quad \delta^\star_\alpha A_\mu = \partial_\mu \alpha. \tag{2.60}$$

Covariant derivative:

$$D^\star_\mu = (\partial_\mu - iA_\mu \star), \quad D^\star_\mu \psi = (\partial_\mu - iA_\mu \star)\psi. \tag{2.61}$$

Field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \overset{\star}{,} A_\nu]. \tag{2.62}$$

Lagrangian:

$$\mathscr{L} = -\frac{1}{4} F^{\mu\nu} \star F_{\mu\nu} + \bar\psi \star \gamma^\mu (i\partial_\mu \psi + A_\mu \star \psi) - m\bar\psi \star \psi. \tag{2.63}$$

Field equations:

$$\begin{aligned} &\partial_\mu F^{\mu\rho} + i[A_\mu \overset{\star}{,} F^{\rho\mu}] + \gamma^\rho \psi \star \bar\psi = 0, \\ &\gamma^\mu(\partial_\mu \psi) - i\gamma^\mu A_\mu \star \psi + im\psi = 0, \\ &(\partial_\mu \bar\psi)\gamma^\mu + i\bar\psi\gamma^\mu \star A_\nu - im\bar\psi = 0. \end{aligned} \tag{2.64}$$

Consistency condition:

$$\partial_\rho([A_\mu \overset{\star}{,} F^{\rho\mu}] + \gamma^\rho \psi \star \bar{\psi}) = 0. \tag{2.65}$$

Proof: As before, the spinor terms have to be added in the current and the field equations.

Current:

$$j^\rho = [A_\nu \overset{\star}{,} F^{\rho\nu}] + \gamma^\rho \psi \star \bar{\psi}, \quad \partial_\nu j^\nu = 0. \tag{2.66}$$

3) Electrodynamics with several charged fields

We try to formulate a model with one vector potential and differently charged matter fields as we do in the undeformed situation. This amounts to introduce an $U(1)$ gauge-invariant action for the gauge potential and for the matter fields.

Let us consider the part of the vector potential first.

The transformation law is

$$\delta_\alpha^\star A_\mu = \partial_\mu \alpha. \tag{2.67}$$

The covariant derivative

$$D_\mu^\star = (\partial_\mu - iA_\mu \star) \tag{2.68}$$

gives the following field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \overset{\star}{,} A_\nu]. \tag{2.69}$$

As an invariant Lagrangian we choose

$$\mathcal{L}_A = -\frac{1}{4} F^{\mu\nu} \star F_{\mu\nu}. \tag{2.70}$$

Next we consider the matter fields ψ^r with charges g_r, $r = 1, \ldots, n$. They transform as follows:

$$\delta_\alpha^\star \psi^r = i g_r \alpha \psi^r. \tag{2.71}$$

The covariant derivative depends on the charge of the field it acts on:

$$D_\mu^\star \psi^r = (\partial_\mu - i g_r A_\mu \star) \psi^r. \tag{2.72}$$

The $U(1)$ gauge-invariant action can be chosen as follows:

$$\mathcal{L}_\psi = \sum_r \bar{\psi}^r \star \gamma^\mu \left(i(\partial_\mu \psi) + g_r A_\mu \star \psi^r \right) - m_r \bar{\psi}^r \star \psi^r. \tag{2.73}$$

As the total Lagrangian we take the sum

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_\psi. \tag{2.74}$$

It is $U(1)$ gauge invariant and it is a deformation of the usual electrodynamics with different charged fields. This Lagrangian now leads to the field equations:

$$
\begin{aligned}
&\partial_\mu F^{\mu\rho} + i[A_\mu \stackrel{\star}{,} F^{\rho\mu}] + \sum_r g_r \gamma^\rho \psi^r \star \bar{\psi}^r = 0,\\
&\gamma^\mu (\partial_\mu \psi) - i g_r \gamma^\mu A_\mu \star \psi + i m_r \psi^r = 0, \qquad (2.75)\\
&\partial_\mu \bar{\psi}^r \gamma^\mu + i \bar{\psi}^r \gamma^\mu \star g^r A_\nu - i m_r \bar{\psi}^r = 0.
\end{aligned}
$$

The first of these equations gives rise to a consistency condition:

$$
\partial_\rho \left(i[A_\nu \stackrel{\star}{,} F^{\rho\nu}] + \sum_r g_r \gamma^\rho \psi^r \star \bar{\psi}^r \right) = 0. \qquad (2.76)
$$

From a direct calculation, using the field equations, follows:

$$
\partial_\rho \left(i[A_\nu \stackrel{\star}{,} F^{\rho\nu}] + \sum_r g_r \gamma^\rho \psi^r \star \bar{\psi}^r \right) \qquad (2.77)
$$

$$
= - \sum_r (g_r^2 - g_r)[A_\mu \stackrel{\star}{,} \gamma^\mu \psi^r \star \bar{\psi}^r]. \qquad (2.78)
$$

The consistency condition is only satisfied if $g_r = g_r^2$ or $g_r = 1$. With one vector potential we can in a U(1) model only describe particles with one charge. There can be an arbitrary number of matter fields with this charge. This is different from the usual undeformed situation. There the commutator in (2.69) vanishes and does not give rise to an inconsistency.

This is not surprising, we forgot that the vector potential has at least to be enveloping algebra valued. This is demonstrated in the next example.

4) Electrodynamics of a positive and a negative charged matter field

The gauge group is supposed to be $U(1)$ and the matter fields are in the multiplet that transforms as follows:

$$
\delta_\alpha^\star \psi = i\alpha Q \psi, \qquad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2.79)
$$

As outlined in Sect. 2.5, the gauge potential has to be in the same representation of the enveloping algebra as the matter fields are.

The enveloping algebra has two elements:

$$
I \text{ and } Q, \quad Q^2 = 1. \qquad (2.80)
$$

We generalize the transformation law (2.79) to be enveloping algebra valued

$$
\delta_\Lambda \psi = i\Lambda \psi, \qquad \Lambda = \lambda_0(x) I + \lambda_1(x) Q. \qquad (2.81)
$$

The vector potential A_μ has the analogous decomposition

$$
\mathscr{A}_\mu = A_\mu(x) I + B_\mu(x) Q. \qquad (2.82)
$$

The covariant derivative is

$$D^\star_\mu \psi = (\partial_\mu - i\mathscr{A}_\mu \star)\psi = (\partial_\mu - iA_\mu(x)\star I - iB_\mu(x)\star Q)\psi. \tag{2.83}$$

The field strength can also be decomposed in the enveloping algebra

$$\mathscr{F}_{\mu\nu} = F_{\mu\nu} I + G_{\mu\nu} Q. \tag{2.84}$$

From the definition of the field strength

$$\mathscr{F}_{\mu\nu} = \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu - i[\mathscr{A}_\mu \stackrel{\star}{,} \mathscr{A}_\nu], \tag{2.85}$$

follows

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu] - i[B_\mu \stackrel{\star}{,} B_\nu], \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu - i[A_\mu \stackrel{\star}{,} B_\nu] - i[B_\mu \stackrel{\star}{,} A_\nu]. \end{aligned} \tag{2.86}$$

The matter fields couple to the vector potential via the covariant derivative

$$\begin{aligned} D^\star_\mu \psi &= (\partial_\mu - i\mathscr{A}_\mu \star)\psi \\ &= (\partial_\mu - iA_\mu(x)\star I - iB_\mu(x)\star Q)\psi. \end{aligned} \tag{2.87}$$

This leads to the Lagrangian

$$\mathscr{L} = -\frac{1}{4}\mathscr{F}^{\mu\nu}\star\mathscr{F}_{\mu\nu} + \bar{\psi}\star\gamma^\mu\left(i(\partial_\mu\psi) + \mathscr{A}_\mu\star\psi\right) - m\bar{\psi}\star\psi \tag{2.88}$$

and the field equations

$$\begin{aligned} \frac{\delta\mathscr{L}}{\delta A_\rho}: &\quad \partial_\mu F^{\mu\rho} + i[A_\mu \stackrel{\star}{,} F^{\rho\mu}] + i[B_\mu \stackrel{\star}{,} G^{\rho\mu}] + i\gamma^\rho\psi\star\bar{\psi} = 0, \\ \frac{\delta\mathscr{L}}{\delta B_\rho}: &\quad \partial_\mu G^{\mu\rho} + i[B_\mu \stackrel{\star}{,} F^{\rho\mu}] + i[A_\mu \stackrel{\star}{,} G^{\rho\mu}] + i\gamma^\rho\psi_A\star\bar{\psi}_B Q^{AB} = 0, \\ \frac{\delta\mathscr{L}}{\delta\bar{\psi}}: &\quad \gamma^\mu(\partial_\mu\psi) - i\gamma^\mu\mathscr{A}_\mu\star\psi + m\psi = 0, \\ \frac{\delta\mathscr{L}}{\delta\psi}: &\quad \partial_\mu\bar{\psi}\gamma^\mu + i\bar{\psi}\gamma^\mu\star\mathscr{A}_\mu - m\bar{\psi} = 0. \end{aligned} \tag{2.89}$$

We obtain two consistency equations that render two transformation laws, in agreement with the extended symmetry (2.81)

$$j^\rho_A = i[A_\mu \stackrel{\star}{,} F^{\rho\mu}] + i[B_\mu \stackrel{\star}{,} G^{\rho\mu}] + \gamma^\rho\psi_A\star\bar{\psi}_A, \tag{2.90}$$

with

$$\partial_\rho j^\rho_A = 0 \tag{2.91}$$

and

$$j^\rho_B = i[B_\mu \stackrel{\star}{,} F^{\rho\mu}] + i[A_\mu \stackrel{\star}{,} G^{\rho\mu}] - i\gamma^\rho\psi_A\star\bar{\psi}_B Q^{AB}. \tag{2.92}$$

We learn that the deformed gauge theory leads to a theory with a larger symmetry structure, the enveloping algebra structure. This structure survives in the limit $\theta \to 0$. We find the corresponding conservation laws and gauge transformations needed for a consistent gauge theory.

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