Chapter 2
Waves in Piles

In this chapter the problem of the propagation of compression waves in piles is studied. This problem is of importance when considering the behaviour of a foundation pile and the soil during pile driving, and under dynamic loading, such as the behaviour of a pile in the foundation of a railway bridge. Because of the one-dimensional character of the problem, and the simple shape of the pile, usually having a constant cross section and a long length, this is one of the simplest problems of wave propagation in a mathematical sense, and therefore it may be used to illustrate some of the main characteristics of engineering dynamics. Several methods of analysis will be used: the Laplace transform method, separation of variables, the method of characteristics, and numerical solution methods.

2.1 One-Dimensional Wave Equation

First, the case of a free standing pile will be considered, ignoring the interaction with the soil. In later sections the friction interaction with the surrounding soil, and the interaction with the soil at the base will be considered.

Consider a pile of constant cross sectional area A, consisting of a linear elastic material, with modulus of elasticity E. If there is no friction along the shaft of the pile the equation of motion of an element is

$$\frac{\partial N}{\partial z} = \rho A \frac{\partial^2 w}{\partial t^2}, \tag{2.1}$$

where ρ is the mass density of the material, and w is the displacement in axial direction. The normal force N is related to the stress by

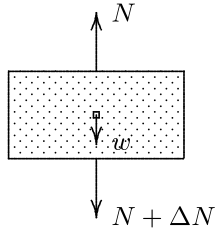
$$N = \sigma A,$$

and the stress is related to the strain by Hooke's law for the pile material

$$\sigma = E\varepsilon.$$

A. Verruijt, *An Introduction to Soil Dynamics*,
Theory and Applications of Transport in Porous Media 24,

Fig. 2.1 Element of pile



Finally, the strain is related to the vertical displacement w by the relation

$$\varepsilon = \partial w/\partial z.$$

Thus the normal force N is related to the vertical displacement w by the relation

$$N = EA\frac{\partial w}{\partial z}. \tag{2.2}$$

Substitution of (2.2) into (2.1) gives

$$E\frac{\partial^2 w}{\partial z^2} = \rho\frac{\partial^2 w}{\partial t^2}. \tag{2.3}$$

This is the *wave equation*. It can be solved analytically, for instance by the Laplace transform method, separation of variables, or by the method of characteristics, or it can be solved numerically. All these techniques are presented in this chapter. The analytical solution will give insight into the behaviour of the solution. A numerical model is particularly useful for more complicated problems, involving friction along the shaft of the pile, and non-uniform properties of the pile and the soil.

2.2 Solution by Laplace Transform Method

Many problems of one-dimensional wave propagation can be solved conveniently by the Laplace transform method (Churchill, 1972), see also Appendix A. Some examples of this technique are given in this section.

2.2.1 Pile of Infinite Length

The Laplace transform of the displacement w is defined by

$$\overline{w}(z,s) = \int_0^\infty w(z,t)\,\exp(-st)\,dt, \tag{2.4}$$

where s is the Laplace transform parameter, which can be assumed to have a positive real part. Now consider the problem of a pile of infinite length, which is initially at rest, and on the top of which a constant pressure is applied, starting at time $t = 0$. The Laplace transform of the differential equation (2.3) now is

$$\frac{d^2\overline{w}}{dz^2} = \frac{s^2}{c^2}\overline{w}, \tag{2.5}$$

where c is the wave velocity,

$$c = \sqrt{E/\rho}. \tag{2.6}$$

The solution of the ordinary differential equation (2.5) that vanishes at infinity is

$$\overline{w} = A\exp(-sz/c). \tag{2.7}$$

The integration constant A, which may depend upon the transformation parameter s, can be obtained from the boundary condition. For a constant pressure p_0 applied at the top of the pile this boundary condition is

$$z = 0,\ t > 0\ :\ E\frac{\partial w}{\partial z} = -p_0. \tag{2.8}$$

The Laplace transform of this boundary condition is

$$z = 0\ :\ E\frac{d\overline{w}}{dz} = -\frac{p_0}{s}. \tag{2.9}$$

With (2.7) the value of the constant A can now be determined. The result is

$$A = \frac{pc}{Es^2}, \tag{2.10}$$

so that the final solution of the transformed problem is

$$\overline{w} = \frac{pc}{Es^2}\exp(-sz/c). \tag{2.11}$$

The inverse transform of this function can be found in elementary tables of Laplace transforms, see for instance Abramowitz and Stegun (1964) or Churchill (1972). The final solution now is

$$w = \frac{pc(t - z/c)}{E}H(t - z/c), \tag{2.12}$$

where $H(t - t_0)$ is Heaviside's unit step function, defined as

$$H(t - t_0) = \begin{cases} 0, & \text{if } t < t_0, \\ 1, & \text{if } t > t_0. \end{cases} \tag{2.13}$$

The solution (2.12) indicates that a point in the pile remains at rest as long as $t < z/c$. From that moment on (this is the moment of arrival of the wave) the point starts to move, with a linearly increasing displacement, which represents a constant velocity.

It may seem that this solution is in disagreement with Newton's second law, which states that the velocity of a mass point will linearly increase in time when a constant force is applied. In the present case the velocity is constant. The moving mass gradually increases, however, so that the results are really in agreement with Newton's second law: the momentum (mass times velocity) linearly increases with time. Actually, Newton's second law is the basic principle involved in deriving the basic differential equation (2.3), so that no disagreement is possible, of course.

2.2.2 *Pile of Finite Length*

The Laplace transform method can also be used for the analysis of waves in piles of finite length. Many solutions can be found in the literature (Churchill, 1972; Carslaw and Jaeger, 1948). An example will be given below.

Consider the case of a pile of finite length, say h, see Fig. 2.2. The boundary $z = 0$ is free of stress, and the boundary $z = h$ undergoes a sudden displacement, at time $t = 0$. Thus the boundary conditions are

$$z = 0,\ t > 0 : \frac{\partial w}{\partial z} = 0, \tag{2.14}$$

and

$$z = h,\ t > 0 : w = w_0. \tag{2.15}$$

The general solution of the transformed differential equation

$$\frac{d^2\overline{w}}{dz^2} = \frac{s^2}{c^2}\,\overline{w}, \tag{2.16}$$

is

$$\overline{w} = A\exp(sz/c) + B\exp(-sz/c). \tag{2.17}$$

The constants A and B (which may depend upon the Laplace transform parameter s) can be determined from the transforms of the boundary conditions (2.14) and (2.15). The result is

$$\overline{w} = \frac{w_0}{s}\frac{\cosh(sz/c)}{\cosh(sh/c)}. \tag{2.18}$$

The mathematical problem now remaining is to find the inverse transform of this expression. This can be accomplished by using the complex inversion integral

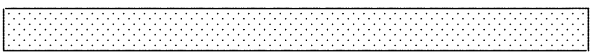


Fig. 2.2 Pile of finite length

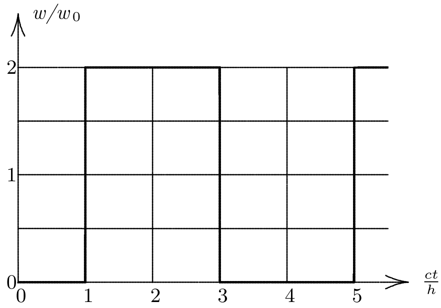


Fig. 2.3 Displacement of free end

(Churchill, 1972), or its simplified form, the Heaviside expansion theorem, see Appendix A. This gives, after some elementary mathematical analysis,

$$\frac{w}{w_0} = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\cos\left[(2k+1)\frac{\pi z}{2h}\right]\cos\left[(2k+1)\frac{\pi ct}{2h}\right]. \tag{2.19}$$

As a special case one may consider the displacement of the free end $z = 0$. This is found to be

$$\frac{w}{w_0} = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\cos\left[(2k+1)\frac{\pi ct}{2h}\right]. \tag{2.20}$$

This expression is of the form of a Fourier series. Actually, it is the same series as the one given in the example in Appendix A, except for a constant factor and some changes in notation. The summation of the series is shown in Fig. 2.3.

It appears that the free end remains at rest for a time h/c, then suddenly shows a displacement $2w_0$ for a time span $2h/c$, and then switches continuously between zero displacement and $2w_0$. The physical interpretation, which may become more clear after considering the solution of the problem by the method of characteristics in a later section, is that a compression wave starts to travel at time $t = 0$ towards the free end, and then is reflected as a tension wave in order that the end remains free. The time h/c is the time needed for a wave to travel through the entire length of the pile.

2.3 Separation of Variables

For certain problems, especially problems of continuous vibrations, the differential equation (2.3) can be solved conveniently by a method known as *separation of variables*. Two examples will be considered in this section.

2.3.1 Shock Wave in Finite Pile

As an example of the general technique used in the method of separation of variables the problem of a pile of finite length loaded at time $t = 0$ by a constant displacement at one of its ends will be considered once more. The differential equation is

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial z^2}, \tag{2.21}$$

with the boundary conditions

$$z = 0,\ t > 0\ :\ \frac{\partial w}{\partial z} = 0, \tag{2.22}$$

and

$$z = h,\ t > 0\ :\ w = w_0. \tag{2.23}$$

The first condition expresses that the boundary $z = 0$ is a free end, and the second condition expresses that the boundary $z = h$ is displaced by an amount w_0 at time $t = 0$. The initial conditions are supposed to be that the pile is at rest at $t = 0$.

The solution of the problem is now sought in the form

$$w = w_0 + Z(z)T(t). \tag{2.24}$$

The basic assumption here is that solutions can be written as a product of two functions, a function $Z(z)$, which depends upon z only, and another function $T(t)$, which depends only on t. Substitution of (2.24) into the differential equation (2.21) gives

$$\frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2} = \frac{1}{Z}\frac{d^2 Z}{dz^2}. \tag{2.25}$$

The left hand side of this equation depends upon t only, the right hand side depends upon z only. Therefore the equation can be satisfied only if both sides are equal to a certain constant. This constant may be assumed to be negative or positive. If it is assumed that this constant is negative one may write

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\lambda^2, \tag{2.26}$$

where λ is an unknown constant. The general solution of (2.26) is

$$Z = C_1 \cos(\lambda z) + C_2 \sin(\lambda z), \tag{2.27}$$

where C_1 and C_2 are constants. They can be determined from the boundary conditions. Because dZ/dz must be 0 for $z = 0$ it follows that $C_2 = 0$. If now it is required that $Z = 0$ for $z = h$, in order to satisfy the boundary condition (2.23), it follows that a non-zero solution can be obtained only if $\cos(\lambda h) = 0$, which can be satisfied if

$$\lambda = \lambda_k = (2k+1)\frac{\pi}{2h}, \quad k = 0, 1, 2, \dots . \tag{2.28}$$

On the other hand, one obtains for the function T

$$\frac{1}{T}\frac{d^2T}{dt^2} = -c^2\lambda^2, \tag{2.29}$$

with the general solution

$$T = A\cos(\lambda ct) + B\sin(\lambda ct). \tag{2.30}$$

The solution for the displacement w can now be written as

$$w = w_0 + \sum_{k=0}^{\infty}\big[A_k\cos(\lambda_k ct) + B_k\sin(\lambda_k ct)\big]\cos(\lambda_k z). \tag{2.31}$$

The velocity now is

$$\frac{\partial w}{\partial t} = \sum_{k=0}^{\infty}\big[-A_k\lambda_k c\sin(\lambda_k ct) + B_k\lambda_k c\cos(\lambda_k ct)\big]\cos(\lambda_k z). \tag{2.32}$$

Because this must be zero for $t = 0$ and all values of z, to satisfy the initial condition of rest, it follows that $B_k = 0$. Furthermore, the initial condition that the displacement must also be zero for $t = 0$, now leads to the equation

$$\sum_{k=0}^{\infty} A_k\cos(\lambda_k z) = -w_0, \tag{2.33}$$

which must be satisfied for all values of z in the range $0 < z < h$. This is the standard problem from Fourier series analysis, see Appendix A. It can be solved by multiplication of both sides by $\cos(\lambda_j z)$, and then integrating both sides over z from $z = 0$ to $z = h$. The result is

$$A_k = \frac{4}{\pi}\frac{w_0}{(2k+1)}(-1)^k. \tag{2.34}$$

Substitution of this result into the solution (2.31) now gives finally, with $B_k = 0$,

$$\frac{w}{w_0} = 1 + \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}\cos\left[(2k+1)\frac{\pi z}{2h}\right]\cos\left[(2k+1)\frac{\pi ct}{2h}\right]. \tag{2.35}$$

This is exactly the same result as found earlier by using the Laplace transform method, see (2.19). It may give some confidence that both methods lead to the same result.

The solution (2.35) can be seen as a summation of periodic solutions, each combined with a particular shape function. Usually a periodic function is written as $\cos(\omega t)$. In this case it appears that the possible frequencies are

$$\omega = \omega_k = (2k+1)\frac{\pi c}{2h}, \quad k = 0, 1, 2, \ldots. \tag{2.36}$$

These are usually called the *characteristic frequencies*, or *eigen frequencies* of the system. The corresponding shape functions

$$\psi_k(z) = \cos\left[(2k+1)\frac{\pi z}{2h}\right], \quad k = 0, 1, 2, \ldots, \tag{2.37}$$

are the *eigen functions* of the system.

2.3.2 Periodic Load

The solution is much simpler if the load is periodic, because then it can be assumed that all displacements are periodic. As an example the problem of a pile of finite length, loaded by a periodic load at one end, and rigidly supported at its other end, will be considered, see Fig. 2.4. In this case the boundary conditions at the left side boundary, where the pile is supported by a rigid wall or foundation, is

$$z = 0 \; : \; w = 0. \tag{2.38}$$

The boundary condition at the other end is

$$z = h \; : \; \sigma = E\frac{\partial w}{\partial z} = -p_0 \sin(\omega t), \tag{2.39}$$

where h is the length of the pile, and ω is the frequency of the periodic load.

It is again assumed that the solution of the partial differential equation (2.3) can be written as the product of a function of z and a function of t. In particular, because the load is periodic, it is now assumed that

$$w = W(z)\sin(\omega t). \tag{2.40}$$

Substitution into the differential equation (2.3) shows that this equation can indeed be satisfied, provided that the function $W(z)$ satisfies the ordinary differential equation

$$\frac{d^2W}{dz^2} = -\frac{\omega^2}{c^2}W, \tag{2.41}$$

where $c = \sqrt{E/\rho}$, the wave velocity.

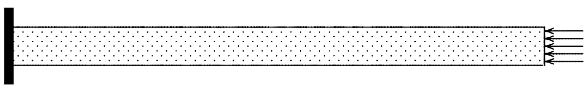


Fig. 2.4 Pile loaded by periodic pressure

The solution of the differential equation (2.41) that also satisfies the two boundary conditions (2.38) and (2.39) is

$$W(z) = -\frac{p_0 c}{E\omega}\frac{\sin(\omega z/c)}{\cos(\omega h/c)}. \tag{2.42}$$

This means that the final solution of the problem is, with (2.42) and (2.40),

$$w(z,t) = -\frac{p_0 c}{E\omega}\frac{\sin(\omega z/c)}{\cos(\omega h/c)}\sin(\omega t). \tag{2.43}$$

It can easily be verified that this solution satisfies all requirements, because it satisfies the differential equation, and both boundary conditions. Thus a complete solution has been obtained by elementary procedures. Of special interest is the motion of the free end of the pile. This is found to be

$$w(h,t) = w_0 \sin(\omega t), \tag{2.44}$$

where

$$w_0 = -\frac{p_0 c}{E\omega}\tan(\omega h/c). \tag{2.45}$$

The amplitude of the total force, $F_0 = -p_0 A$, can be written as

$$F_0 = \frac{EA}{c}\frac{\omega}{\tan(\omega h/c)} w_0. \tag{2.46}$$

Resonance

It may be interesting to consider the case that the frequency ω is equal to one of the eigen frequencies of the system,

$$\omega = \omega_k = (2k+1)\frac{\pi c}{2h}, \quad k = 0, 1, 2, \ldots. \tag{2.47}$$

In that case $\cos(\omega h/c) = 0$, and the amplitude of the displacement, as given by (2.45), becomes infinitely large. This phenomenon is called *resonance* of the system. If the frequency of the load equals one of the eigen frequencies of the system, this may lead to very large displacements, indicating resonance.

In engineering practice the pile may be a concrete foundation pile, for which the order of magnitude of the wave velocity c is about 3000 m/s, and for which a normal length h is 20 m. In civil engineering practice the frequency ω is usually not very large, at least during normal loading. A relatively high frequency is say $\omega = 20\ \mathrm{s}^{-1}$. In that case the value of the parameter $\omega h/c$ is about 0.13, which is rather small, much smaller than all eigen frequencies (the smallest of which occurs for $\omega h/c = \pi/2$). The function $\tan(\omega h/c)$ in (2.46) may now be approximated by its argument, so that this expression reduces to

$$\omega h/c \ll 1 \; : \; F_0 \approx \frac{EA}{h} w_0. \tag{2.48}$$

This means that the pile can be considered to behave, as a first approximation, as a spring, without mass, and without damping. In many situations in civil engineering practice the loading is so slow, and the elements are so stiff (especially when they consist of concrete or steel), that the dynamic analysis can be restricted to the motion of a single spring.

It must be noted that the approximation presented above is not always justified. When the material is soft (e.g. soil) the velocity of wave propagation may not be that high. And loading conditions with very high frequencies may also be of importance, for instance during installation (pile driving). In general one may say that in order for dynamic effects to be negligible, the loading must be so slow that the frequency is considerably smaller than the smallest eigen frequency.

2.4 Solution by Characteristics

A powerful method of solution for problems of wave propagation in one dimension is provided by the method of characteristics. This method is presented in this section.

The wave equation (2.3) has solutions of the form

$$w = f_1(z - ct) + f_2(z + ct), \tag{2.49}$$

where f_1 and f_2 are arbitrary functions, and c is the velocity of propagation of waves,

$$c = \sqrt{E/\rho}. \tag{2.50}$$

In mathematics the directions $z = ct$ and $z = -ct$ are called the *characteristics*. The solution of a particular problem can be obtained from the general solution (2.49) by using the initial conditions and the boundary conditions.

A convenient way of constructing solutions is by writing the basic equations in the following form

$$\frac{\partial \sigma}{\partial z} = \rho \frac{\partial v}{\partial t}, \tag{2.51}$$

$$\frac{\partial \sigma}{\partial t} = E \frac{\partial v}{\partial z}, \tag{2.52}$$

where v is the velocity, $v = \partial w/\partial t$, and σ is the stress in the pile.

In order to simplify the basic equations two new variables ξ and η are introduced, defined by

$$\xi = z - ct, \qquad \eta = z + ct. \tag{2.53}$$

Equations (2.51) and (2.52) can now be transformed into

$$\frac{\partial \sigma}{\partial \xi} + \frac{\partial \sigma}{\partial \eta} = \rho c\Big(-\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\Big), \tag{2.54}$$

$$\frac{\partial \sigma}{\partial \xi} - \frac{\partial \sigma}{\partial \eta} = \rho c\Big(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\Big), \tag{2.55}$$

from which it follows, by addition or subtraction of the two equations, that

$$\frac{\partial(\sigma - Jv)}{\partial \eta} = 0, \tag{2.56}$$

$$\frac{\partial(\sigma + Jv)}{\partial \xi} = 0, \tag{2.57}$$

where J is the impedance,

$$J = \rho c = \sqrt{E\rho}. \tag{2.58}$$

In terms of the original variables z and t the equations are

$$\frac{\partial(\sigma - Jv)}{\partial(z + ct)} = 0, \tag{2.59}$$

$$\frac{\partial(\sigma + Jv)}{\partial(z - ct)} = 0. \tag{2.60}$$

These equations mean that the quantity $\sigma - Jv$ is independent of $z + ct$, and $\sigma + Jv$ is independent of $z - ct$. This means that

$$\sigma - Jv = f_1(z - ct), \tag{2.61}$$

$$\sigma + Jv = f_2(z + ct). \tag{2.62}$$

These equations express that the quantity $\sigma - Jv$ is a function of $z - ct$ only, and that $\sigma + Jv$ is a function of $z + ct$ only. This means that $\sigma - Jv$ is constant when $z - ct$ is constant, and that $\sigma + Jv$ is constant when $z + ct$ is constant. These properties enable to construct solutions, either in a formal analytical way, or graphically, by mapping the solution, as represented by the variables σ and Jv, onto the plane of the independent variables z and ct.

As an example let there be considered the case of a free pile, which is hit at its upper end $z = 0$ at time $t = 0$ such that the stress at that end is $-p$. The other end, $z = h$, is free, so that the stress is zero there. The initial state is such that all velocities are zero. The solution is illustrated in Fig. 2.5. In the upper figure, the diagram of z and ct has been drawn, with lines of constant $z - ct$ and lines of constant $z + ct$. Because initially the velocity v and the stress σ are zero throughout the pile, the condition in each point of the pile is represented by the point 1 in the lower figure, the diagram of σ and Jv. The points in the lower left corner of the upper diagram (this region is marked 1) can all be reached from points on the axis

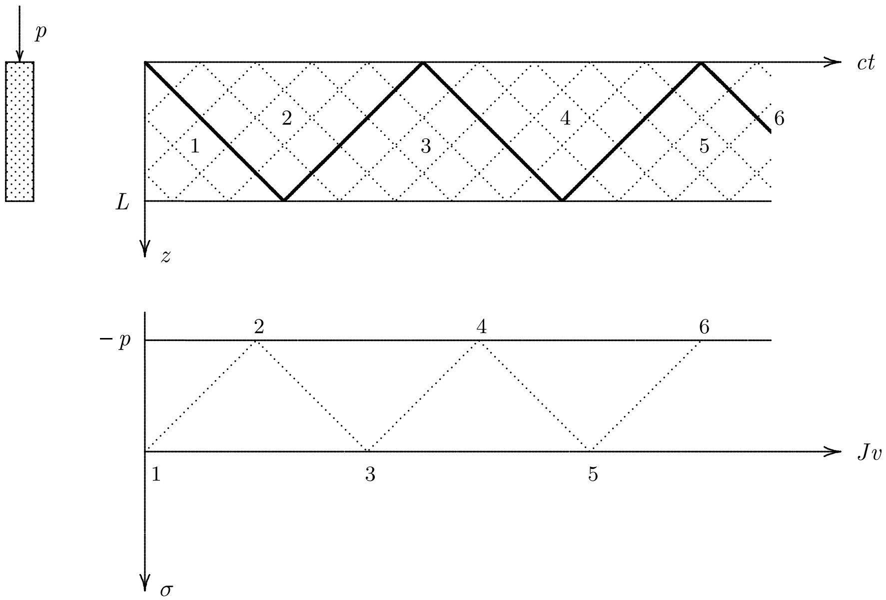


Fig. 2.5 The method of characteristics

$ct = 0$ (for which $\sigma = 0$ and $Jv = 0$) by a downward going characteristic, i.e. lines $z - ct =$ constant. Thus in all these points $\sigma - Jv = 0$. At the bottom of the pile the stress is always zero, $\sigma = 0$. Thus in the points in region 1 for which $z = 0$ the velocity is also zero, $Jv = 0$. Actually, in the entire region 1 : $\sigma = Jv = 0$, because all these points can be reached by an upward going characteristic and a downward going characteristic from points where $\sigma = Jv = 0$. The point 1 in the lower diagram thus is representative for all points in region 1 in the upper diagram.

For $t > 0$ the value of the stress σ at the upper boundary $z = 0$ is $-p$, for all values of t. The velocity is unknown, however. The axis $z = 0$ in the upper diagram can be reached from points in the region 1 along lines for which $z + ct =$ constant. Therefore the corresponding point in the diagram of σ and Jv must be located on the line for which $\sigma + Jv =$ constant, starting from point 1. Because the stress σ at the top of the pile must be $-p$ the point in the lower diagram must be point 2. This means that the velocity is $Jv = p$, or $v = p/J$. This is the velocity of the top of the pile for a certain time, at least for $ct = 2h$, if h is the length of the pile, because all points for which $z = 0$ and $ct < 2h$ can be reached from region 1 along characteristics $z + ct =$ constant.

At the lower end of the pile the stress σ must always be zero, because the pile was assumed to be not supported. Points in the upper diagram on the line $z = h$ can be reached from region 2 along lines of constant $x - ct$. Therefore they must be located on a line of constant $N - Jv$ in the lower diagram, starting from point 2. This gives point 3, which means that the velocity at the lower end of the pile is now $v = 2p/J$. This velocity applies to all points in the region 3 in the upper diagram.

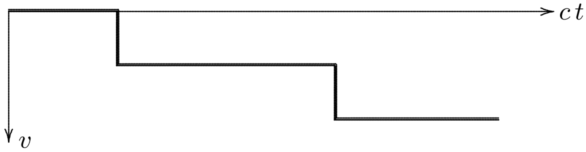


Fig. 2.6 Velocity of the bottom of the pile

In this way the velocity and the stress in the pile can be analyzed in successive steps. The thick lines in the upper diagram are the boundaries of the various regions. If the force at the top continues to be applied, as is assumed in Fig. 2.5, the velocity of the pile increases continuously. Figure 2.6 shows the velocity of the bottom of the pile as a function of time. The velocity gradually increases with time, because the pressure p at the top of the pile continues to act. This is in agreement with Newton's second law, which states that the velocity will increase linearly under the influence of a constant force.

2.5 Reflection and Transmission of Waves

An interesting aspect of wave propagation in continuous media is the behaviour of waves at surfaces of discontinuity of the material properties. In order to study this phenomenon let us consider the propagation of a short shock wave in a pile consisting of two materials, see Fig. 2.7. A compression wave is generated in the pile by a pressure of short duration at the left end of the pile. The pile consists of two materials: first a stiff section, and then a very long section of smaller stiffness.

The solution of the basic equations in the first section can be written as

$$v = v_1 = f_1(z - c_1 t) + f_2(z + c_1 t), \tag{2.63}$$

$$\sigma = \sigma_1 = -\rho_1 c_1 f_1(z - c_1 t) + \rho_1 c_1 f_2(z + c_1 t), \tag{2.64}$$

where ρ_1 is the density of the material in that section, and c_1 is the wave velocity, $c_1 = \sqrt{E_1/\rho_1}$. It can easily be verified that this solution satisfies the two basic differential equations (2.51) and (2.52).

In the second part of the pile the solution is

$$v = v_2 = g_1(z - c_2 t) + g_2(z + c_2 t), \tag{2.65}$$

$$\sigma = \sigma_2 = -\rho_2 c_2 g_1(z - c_2 t) + \rho_2 c_2 g_2(z + c_2 t), \tag{2.66}$$

where ρ_2 and c_2 are the density and the wave velocity in that part of the pile.

At the interface of the two materials the value of z is the same in both solutions, say $z = h$, and the condition is that both the velocity v and the normal stress σ must

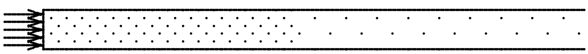


Fig. 2.7 Non-homogeneous pile

be continuous at that point, at all values of time. Thus one obtains

$$f_1(h-c_1t)+f_2(h+c_1t)=g_1(h-c_2t)+g_2(h+c_2t), \tag{2.67}$$

$$-\rho_1c_1f_1(h-c_1t)+\rho_1c_1f_2(h+c_1t)$$
$$=-\rho_2c_2g_1(h-c_2t)+\rho_2c_2g_2(h+c_2t). \tag{2.68}$$

If we write

$$f_1(h-c_1t)=F_1(t), \tag{2.69}$$

$$f_2(h+c_1t)=F_2(t), \tag{2.70}$$

$$g_1(h-c_2t)=G_1(t), \tag{2.71}$$

$$g_2(h+c_2t)=G_2(t), \tag{2.72}$$

then the continuity conditions are

$$F_1(t)+F_2(t)=G_1(t)+G_2(t), \tag{2.73}$$

$$-\rho_1c_1F_1(t)+\rho_1c_1F_2(t)=-\rho_2c_2G_1(t)+\rho_2c_2G_2(t). \tag{2.74}$$

In general these equations are, of course, insufficient to solve for the four functions. However, if it is assumed that the pile is very long (or, more generally speaking, when the value of time is so short that the wave reflected from the end of the pile has not yet arrived), it may be assumed that the solution representing the wave coming from the end of the pile is zero, $G_2(t)=0$. In that case the solutions F_2 and G_1 can be expressed in the first wave, F_1, which is the wave coming from the top of the pile. The result is

$$F_2(t)=\frac{\rho_1c_1-\rho_2c_2}{\rho_1c_1+\rho_2c_2}\,F_1(t), \tag{2.75}$$

$$G_1(t)=\frac{2\rho_1c_1}{\rho_1c_1+\rho_2c_2}\,F_1(t), \tag{2.76}$$

This means, for instance, that whenever the first wave $F_1(t)=0$ at the interface, then there is no reflected wave, $F_2(t)=0$, and there is no transmitted wave either, $G_1(t)=0$. On the other hand, when the first wave has a certain value at the interface, then the values of the reflected wave and the transmitted wave at that point may be calculated from the relations (2.75) and (2.76). If the values are known the values at later times may be calculated using the relations (2.69)–(2.72).

The procedure may be illustrated by an example. Therefore let it be assumed that the two parts of the pile have the same density, $\rho_1=\rho_2$, but the stiffness in the first section is 9 times the stiffness in the rest of the pile, $E_1=9E_2$. This means that the wave velocities differ by a factor 3, $c_1=3c_2$. The reflection coefficient and the transmission coefficient now are, with (2.75) and (2.76),

$$R_v=\frac{\rho_1c_1-\rho_2c_2}{\rho_1c_1+\rho_2c_2}=0.5, \tag{2.77}$$

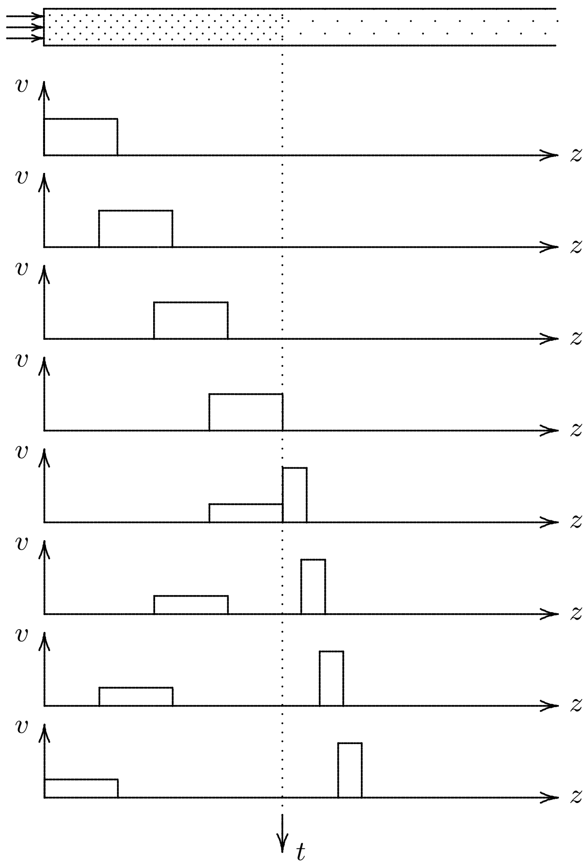


Fig. 2.8 Reflection and transmission (velocity)

$$T_v = \frac{2\rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2} = 1.5. \tag{2.78}$$

The behaviour of the solution is illustrated graphically in Fig. 2.8, which shows the velocity profile at various times. In the first four diagrams the incident wave travels toward the interface. During this period there is no reflected wave, and no transmitted wave in the second part of the pile. As soon as the incident wave hits the interface a reflected wave is generated, and a wave is transmitted into the second part of the pile. The magnitude of the velocities in this transmitted wave is 1.5 times the original wave, and it travels a factor 3 slower. The magnitude of the velocities in the reflected wave is 0.5 times those in the original wave.

The stresses in the two parts of the pile are shown in graphical form in Fig. 2.9. The reflection coefficient and the transmission coefficient for the stresses can be obtained using (2.64) and (2.66). The result is

$$R_\sigma = -\frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2} = -0.5, \tag{2.79}$$

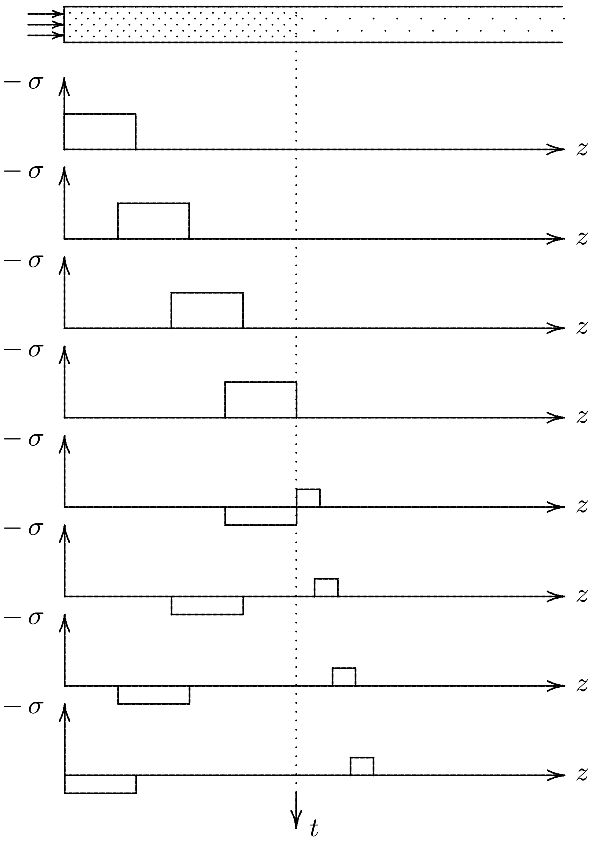


Fig. 2.9 Reflection and transmission (stress)

$$T_\sigma = \frac{2\rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2} = 0.5, \tag{2.80}$$

where it has been taken into account that the form of the solution for the stresses, see (2.64) and (2.66), involves factors ρc, and signs of the terms different from those in the expressions for the velocity. In the case considered here, where the first part of the pile is 9 times stiffer than the rest of the pile, it appears that the reflected wave leads to stresses of the opposite sign in the first part. Thus a compression wave in the pile is reflected in the first part by tension.

It may be interesting to note the two extreme cases of reflection. When the second part of the pile is so soft that it can be entirely disregarded (or, when the pile consists only of the first part, which is free to move at its end), the reflection coefficient for the velocity is $R_v = 1$, and for the stress it is $R_\sigma = -1$. This means that in this case a compression wave is reflected as a tension wave of equal magnitude. The velocity in the reflected wave is in the same direction as in the incident wave.

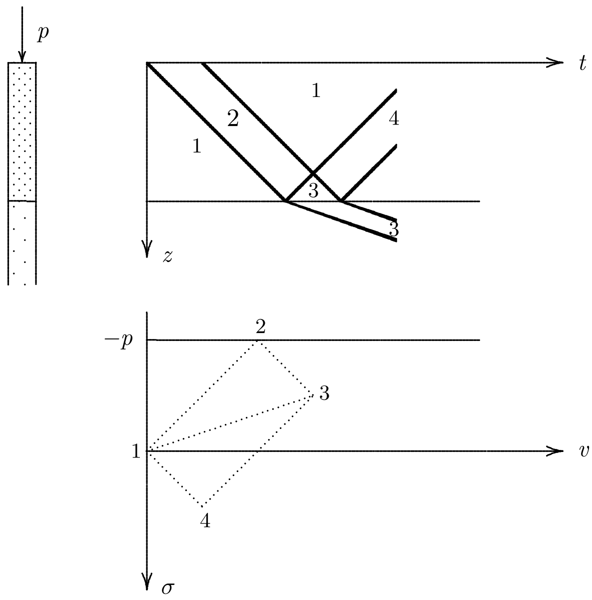


Fig. 2.10 Graphical solution using characteristics

If the second part of the pile is infinitely stiff (or, if the pile meets a rigid foundation after the first part) the reflection coefficient for the velocity is $R_v = -1$, and for the stresses it is $R_\sigma = 1$. Thus, in this case a compression wave is reflected as a compressive wave of equal magnitude. These results are of great importance in pile driving. When a pile hits a very soft layer, a tension wave may be reflected from the end of the pile, and a concrete pile may not be able to withstand these tensile stresses. Thus, the energy supplied to the pile must be reduced in this case, for instance by reducing the height of fall of the hammer. When the pile hits a very stiff layer the energy of the driving equipment may be increased without the risk of generating tensile stresses in the pile, and this may help to drive the pile through this stiff layer. Of course, great care must be taken when the pile tip suddenly passes from the very stiff layer into a soft layer. Experienced pile driving operators use these basic principles intuitively.

It may be noted that tensile stresses may also be generated in a pile when an upward traveling (reflected) wave reaches the top of the pile, which by that time may be free of stress. This phenomenon has caused severe damage to concrete piles, in which cracks developed near the top of the pile, because concrete cannot withstand large tensile stresses. In order to prevent this problem, driving equipment has been developed that continues to apply a compressive force at the top of the pile for a relatively long time. Also, the use of prestressed concrete results in a considerable tensile strength of the material.

The problem considered in this section can also be analyzed graphically, by using the method of characteristics, see Fig. 2.10. The data given above imply that the wave velocity in the second part of the pile is 3 times smaller than in the first part, and that the impedance in the second part is also 3 times smaller than in the first part. This means that in the lower part of the pile the slope of the characteristics is 3 times smaller than the slope in the upper part. In the figure these slopes have been taken as 1:3 and 1:1, respectively. Starting from the knowledge that the pile is initially at rest (1), and that at the top of the pile a compression wave of short duration is generated (2), the points in the v,σ-diagram, and the regions in the z,t-diagram can be constructed, taking into account that at the interface both v and σ must be continuous.

2.6 The Influence of Friction

In soil mechanics piles in the ground usually experience friction along the pile shaft, and it may be illuminating to investigate the effect of this friction on the mechanical behaviour of the pile. For this purpose consider a pile of constant cross sectional area A and modulus of elasticity E, standing on a rigid base, and supported along its shaft by shear stresses that are generated by an eventual movement of the pile, see Fig. 2.11.

The basic differential equation is

$$EA\frac{\partial^2 w}{\partial z^2} - C\tau = \rho A\frac{\partial^2 w}{\partial t^2}, \tag{2.81}$$

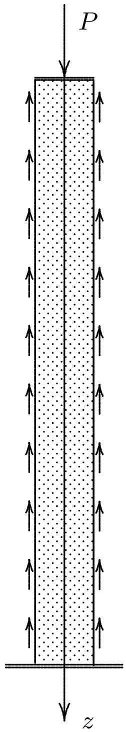


Fig. 2.11 Pile in soil, with friction

where C is the circumference of the pile shaft, and τ is the shear stress. It is assumed, as a first approximation, that the shear stress is linearly proportional to the vertical displacement of the pile,

$$\tau = kw, \tag{2.82}$$

where the constant k has the character of a subgrade modulus. The differential equation (2.81) can now be written as

$$\frac{\partial^2 w}{\partial z^2} - \frac{w}{H^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}, \tag{2.83}$$

where H is a length parameter characterizing the ratio of the axial pile stiffness to the friction constant,

$$H^2 = \frac{EA}{kC}, \tag{2.84}$$

and c is the usual wave velocity, defined by

$$c^2 = E/\rho. \tag{2.85}$$

The boundary conditions are supposed to be

$$z = 0 \;:\; N = EA\frac{\partial w}{\partial z} = -P\,\sin(\omega t), \tag{2.86}$$

$$z = L \;:\; w = 0. \tag{2.87}$$

The first boundary condition expresses that at the top of the pile it is loaded by a periodic force, of amplitude P and circular frequency ω. The second boundary condition expresses that at the bottom of the pile no displacement is possible, indicating that the pile is resting upon solid rock.

The problem defined by the differential equation (2.83) and the boundary conditions (2.86) and (2.87) can easily be solved by the method of separation of variables. In this method it is assumed that the solution can be written as the product of a function of z and a factor $\sin(\omega t)$. It turns out that all the conditions are met by the solution

$$w = \frac{PH}{EA\alpha}\frac{\sinh[\alpha(L-z)/H]}{\cosh(\alpha L/H)}\sin(\omega t), \tag{2.88}$$

where α is given by

$$\alpha = \sqrt{1 - \omega^2 H^2/c^2}. \tag{2.89}$$

The displacement at the top of the pile, w_t, is of particular interest. If this is written as

$$w_t = \frac{P}{K}\sin(\omega t), \tag{2.90}$$

the spring constant K appears to be

$$K = \frac{EA}{L} \frac{\alpha L/H}{\tanh(\alpha L/H)}. \tag{2.91}$$

The first term in the right hand side is the spring constant in the absence of friction, when the elasticity is derived from the deformation of the pile only.

The behaviour of the second term in (2.91) depends upon the frequency ω through the value of the parameter α, see (2.89). It should be noted that for values of $\omega H/c > 1$ the parameter α becomes imaginary, say $\alpha = i\beta$, where now

$$\beta = \sqrt{\omega^2 H^2/c^2 - 1}. \tag{2.92}$$

The spring constant can then be written more conveniently as

$$\omega H/c > 1 \ : \ K = \frac{EA}{L} \frac{\beta L/H}{\tan(\beta L/H)}. \tag{2.93}$$

This formula implies that for certain values of $\omega H/c$ the spring constant will be zero, indicating resonance. These values correspond to the eigen values of the system. For certain other values the spring constant is infinitely large. For these values of the frequency the system appears to be very stiff. In such a case part of the pile is in compression and another part is in tension, such that the total strains from bottom to top just cancel.

The value of the spring constant is shown, as a function of the frequency, in Fig. 2.12, for $H/L = 1$. This figure contains data for both ranges of the parameters.

It is interesting to consider the probable order of magnitude of the parameters in engineering practice. For this purpose the value of the subgrade modulus k must first be evaluated. This parameter can be estimated to be related to the soil stiffness by a formula of the type $k = E_s/D$, where E_s is the modulus of elasticity of the soil (assuming that the deformations are small enough to justify the definition of such a quantity), and D is the width of the pile. For a circular concrete pile of diameter D the value of the characteristic length H now is, with (2.84),

$$H^2 = \frac{EA}{kC} = \frac{E_c D^2}{2E_s}. \tag{2.94}$$

Under normal conditions, with a pile being used in soft soil, the ratio of the elastic moduli of concrete and soil is about 1000, and most piles have diameters of about 0.40 m. This means that $H \approx 10$ m. Furthermore the order of magnitude of the wave propagation velocity c in concrete is about 3000 m/s. This means that the parameter $\omega H/c$ will usually be small compared to 1, except for phenomena of very high frequency, such as may occur during pile driving. In many civil engineering problems, where the fluctuations originate from wind or wave loading, the frequency is usually about 1 s^{-1} or smaller, so that the order of magnitude of the parameter $\omega H/c$ is about 0.01. In such cases the value of α will be very close to 1, see (2.89). This indicates that the response of the pile is practically static.

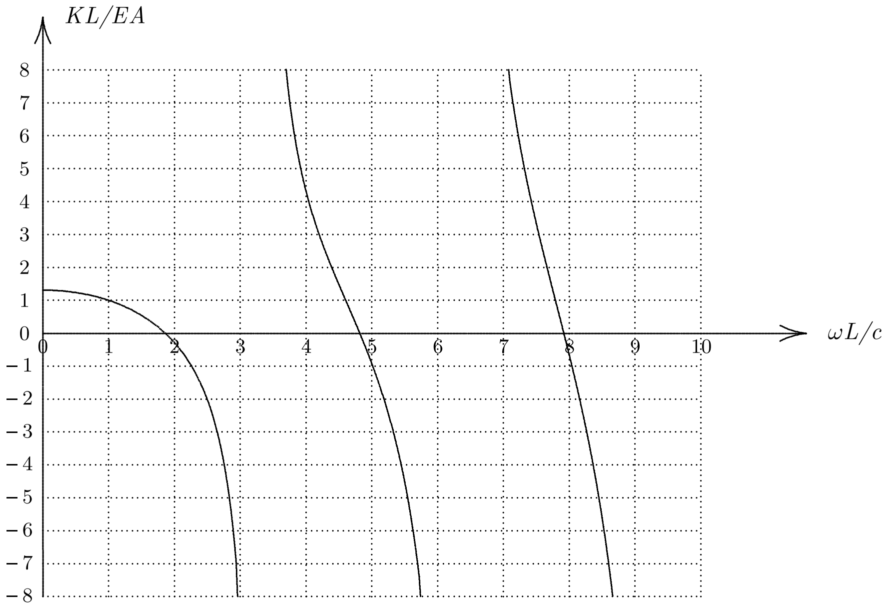


Fig. 2.12 Spring constant ($H/L = 1$)

If the loading is due to the passage of a heavy train, at a velocity of 100 km/h, and with a distance of the wheels of 5 m, the period of the loading is about 1/6 s, and thus the frequency is about 30 s^{-1}. In such cases the parameter $\omega H/c$ may not be so small, indicating that dynamic effects may indeed be relevant.

Infinitely Long Pile

A case of theoretical interest is that of an infinitely long pile, $L \to \infty$. If the frequency is low this limiting case can immediately be obtained from the general solution (2.91), because then the function $\tanh(\alpha L/H)$ can be approximated by its asymptotic value 1. The result is

$$L \to \infty, \ \omega H/c < 1 \ : \ K = \frac{EA\alpha}{H}. \tag{2.95}$$

This solution degenerates when the dimensionless frequency $\omega H/c = 1$, because then $\alpha = 0$, see (2.89). Such a zero spring constant indicates resonance of the system.

For frequencies larger than this resonance frequency the solution (2.93) can not be used, because the function $\tan(\beta L/H)$ continues to fluctuate when its argument tends towards infinity. Therefore the problem must be studied again from the beginning, but now for an infinitely long pile. The general solution of the differential equation now appears to be

$$w = [C_1 \sin(\beta z/H) + C_2 \cos(\beta z/H)] \sin(\omega t)$$
$$+ [C_3 \sin(\beta z/H) + C_4 \cos(\beta z/H)] \cos(\omega t), \tag{2.96}$$

and there is no combination of the constants C_1, C_2, C_3 and C_4 for which this solution tends towards zero as $z \to \infty$. This dilemma can be solved by using the *radiation condition* (Sommerfeld, 1949), which states that it is not to be expected that waves travel from infinity towards the top of the pile. Therefore the solution (2.96) is first rewritten as

$$w = D_1 \sin(\omega t - \beta z/H) + D_2 \cos(\omega t - \beta z/H)$$
$$+ D_3 \sin(\omega t + \beta z/H) + D_4 \cos(\omega t + \beta z/H). \tag{2.97}$$

Written in this form it can be seen that the first two solutions represent waves traveling from the top of the pile towards infinity, whereas the second two solutions represent waves traveling from infinity up to the top of the pile. If the last two are excluded, by assuming that there is no agent at infinity which generates such incoming waves, it follows that $C_3 = C_4 = 0$. The remaining two conditions can be determined from the boundary condition at the top of the pile, (2.86). The final result is

$$L \to \infty,\ \omega H/c > 1\ :\ w = \frac{PH}{EA\beta} \sin(\omega t - \beta z/H). \tag{2.98}$$

This solution applies only if the frequency is larger than the eigen frequency of the system, which is defined by $\omega H/c = 1$. It may be noted that the solution (2.98) also degenerates for $\omega H/c = 1$ because then $\beta = 0$, see (2.92).

2.7 Numerical Solution

In order to construct a numerical model for the solution of wave propagation problems the basic equations are written in a numerical form. For this purpose the pile is subdivided into n elements, all of the same length Δz. The displacement w_i and the velocity v_i of an element are defined in the centroid of element i, and the normal forces N_i are defined at the boundary between elements i and $i+1$, see Fig. 2.13. The friction force acting on element i is denoted by F_i. This particular choice for

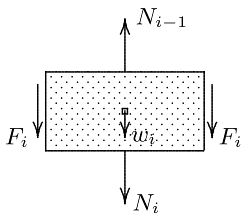


Fig. 2.13 Element of pile

the definition of the various quantities either at the centroid of the elements or at their boundaries, has a physical background. The velocity derives its meaning from a certain mass, whereas the normal force is an interaction between the material on both sides of a section. It is interesting to note, however, that this way of modeling, sometimes denoted as *leap frog* modeling, also has distinct mathematical advantages, with respect to accuracy and stability.
The equation of motion of an element is

$$N_i - N_{i-1} + F_i = \rho A \Delta z \frac{v_i(t+\Delta t) - v_i(t)}{\Delta t} \quad (i = 1, \ldots, n). \tag{2.99}$$

It should be noted that there are $n+1$ normal forces, from N_0 to N_n. The force N_0 can be considered to be the force at the top of the pile, and N_n is the force at the bottom end of the pile.
The displacement w_i is related to the velocity v_i by the equation

$$v_i = \frac{w_i(t+\Delta t) - w_i(t)}{\Delta t} \quad (i = 1, \ldots, n). \tag{2.100}$$

The deformation is related to the normal force by Hooke's law, which can be formulated as

$$N_i = EA \frac{w_{i+1} - w_i}{\Delta z} \quad (i = 1, \ldots, n-1). \tag{2.101}$$

Here EA is the product of the modulus of elasticity E and the area A of the cross section.

The values of the normal force at the top and at the bottom of the pile, N_0 and N_n are supposed to be given by the boundary conditions.

Example

A simple example may serve to illustrate the numerical algorithm. Suppose that the pile is initially at rest, and let a constant force P be applied at the top of the pile, with the bottom end being free. In this case the boundary conditions are

$$N_0 = -P, \tag{2.102}$$

and

$$N_n = 0. \tag{2.103}$$

The friction forces are supposed to be zero.

At time $t = 0$ all quantities are zero, except N_0. A new set of velocities can now be calculated from (2.99). Actually, this will make only one velocity non-zero, namely v_1, which will then be

$$v_1 = \frac{P \Delta t}{\rho A \Delta z}. \tag{2.104}$$

Next, a new set of values for the displacements can be calculated from (2.100). Again, in the first time step, only one value will be non-zero, namely

$$w_1 = v_1 \Delta t = \frac{P(\Delta t)^2}{\rho A \Delta z}. \tag{2.105}$$

Finally, a new set of values for the normal force can be calculated from (2.101). This will result in N_1 getting a value, namely

$$N_1 = -EA\frac{w_1}{\Delta z} = -P\frac{c^2(\Delta t)^2}{(\Delta z)^2}. \tag{2.106}$$

This process can now be repeated, using the equations in the same order.

An important part of the numerical process is the value of the time step used. The description of the process given above indicates that in each time step the non-zero values of the displacements, velocities and normal forces increase by 1 in downward direction. This suggests that in each time step a wave travels into the pile over a distance Δz. In the previous section, when considering the analytical solution of a similar problem (actually, the same problem), it was found that waves travel in the pile at a velocity

$$c = \sqrt{E/\rho}. \tag{2.107}$$

Combining these findings suggests that the ratio of spatial step and time step should be

$$\Delta z = c\Delta t. \tag{2.108}$$

It may be noted that this means that (2.104) reduces to

$$v_1 = \frac{P}{\rho A c}. \tag{2.109}$$

The expression in the denominator is precisely what was defined as the impedance in the previous section, see (2.58), and the value P/J corresponds exactly to what was found in the analytical solution. Equation (2.105) now gives

$$w_1 = \frac{P\Delta t}{\rho A c}, \tag{2.110}$$

and the value of N_1 after one time step is found to be, from (2.106),

$$N_1 = -P. \tag{2.111}$$

Again this corresponds exactly with the analytical solution. If the time step is chosen different from the critical time step the numerical solution will show considerable deviations from the correct analytical solution. This is usually denoted as *numerical diffusion*.

All this confirms the propriety of the choice (2.108) for the relation between time step and spatial step. In a particular problem the spatial step is usually chosen first, by subdividing the pile length into a certain number of elements. Then the time step may be determined from (2.108).

It should be noted that the choice of the time step is related to the algorithm proposed here. When using a different algorithm it may be more appropriate to use a different (usually smaller) time step than the critical time step used here (Bowles, 1974).

The calculations described above can be performed by the program IMPACT, for the case of a pile loaded at its top by a constant force, for a short time. The main function in this program is given below, with the quantities S, V and W denoting the stress, the velocity and the displacement.

```
void Calculate(void)
 {
  int j;
  if (T>TT) S[0]=0;else S[0]=1;
  for (j=1;j<=N;j++) V[j]+=(S[j]-S[j-1])/(RHO*C);
  for (j=1;j<=N;j++) W[j]+=V[j]*DT;
  for (j=1;j<N;j++) S[j]=E*(W[j+1]-W[j])/DX;
 }
```

The main function of the program IMPACT.

The program uses interactive input, in which the user may edit the input data before the calculations are started. The program will show the stresses in the pile on the screen, in graphical form. An example is shown in Fig. 2.14. In this case the pile has been subdivided into 500 elements, and the figure shows the stresses in the pile after 200, 400, 600, 800 and 1000 time steps. It appears that the block wave is traveling through the pile without any deformation, and it is reflected at the free bottom as a tensile wave of the same magnitude. All this is in agreement with the general theory presented in earlier sections of this chapter.

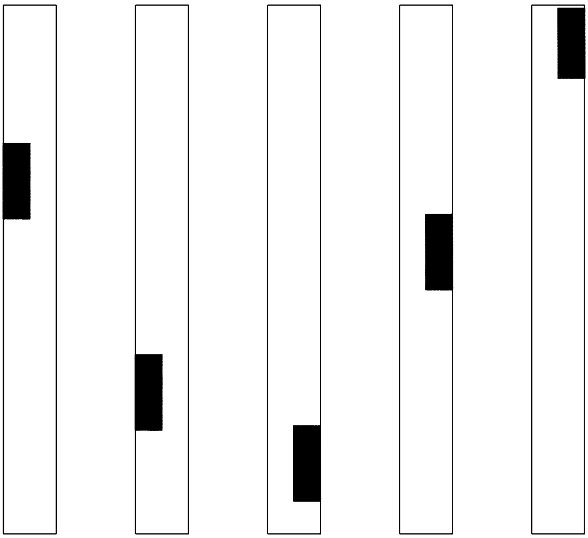


Fig. 2.14 Block wave in pile

2.8 A Simple Model for a Pile with Friction

When there is friction along the shaft of the pile, this can be introduced through the variables F_i, see (2.99). It should then be known how the friction force depends upon variables such as the local displacement and the local velocity. A simple model is to assume that the friction is proportional to the velocity, always acting in the direction opposite to the velocity. The program FRICTION can perform these calculations. The main function of this program is reproduced below, for the case of a single sinusoidal wave applied at the top of the pile.

```
void Calculate(void)
 {
  int j;
  if (T>TT) S[0]=0;else S[0]=(F/AREA)*sin(PI*T/TT);
  for (j=1;j<=N;j++) V[j]+=(S[j]-S[j-1]-FR*DX*CIRC*V[j])/(RHO*AREA*C);
  for (j=1;j<=N;j++) W[j]+=V[j]*DT;
  for (j=1;j<N;j++) S[j]=E*AREA*(W[j+1]-W[j])/DX;
 }
```

The main function of the program FRICTION.

The variable FR in this program is the shear stress generated along the shaft of the pile in case of a unit velocity (1 m/s). In professional programs a more sophisticated formula for the friction may be used, in which the friction not only depends upon the velocity but also on the displacement, in a non-linear way. Also a model for the resistance at the point of the pile may be introduced, and the possibility of a layered soil, see for instance Bowles (1974).

Output of the program is shown in Fig. 2.15. The pile has been divided into 200 elements, its length is 20 m, and its cross section is a square of 0.40 m $\times$ 0.40 m. The maximum applied force is 100 kN, and the shear stress by friction is 1 kN/m^2 if the local velocity is 1 m/s.

Results for the stresses in the pile are shown after 100, 20100 and 40100 time steps. This means that between the successive plots in the figure the wave has traveled 100 times through the pile, up and down. It appears from the results that after a

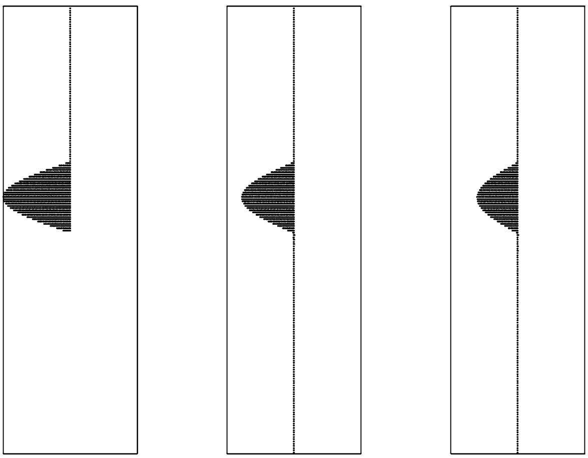


Fig. 2.15 Block wave in pile, with friction

large number of time steps the magnitude of the stresses is indeed decreased by the effect of the friction.

It may be mentioned that the program becomes unstable if the friction constant is taken too large, or if the initial wave is discontinuous, as in the case of a block wave. These unwanted effects can be eliminated by using a more sophisticated numerical method, such as the finite element method, see for instance Brinkgreve and Vermeer (2002).

Problems

2.1 A free pile is hit by a normal force of short duration. Analyze the motion of the pile by the method of characteristics, using a diagram as in Fig. 2.5.

2.2 Extend the diagram shown in Fig. 2.10 towards the right, so that the reflected wave hits the top of the pile, and is again reflected there.

2.3 As a first order approximation of (2.46) the response of a pile may be considered to be equivalent to a spring, see (2.48). Show, by using an approximation of the function $\tan(\omega h/c)$ by its first two terms, that a second order approximation is by a spring and a mass. Show, by comparison with (1.37), that the equivalent mass is $\frac{2}{3}$ of the total mass of the pile.

2.4 Verify some of the characteristic data shown in Fig. 2.12. For instance, check the values for $\omega L/c = 0$ and $\omega L/c = 1$, and check the zeroes of the spring constant.

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