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Hyperbolic Conservation Laws in Continuum Physics

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Entropy and the Stability of Classical Solutions

It is a tenet of continuum physics that the Second Law of thermodynamics is essentially a statement of stability. In the examples discussed in the previous chapters, the Second Law manifests itself in the presence of companion balance laws, to be satisfied identically, as equalities, by classical solutions, and to be imposed as thermodynamic admissibility inequality constraints on weak solutions of the systems of balance laws. A recurring theme in the exposition of the theory of hyperbolic systems of balance laws in this book will be that companion balance laws induce stability under various guises. Here the reader will get a glimpse of the implications of entropy inequalities on the stability of classical solutions.

It will be shown that when the system of balance laws is endowed with a companion balance law induced by a convex entropy, the initial value problem is locally well-posed in the context of classical solutions: sufficiently smooth initial data generate a classical solution defined on a maximal time interval, typically of finite duration. However, in the presence of damping induced by relaxation or other dissipative mechanisms, and when the initial data are sufficiently small, the classical solution exists globally in time. Classical solutions are unique and depend continuously on their initial values, not only within the class of classical solutions but even within the broader class of weak solutions that satisfy the companion balance law as an inequality admissibility constraint.

Similar existence and stability results will be established, even when the entropy fails to be convex, in the following two situations: (a) the entropy is convex only in the direction of a certain cone in state space but the system is equipped with special companion balance laws, called involutions, whose presence compensates for the lack of convexity in complementary directions; or (b) the system is endowed with complementary entropies and the principal entropy is polyconvex. This structure arises in elastodynamics and electromagnetism.

The chapter will close with a brief discussion of the existence of classical solutions to the initial-boundary value problems.

From the standpoint of analytical technique, this chapter presents the aspects of the theory of quasilinear hyperbolic systems of balance laws that can be tackled by the methodology of the linear theory, namely energy estimates and Fourier analysis.

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5.1 Convex Entropy and the Existence of Classical Solutions

As in Chapter IV, we consider here the Cauchy problem

$$(5.1.1) \qquad \partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0, \qquad x \in \mathbb{R}^m,\ t > 0,$$

$$(5.1.2) \qquad U(x,0) = U_0(x), \qquad x \in \mathbb{R}^m,$$

for a homogeneous system of conservation laws in canonical form, endowed with an entropy $\eta(U)$. For any $U \in \mathscr{O}$, we define the $n \times n$ matrices

$$(5.1.3) \qquad A(U) = \mathrm{D}^2\eta(U),$$

$$(5.1.4) \qquad J_\alpha(U) = A(U)\mathrm{D}G_\alpha(U)\,, \qquad \alpha = 1,\cdots,m,$$

which are symmetric, by virtue of (3.2.4).

The aim is to establish local existence of classical solutions, by employing energy-type estimates induced by the entropy. In particular, this section will discuss the case of convex entropy; Section 5.4 will consider the situation where the entropy is merely convex in the direction of a certain cone; and Section 5.5 will deal with entropy satisfying a weaker condition, called polyconvexity. The discussion will be restricted to homogeneous conservation laws just for simplicity, as the extension to general balance laws (3.1.1) is routine.

Throughout this chapter, a *multi-index* r will stand for a m-tuple of nonnegative integers: $r = (r_1,\cdots,r_m)$. We put $|r| = r_1 + \cdots + r_m$ and $\partial^r = \partial_1^{r_1}\cdots\partial_m^{r_m}$. Thus ∂^r is a differential operator of order $|r|$. We also employ the notation $\nabla = (\partial_1,\cdots,\partial_m)$.

For $\ell = 0,1,2,\cdots$, H^ℓ will be the Sobolev space $W^{\ell,2}(\mathbb{R}^m;\mathbb{M}^{n\times m})$ of $n \times m$ matrix-valued functions on $\mathbb{R}^m$. The norm of H^ℓ will be denoted by $\|\cdot\|_\ell$. By the Sobolev embedding theorem, for $\ell > m/2$, H^ℓ is continuously embedded in the space of continuous $n \times m$ matrix-valued functions on $\mathbb{R}^m$.

The main result of this section is the following

5.1.1 Theorem. *Assume the system of conservation laws* (5.1.1) *is endowed with a* C^3 *entropy* $\eta(U)$, *such that* $\mathrm{D}^2\eta(U)$ *is positive definite on* $\mathscr{O}$. *Suppose the initial data* U_0 *are continuously differentiable on* $\mathbb{R}^m$, *take values in some compact subset of* $\mathscr{O}$ *and* $\nabla U_0 \in H^\ell$ *for some* $\ell > m/2$. *Moreover, let* $G \in C^{\ell+2}$. *Then there exists* $T_\infty \le \infty$, *and a unique continuously differentiable function* U *on* $\mathbb{R}^m \times [0,T_\infty)$, *taking values in* $\mathscr{O}$, *which is a classical solution of the Cauchy problem* (5.1.1), (5.1.2) *on* $[0,T_\infty)$. *Furthermore,*

$$(5.1.5) \qquad \nabla U(\cdot,t) \in \bigcap_{k=0}^{\ell} C^k([0,T_\infty);H^{\ell-k}).$$

The interval $[0,T_\infty)$ *is maximal in that if* $T_\infty < \infty$ *then*

(5.1.6) $$\int_0^{T_\infty} \|\nabla U(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} dt = \infty$$

and/or the range of $U(\cdot,t)$ escapes from every compact subset of $\mathscr{O}$, as $t \uparrow T_\infty$.

The traditional proof of the above theorem, presented in the literature cited in Section 5.7, and also in earlier editions of this book, determines the solution of (5.1.1), (5.1.2), in a suitable function space $\mathscr{F}$, as a fixed point of the map that carries $V \in \mathscr{F}$ to the solution $U \in \mathscr{F}$ of the linearized system

(5.1.7) $$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(V(x,t))\partial_\alpha U(x,t) = 0,$$

under initial conditions (5.1.2). This approach is effective when $\eta(U)$ is convex, because in that case multiplication by $\mathrm{D}^2\eta(V)$ renders (5.1.7) symmetric hyperbolic; but it is inapplicable under weaker hypotheses on $\eta(U)$, to be introduced in Sections 4.5 and 4.6, where the a priori estimates are inexorably tied to the geometric structure of (5.1.1) and do not carry over to the linearized form (5.1.7). Accordingly, we shall employ here the vanishing viscosity method, which determines solutions to (5.1.1) as the $\varepsilon \downarrow 0$ limit of solutions of the parabolic system

(5.1.8) $$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = \varepsilon \Delta U(x,t).$$

This approach is lengthier, but it starts at a more elementary level (the sole prerequisite is the solution of the Cauchy problem for the classical heat equation) and, more importantly, it affords a unifying treatment of the various cases to be considered in this chapter.

The first step is to establish local existence for the Cauchy problem for (5.1.8), with fixed $\varepsilon > 0$. The leading term is the Laplacian, so hyperbolicity and entropy will play no role at this stage.

5.1.2 Lemma. *Assume U_0 takes values in a compact subset of an open bounded set $\mathscr{C}$ whose closure $\overline{\mathscr{C}}$ is contained in $\mathscr{O}$, and let $\nabla U_0 \in H^\ell$, with $\ell > m/2$. For any fixed $\omega > \|\nabla U_0\|_\ell$, there exists a solution U of* (5.1.8), (5.1.2) *on a time interval $[0,T_\varepsilon), 0 < T_\varepsilon \leq \infty$, such that, for each fixed $t \in [0,T_\varepsilon)$, $U(\cdot,t)$ takes values in a compact subset of $\mathscr{C}$, and*

(5.1.9) $$\nabla U(\cdot,t) \in C^0([0,T_\varepsilon);H^\ell) \bigcap L^2([0,T_\varepsilon);H^{\ell+1}),$$

with

(5.1.10) $$\|\nabla U(\cdot,t)\|_\ell < \omega,$$

for all $t \in [0,T_\varepsilon)$. The interval $[0,T_\varepsilon)$ is maximal, in that if $T_\varepsilon < \infty$ then, as t tends to T_ε, $\|\nabla U(\cdot,t)\|_\ell \to \omega$ and/or the range of $U(\cdot,t)$ escapes from every compact subset of $\mathscr{C}$.

Proof. With the fixed ω and with T to be selected later, we associate the class $\mathscr{V}$ of Lipschitz functions V defined on $\mathbb{R}^m \times [0,T]$, taking values in $\overline{\mathscr{C}}$, and satisfying

$$V(\cdot,t) - U_0(\cdot) \in L^\infty([0,T];L^2), \qquad \nabla V(\cdot,t) \in L^\infty([0,T];H_\ell), \tag{5.1.11}$$

$$\sup_{[0,T]} \|V(\cdot,t) - U_0(\cdot)\|_{L^2} \leq \omega, \qquad \sup_{[0,T]} \|\nabla V(\cdot,t)\|_\ell \leq \omega. \tag{5.1.12}$$

By standard weak lower semicontinuity of L^p norms, $\mathscr{V}$ is a complete metric space under the metric

$$\rho(V,\bar{V}) = \sup_{[0,T]} \|V(\cdot,t) - \bar{V}(\cdot,t)\|_0 \,. \tag{5.1.13}$$

For given $V \in \mathscr{V}$, we construct the solution U on $\mathbb{R}^m \times [0,T]$ of the linear parabolic system

$$\partial_t U(x,t) - \varepsilon \Delta U(x,t) = -\sum_{\alpha=1}^m \partial_\alpha G_\alpha(V(x,t))\,, \tag{5.1.14}$$

with initial condition (5.1.2). Thus

$$\begin{aligned}(4\pi\varepsilon)^{\frac{m}{2}} U(x,t) &= \int_{\mathbb{R}^m} t^{-\frac{m}{2}} \exp\left[-\frac{|x-y|^2}{4\varepsilon t}\right] U_0(y)dy \\ &\quad - \int_0^t \int_{\mathbb{R}^m} (t-\tau)^{-\frac{m}{2}} \exp\left[-\frac{|x-y|^2}{4\varepsilon(t-\tau)}\right] \sum_{\alpha=1}^m \partial_\alpha G_\alpha(V(y,\tau))dyd\tau\,.\end{aligned} \tag{5.1.15}$$

We proceed to show that if T is sufficiently small, then $U \in \mathscr{V}$ and the map that carries V to U is a contraction. The unique fixed point of that map will then be the desired solution of (5.1.8), (5.1.2) on $[0,T]$.

In what follows, c will stand for a generic constant that may depend solely on ℓ and on bounds of $G(U)$ and its derivatives on $\overline{\mathscr{C}}$.

From (5.1.14), (5.1.11) and $\|\nabla U_0\|_\ell < \omega$, we deduce

$$\sup_{[0,T]} \|U(\cdot,t) - U_0(\cdot)\|_{L^\infty} \leq c\omega(1+\sqrt{T})\sqrt{T}, \tag{5.1.16}$$

$$\sup_{[0,T]} \|U(\cdot,t) - U_0(\cdot)\|_{L^2} \leq c\omega T, \tag{5.1.17}$$

which shows, in particular, that when T is sufficiently small, then, for any fixed $t \in [0,T], U(\cdot,t)$ takes values in a compact subset of $\mathscr{C}$, and $\|U(\cdot,t) - U_0(\cdot)\|_{L^2} < \omega$.

We now fix any multi-index r of order $1 \leq |r| \leq \ell+1$ and apply ∂^r to (5.1.14), which yields

$$\partial_t U_r(x,t) - \varepsilon \Delta U_r(x,t) = -\sum_{\alpha=1}^m \partial_\alpha \partial^r G_\alpha(V(x,t)), \tag{5.1.18}$$

where U_r stands for $\partial^r U$. We also set $U_{0r} = \partial^r U_0$. By virtue of (5.1.12) and standard Moser-type inequalities,

$$\sup_{[0,T]} \|\partial^r G_\alpha(V(\cdot,t))\|_0 \leq c\omega. \tag{5.1.19}$$

By standard theory of the heat equation, the solution U_r of (5.1.18) lies in $C^0([0,T];L^2)$. Furthermore, the energy estimate, obtained formally by multiplying (5.1.18) by $2U_r^\top$, integrating over $\mathbb{R}^m \times [0,t]$, for $t \in [0,T]$, and integrating by parts, is here valid:

$$\begin{aligned} &\int_{\mathbb{R}^m} |U_r(x,t)|^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^m} |\nabla U_r|^2 dx d\tau \\ &\qquad = \int_{\mathbb{R}^m} |U_{0r}(x)|^2 dx + 2\int_0^t \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha U_r^\top \partial^r G_\alpha(V) dx d\tau \\ &\qquad \leq \int_{\mathbb{R}^m} |U_{0r}(x)|^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}^m} |\nabla U_r|^2 dx d\tau + \frac{cT\omega^2}{\varepsilon}. \end{aligned} \tag{5.1.20}$$

Therefore, upon summing over all multi-indices r of order $1 \leq |r| \leq \ell+1$,

$$\sup_{[0,T]} \|\nabla U(\cdot,t)\|_\ell^2 \leq \|\nabla U_0(\cdot)\|_\ell^2 + \frac{cT\omega^2}{\varepsilon}. \tag{5.1.21}$$

Since $\|\nabla U_0\|_\ell < \omega$, when T is sufficiently small, $\|\nabla U(\cdot,t)\|_\ell < \omega$, for all $t \in [0,T]$, and hence $U \in \mathscr{V}$.

We now fix V and $\bar{V}$ and consider the solutions U and $\bar{U}$ of (5.1.14), (5.1.2), induced by them. Then

$$\partial_t(U-\bar{U}) - \varepsilon\Delta(U-\bar{U}) = -\sum_{\alpha=1}^m \partial_\alpha[G_\alpha(V) - G_\alpha(\bar{V})]. \tag{5.1.22}$$

Multiplying (5.1.22) by $2(U-\bar{U})^\top$, integrating over $\mathbb{R}^m \times [0,t]$, and integrating by parts yields

$$\begin{aligned} &\int_{\mathbb{R}^m} |U(x,t) - \bar{U}(x,t)|^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}^m} |\nabla(U-\bar{U})|^2 dx d\tau \\ &\qquad = 2\int_0^t \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha(U-\bar{U})^\top [G_\alpha(V) - G_\alpha(\bar{V})] dx d\tau \\ &\qquad \leq \varepsilon \int_0^t \int_{\mathbb{R}^m} |\nabla(U-\bar{U})|^2 dx d\tau + \frac{cT}{\varepsilon} \sup_{[0,T]} \int_{\mathbb{R}^m} |V(x,\tau) - \bar{V}(x,\tau)|^2 dx. \end{aligned} \tag{5.1.23}$$

Recalling (5.1.13), we conclude that $cT/\varepsilon = \mu^2 < 1$ implies contraction, $\rho(U,\bar{U}) \leq \mu\rho(V,\bar{V})$, which establishes the existence of a single fixed point and thereby the existence of a unique solution U to (5.1.8), (5.1.2) on $[0,T]$.

Since $\|\nabla U(\cdot,T)\|_\ell < \omega$ and $U(\cdot,T)$ takes values in a compact subset of $\mathscr{C}$, we may repeat the above construction and extend U to a longer time interval $[0,T']$. Continuing this process, we end up with a solution U to (5.1.8), (5.1.2) defined on a maximal interval $[0,T_\varepsilon)$, such that if $T_\varepsilon < \infty$ then, as $t \uparrow T_\varepsilon$, $\|\nabla U(\cdot,t)\|_\ell \to \omega$ and/or the range of $U(\cdot,t)$ escapes from every compact subset of $\mathscr{C}$. The proof is complete.

Next we construct solutions to the hyperbolic system (5.1.1). The presence of a convex entropy will now play a crucial role.

5.1.3 Lemma. *Let U_0 and $\mathscr{C}$ be as in Lemma* 5.1.2. *Then there is a positive constant c_0, depending solely on ℓ and $\mathscr{C}$, such that with any fixed $\omega > c_0\|\nabla U_0\|_\ell$ is associated $T > 0$ and a unique classical solution U of* (5.1.1), (5.1.2) *on* $[0,T]$, *having the following properties: For each $t \in [0,T]$, $U(\cdot,t)$ takes values in $\overline{\mathscr{C}}$, $\nabla U(\cdot,t) \in H^\ell$ and $\|\nabla U(\cdot,t)\|_\ell \le \omega$.*

Proof. Assume, temporarily, that $\nabla U_0 \in H^{\ell+2}$. Fix $\varepsilon \in (0,1)$ and consider the solution of (5.1.8), (5.1.2), defined on a maximal time interval $[0,T_\varepsilon)$, as established by Lemma 5.1.2. The aim is to show that T_ε is bounded below by some $T > 0$, uniformly in ε, and establish a priori bounds for U on $[0,T]$, independent of ε, which will allow us to construct the solution to (5.1.1), (5.1.2) by passing to the limit $\varepsilon \downarrow 0$.

For the remainder of this section, c will stand for a generic constant that depends solely on ℓ and on bounds of the functions G_α, η and their derivatives on $\mathscr{C}$.

We fix any multi-index r of order $1 \le |r| \le \ell+1$ and apply ∂^r to (5.1.8), which yields

$$\partial_t U_r(x,t) + \sum_{\alpha=1}^m \partial^r \partial_\alpha G_\alpha(U(x,t)) = \varepsilon \Delta U_r(x,t), \tag{5.1.24}$$

with $U_r = \partial^r U$. Since $\nabla U_0 \in H^{\ell+2}, \partial_t U_r \in L^\infty([0,T_\varepsilon);L^2)$. Multiplying (5.1.24) by $\partial_t U_r^\top$, integrating over $\mathbb{R}^m \times [0,t], 0 < t < T_\varepsilon$, and integrating by parts, we obtain

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^m} |\partial_t U_r|^2 dx d\tau + \frac{\varepsilon}{2}\int_{\mathbb{R}^m} |\nabla U_r(x,t)|^2 dx \\ = \frac{\varepsilon}{2}\int_{\mathbb{R}^m} |\nabla U_{0r}(x)|^2 dx - \int_0^t \int_{\mathbb{R}^m} \partial_t U_r^\top \sum_{\alpha=1}^m \partial^r \partial_\alpha G_\alpha(U) dx d\tau.\end{aligned} \tag{5.1.25}$$

By Moser-type estimates, for any r of order $|r| \le \ell$,

$$\|\partial^r \partial_\alpha G_\alpha(U(\cdot,\tau))\|_{L^2} \le c\|\nabla U(\cdot,\tau)\|_\ell\,. \tag{5.1.26}$$

Hence, applying the Cauchy-Schwarz inequality to (5.1.25) and then summing over all multi-indices r of order $|r| \le \ell$,

$$\int_0^t \|\partial_t U(\cdot,\tau)\|^2_{W^{\ell,2}} d\tau \le \varepsilon \|\nabla U_0(\cdot)\|_\ell^2 + c\int_0^t \|\nabla U(\cdot,\tau)\|_\ell^2 d\tau. \tag{5.1.27}$$

Next we rewrite (5.1.24) as

(5.1.28)
$$\partial_t U_r + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(U)\partial_\alpha U_r = \sum_{\alpha=1}^m \{\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\} + \varepsilon\Delta U_r\,,$$

we multiply by $2U_r^\top A(U)$, where $A(U)$ is the symmetric positive definite matrix defined by (5.1.3), and integrate over $\mathbb{R}^m \times [0,t]$. Notice that

$$2U_r^\top A(U)\partial_t U_r = \partial_t[U_r^\top A(U)U_r] - U_r^\top \partial_t A(U)U_r, \tag{5.1.29}$$

$$U_r^\top A(U)\Delta U_r \tag{5.1.30}$$
$$= \sum_{\alpha=1}^m \partial_\alpha[U_r^\top A(U)\partial_\alpha U_r] - \sum_{\alpha=1}^m \partial_\alpha U_r^\top A(U)\partial_\alpha U_r - \sum_{\alpha=1}^m U_r^\top \partial_\alpha A(U)\partial_\alpha U_r\,.$$

By the Cauchy inequality, since $A(U)$ is positive definite,

$$-\sum_{\alpha=1}^m U_r^\top \partial_\alpha A(U)\partial_\alpha U_r \leq \sum_{\alpha=1}^m \partial_\alpha U_r^\top A(U)\partial_\alpha U_r + c|\nabla U|^2|U_r|^2. \tag{5.1.31}$$

Furthermore, recalling the definition (5.1.4) of the symmetric matrices J_α:
(5.1.32)
$$\sum_{\alpha=1}^m 2U_r^\top A(U)\mathrm{D}G_\alpha(U)\partial_\alpha U_r = \sum_{\alpha=1}^m \partial_\alpha[U_r^\top J_\alpha(U)U_r] - \sum_{\alpha=1}^m U_r^\top \partial_\alpha J_\alpha(U)U_r.$$

We thus obtain

$$\int_{\mathbb{R}^m} U_r^\top(x,t)A(U(x,t))U_r(x,t)dx \leq \int_{\mathbb{R}^m} U_{0r}^\top(x)A(U_0(x))U_{0r}(x)dx \tag{5.1.33}$$
$$+\int_0^t\int_{\mathbb{R}^m} 2U_r^\top A(U)\sum_{\alpha=1}^m\{\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\}dxd\tau$$
$$+\int_0^t\int_{\mathbb{R}^m} U_r^\top\{\partial_t A(U) + \sum_{\alpha=1}^m \partial_\alpha J_\alpha(U)\}U_r dxd\tau + c\varepsilon\int_0^t\int_{\mathbb{R}^m}|\nabla U|^2|U_r|^2 dxd\tau.$$

Notice the Moser-type estimate

$$\|\mathrm{D}G_\alpha(U(\cdot,\tau))\partial_\alpha U_r(\cdot,\tau) - \partial^r[\mathrm{D}G_\alpha(U(\cdot,\tau))\partial_\alpha U(\cdot,\tau)]\|_{L^2} \tag{5.1.34}$$
$$\leq c\|\nabla U(\cdot,\tau)\|_{L^\infty}\|\nabla U(\cdot,\tau)\|_\ell\,.$$

Therefore, upon summing (5.1.33) over all multi-indices r of order $1 \leq |r| \leq \ell+1$, we obtain the estimate

$$\|\nabla U(\cdot,t)\|_\ell^2 \leq c\|\nabla U_0(\cdot)\|_\ell^2 + \int_0^t g(\tau)\|\nabla U(\cdot,\tau)\|_\ell^2 d\tau\,, \tag{5.1.35}$$

where

(5.1.36) $$0 \le g(\tau) \le c\{\|\nabla U(\cdot,\tau)\|_{L^\infty} + \|\partial_t U(\cdot,\tau)\|_{L^\infty} + \varepsilon\|\nabla U(\cdot,\tau)\|^2_{L^\infty}\}.$$

By Gronwall's inequality,

(5.1.37) $$\|\nabla U(\cdot,t)\|^2_\ell \le c\|\nabla U_0(\cdot)\|^2_\ell \exp\int_0^t g(\tau)d\tau.$$

On the other hand, by virtue of (5.1.36), (5.1.27), Schwarz's inequality and the Sobolev lemma,

(5.1.38)
$$\left\{\int_0^t g(\tau)d\tau\right\}^2 \le ct\left\{\varepsilon\|\nabla U_0(\cdot)\|^2_\ell + \int_0^t \|\nabla U(\cdot,\tau)\|^2_\ell d\tau\right\} + c\varepsilon^2\left\{\int_0^t \|\nabla U(\cdot,\tau)\|^2_\ell d\tau\right\}^2.$$

It is now clear that there is c_0 such that if one fixes $\omega > c_0\|\nabla U_0\|_\ell$, then, for T sufficiently small, (5.1.16) implies that $U(\cdot,t)$ takes values in $\mathscr{C}$ and (5.1.37), (5.1.38) together imply that $\|\nabla U(\cdot,t)\|_\ell < \omega$, for any $t \in [0,T]$. Moreover, by (5.1.27),

(5.1.39) $$\int_0^T \|\partial_t U(\cdot,t)\|^2_{W^{\ell,2}} dt \le c(T+1)\omega^2.$$

It should be emphasized that some T with the above specifications may be selected independently of ε and it is thus a common lower bound of T_ε , for all $\varepsilon \in (0,1)$.

Suppose now $\nabla U_0 \in H^\ell$. We fix a sequence $\{U_{0\nu}\}$, with $\nabla U_{0\nu} \in H^{\ell+2}$, and a sequence $\{\varepsilon_\nu\}$ in $(0,1)$, such that $U_{0\nu} \to U_0$, pointwise, $\nabla U_{0\nu} \to \nabla U_0$, in H^ℓ, and $\varepsilon_\nu \to 0$, as $\nu \to \infty$. Let U_ν denote the solution of (5.1.8), with $\varepsilon = \varepsilon_\nu$ and initial data $U_\nu(\cdot,0) = U_{0\nu}(\cdot)$, restricted to the time interval $[0,T]$, with T as above. By virtue of the above estimates, $\{\nabla U_\nu\}$ is contained in a bounded set of $L^\infty([0,T];H^\ell)$ and $\{\partial_t U_\nu\}$ is contained in a bounded set of $L^2([0,T];W^{\ell,2})$. By standard embedding theorems, $\{U_\nu\}$ is then uniformly equicontinuous on $\mathbb{R}^m \times [0,T]$. Therefore, we may extract a subsequence, again denoted by $\{U_\nu\}$, that converges to a continuous function U, uniformly on compact subsets of $\mathbb{R}^m \times [0,T]$. In particular, $U(\cdot,t)$ takes values in $\overline{\mathscr{C}}$, for all $t \in [0,T]$. Clearly, U satisfies the system (5.1.1) on $[0,T]$, in the sense of distributions, and also satisfies the initial conditions (5.1.2). Furthermore, for any fixed $t \in [0,T], \{\nabla U_\nu(\cdot,t)\}$ converges to $\nabla U(\cdot,t)$, weakly in H^ℓ, and hence $\nabla U(\cdot,t) \in L^\infty([0,T];H^\ell)$, with $\|\nabla U(\cdot,t)\|_\ell \le \omega$. It then follows from (5.1.1) that $\partial_t U(\cdot,t) \in L^\infty([0,T];W^{\ell,2})$. In particular, since $\ell > m/2, \nabla U$ and $\partial_t U$ are bounded, $|\nabla U| \le c\omega, |\partial_t U| \le c\omega$, so that U is a classical solution of (5.1.1), (5.1.2).

The uniqueness of the above solution will be established, in a far more general setting, in Section 5.3, and so it will be taken henceforth for granted. The proof is complete.

The next proposition highlights the local dependence property for solutions of the hyperbolic system (5.1.1). In what follows, for $\rho > 0$ and $\ell = 0,1,2,\cdots, H^\ell(\mathscr{B}_\rho)$ stands for the Sobolev space $W^{\ell,2}(\mathscr{B}_\rho;\mathbb{M}^{n\times m})$ on the ball $\mathscr{B}_\rho$.

5.1.4 Lemma. *In the setting and under the assumptions of Lemma* 5.1.3, *there are positive constants a,b,λ, depending solely on ℓ and $\mathscr{C}$, such that*

(5.1.40)
$$\|\nabla U(\cdot,t)\|^2_{H^\ell(\mathscr{B}_\rho)} \leq a\|\nabla U_0(\cdot)\|^2_{H^\ell(\mathscr{B}_{\rho+\lambda t})}\exp\{b\int_0^t \|\nabla U(\cdot,\tau)\|_{L^\infty(\mathscr{B}_{\rho+\lambda(t-\tau)})}d\tau\}.$$

In particular, letting $\rho \to \infty$,

$$\|\nabla U(\cdot,t)\|^2_\ell \leq a\|\nabla U_0(\cdot)\|^2_\ell \exp\{b\int_0^t \|\nabla U(\cdot,\tau)\|_{L^\infty(\mathbb{R}^m)}d\tau\}. \tag{5.1.41}$$

Proof. Assume, temporarily, $\nabla U_0 \in H^{\ell+2}$, so that $\nabla U(\cdot,t) \in L^\infty([0,T];H^{\ell+2})$. For any multi-index r of order $1 \leq |r| \leq \ell+1$ we apply ∂^r to (5.1.1) to get

$$\partial_t U_r + \sum_{\alpha=1}^m \mathrm{D}G_\alpha(U)\partial_\alpha U_r = \sum_{\alpha=1}^m \{\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\}, \tag{5.1.42}$$

where $U_r = \partial^r U$. All the terms in (5.1.42) are at least in $W^{1,2}(\mathbb{R}^m)$, so we may multiply by $2U_r^\top A(U)$ and use (5.1.29) and (5.1.32), which yields

$$\partial_t[U_r^\top A(U)U_r] + \sum_{\alpha=1}^m \partial_\alpha[U_r^\top J_\alpha(U)U_r] = I_r\,, \tag{5.1.43}$$

where

$$\begin{aligned} I_r = 2U_r^\top A(U)\sum_{\alpha=1}^m \{\mathrm{D}G_\alpha(U)\partial_\alpha U_r - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\} \\ +U_r^\top\{\partial_t A(U) + \sum_{\alpha=1}^m \partial_\alpha J_\alpha(U)\}U_r\,. \end{aligned} \tag{5.1.44}$$

We fix $\lambda > 0$ so large that the matrices

$$\lambda A(U) + \sum_{\alpha=1}^m \xi_\alpha J_\alpha(U) \tag{5.1.45}$$

are positive definite, for all $\xi \in S^{m-1}$ and $U \in \overline{\mathscr{C}}$. Then we integrate (5.1.43) over the frustum $\{(x,\tau) : 0 \leq \tau \leq \sigma, |x| \leq \rho + \lambda(t-\tau)\}$, for σ in $(0,t]$, and apply the divergence theorem. The resulting integral over the lateral surface of the frustum is nonnegative, because the matrices (5.1.45) are positive definite. We thus end up with the estimate

(5.1.46)
$$\begin{aligned} \int_{\mathscr{B}_{\rho+\lambda(t-\sigma)}} U_r^\top(x,\sigma)A(U(x,\sigma))U_r(x,\sigma)dx - \int_{\mathscr{B}_{\rho+\lambda t}} U_{0r}^\top(x)A(U_0(x))U_{0r}(x)dx \\ \leq \int_0^\sigma \int_{\mathscr{B}_{\rho+\lambda(t-\tau)}} I_r(x,\tau)dxd\tau. \end{aligned}$$

Using the estimate (5.1.34), albeit for the ball $\mathscr{B}_{\rho+\lambda(t-\tau)}$ rather than for the whole $\mathbb{R}^m$, we deduce

$$\int_{\mathscr{B}_{\rho+\lambda(t-\tau)}} I_r(x,\tau)d\tau \leq c\|\nabla U(\cdot,\tau)\|_{L^\infty(\mathscr{B}_{\rho+\lambda(t-\tau)})}\|\nabla U(\cdot,\tau)\|^2_{H^\ell(\mathscr{B}_{\rho+\lambda(t-\tau)})}. \tag{5.1.47}$$

Let us sum (5.1.46) over all r with $1 \leq |r| \leq \ell$. Since $A(U)$ is positive definite, setting $\|\nabla U(\cdot,\tau)\|_{L^\infty(\mathscr{B}_{\rho+\lambda(t-\tau)})} = g(\tau), \|\nabla U(\cdot,\tau)\|^2_{H^\ell(\mathscr{B}_{\rho+\lambda(t-\tau)})} = u(\tau)$, and combining (5.1.46) with (5.1.47), we end up with an estimate of the form

$$u(\sigma) \leq au(0) + \int_0^\sigma g(\tau)u(\tau)d\tau, \qquad 0 \leq \sigma \leq t, \tag{5.1.48}$$

whence (5.1.40) follows, by virtue of Gronwall's inequality.

Assume now U is the solution of (5.1.1), (5.1.2), with $\nabla U_0 \in H^\ell$. We fix a sequence $\{U_{0\nu}\}$, with $\nabla U_{0\nu} \in H^{\ell+2}$, such that $U_{0\nu} \to U_0$ uniformly on compact subsets of $\mathbb{R}^m$, and $\nabla U_{0\nu} \to \nabla U_0$ in H^ℓ, as $\nu \to \infty$. Let U_ν be the solution of (5.1.1) with initial value $U_{0\nu}$. We write (5.1.40) for the solutions U_ν and let $\nu \to \infty$. As $\{\nabla U_\nu\}$ is confined in a bounded set of $L^\infty([0,\tau];H^\ell)$, $\{\nabla U_\nu(\cdot,t)\}$ converges to $\nabla U(\cdot,t)$, weakly in H^ℓ, for any fixed $t \in [0,T]$. Since $\ell > m/2, \nabla U_\nu(\cdot,t) \to \nabla U(\cdot,t)$, uniformly on $\mathscr{B}_{\rho+\lambda(t-\tau)}$, for all $\tau \in [0,t]$. Therefore, (5.1.40) holds even when ∇U_0 is merely in H^ℓ. This completes the proof.

The remaining ingredient is the following regularity result.

5.1.5 Lemma. *Let U be the solution of* (5.1.1), (5.1.2) *on* $[0,T]$, *derived in Lemma* 5.1.3. *Then $t \mapsto \nabla U(\cdot,t)$ is continuous in H^ℓ on* $[0,T]$.

Proof. Since (5.1.1) is invariant under time translations, $(x,t) \mapsto (x,t+\tau)$, as well as under reflections, $(x,t) \mapsto (-x,-t)$, it will suffice to show that $t \mapsto \nabla U(\cdot,t)$ is right-continuous in H^ℓ at $t=0$, i.e., for any multi-index r of order $1 \leq |r| \leq \ell+1$, we have $U_r(\cdot,t) \to U_{0r}(\cdot)$ in $L^2(\mathbb{R}^m)$, as $t \to 0$. This will be attained with the help of the identity

$$\begin{aligned}(\bar{V}-V)^\top A(U)(\bar{V}-V) &= \bar{V}^\top A(\bar{U})\bar{V} - V^\top A(U)V \\ &\quad -\bar{V}^\top[A(\bar{U})-A(U)]\bar{V} - 2(\bar{V}-V)^\top A(U)V.\end{aligned} \tag{5.1.49}$$

As in the proof of Lemma 5.1.4, we fix some sequence $\{U_{0\nu}\}$, with $\nabla U_{0\nu} \in H^{\ell+2}$, such that $U_{0\nu} \to U_0$, uniformly on $\mathbb{R}^m$, and $\nabla U_{0\nu} \to \nabla U_0$ in H^ℓ, as $\nu \to \infty$. Let U_ν denote the solution of (5.1.1) with initial value $U_{0\nu}$. In particular, $U_{\nu r}$ should satisfy (5.1.46) and so, for any $\rho > 0$,

(5.1.50)
$$\int_{\mathscr{B}_\rho} U^\top_{\nu r}(x,t)A(U_\nu(x,t))U_{\nu r}(x,t)dx \leq \int_{\mathscr{B}_{\rho+\lambda t}} U^\top_{0\nu r}(x)A(U_{0\nu}(x))U_{0\nu r}(x)dx + c\omega^3 t,$$

with c independent of ρ.

We now write (5.1.49) for $U = U(x,t), \bar{U} = U_\nu(x,t), V = U_r(x,t), \bar{V} = U_{\nu r}(x,t)$ and integrate with respect to x over the ball $\mathscr{B}_\rho$. In the resulting equation, the left-hand side is nonnegative, as $A(U)$ is positive definite, while the last two terms on the right-hand side tend to zero, as $\nu \to \infty$, the first one because $U_\nu(\cdot,t) \to U(\cdot,t)$, uniformly on $\mathscr{B}_\rho$, and the second because $U_{\nu r}(\cdot,t) \to U_r(\cdot,t)$, weakly in L^2. Thus,

(5.1.51)
$$\int_{\mathscr{B}_\rho} U_r^\top(x,t)A(U(x,t))U_r(x,t)dx \leq \liminf_{\nu\to\infty} \int_{\mathscr{B}_\rho} U_{\nu r}^\top(x,t)A(U_\nu(x,t))U_{\nu r}(x,t)dx.$$

Combining (5.1.50) with (5.1.51), and then letting $\rho \to \infty$,

$$\int_{\mathbb{R}^m} U_r^\top(x,t)A(U(x,t))U_r(x,t)dx \leq \int_{\mathbb{R}^m} U_{0r}^\top(x)A(U_0(x))U_{0r}(x)dx + c\omega^3 t. \tag{5.1.52}$$

We return to (5.1.49), set $U = U_0(x), \bar{U} = U(x,t), V = U_{0r}(x), \bar{V} = U_r(x,t)$ and integrate with respect to x over $\mathbb{R}^m$. On the right-hand side of the resulting equation, the difference of the first two terms is bounded from above by $c\omega^3 t$, by virtue of (5.1.52); the third (penultimate) term is also bounded by $c\omega^3 t$, because $|U(x,t) - U_0(x)| \leq c\omega t$, for all $x \in \mathbb{R}^m$; finally, the last term tends to zero, as $t \to 0$, since $U_r(\cdot,t) \to U_{0r}(\cdot)$, in the sense of distributions and thus also weakly in $L^2(\mathbb{R}^m)$. Therefore,

$$\limsup_{t\to 0} \int_{\mathbb{R}^m} [U_r(x,t) - U_{0r}(x)]^\top A(U_0(x))[U_r(x,t) - U_{0r}(x)]dx \leq 0, \tag{5.1.53}$$

whence $U_r(\cdot,t) \to U_{0r}(\cdot)$ (strongly) in $L^2(\mathbb{R}^m)$, as $t \to 0$. This completes the proof.

Proof of Theorem 5.1.1. By Lemma 5.1.3, a classical solution U to (5.1.1), (5.1.2) exists on a time interval $[0,T]$, such that $U(\cdot,T)$ takes values in a compact subset of $\mathscr{O}$ and $\nabla U(\cdot,T) \in H^\ell$. We may thus repeat the above construction and extend U to a longer interval $[0,T']$. Continuing this process, we end up with a solution U defined on a maximal interval $[0,T_\infty)$, and if $T_\infty < \infty$ then, as $t \to T_\infty$, $\|\nabla U(\cdot,t)\|_\ell \to \infty$ and/or the range of U escapes from every compact subset of $\mathscr{O}$. However, (5.1.41) implies that $\|\nabla U(\cdot,t)\|_\ell$ cannot blow up as $t \to T_\infty$ unless (5.1.6) holds.

Lemma 5.1.5 implies $\nabla U(\cdot,t) \in C^0([0,T_\infty);H^\ell)$. Applying to (5.1.1) the operator ∂_t^r, for $r = 0,\cdots,\ell-1$, one shows by induction that $\partial_t^{r+1}U(\cdot,t) \in C^0([0,T_\infty);H^{\ell-r})$ and this establishes (5.1.5). The proof is complete.

As we saw in Sections 3.3.5 and 3.3.6, for the systems governing the isentropic or nonisentropic gas flow (Euler equations) the entropy is convex and hence Theorem 5.1.1 establishes the local existence of classical solutions to the Cauchy problem, under smooth initial data with positive density. At the same time, as shown in Chapter IV, finite life span is the rule rather than the exception for classical solutions of these systems.

5.1.6 Remark. The proof of Theorem 5.1.1 hinges on the existence of a symmetric positive definite matrix-valued function $A(U)$ that acts as a *symmetrizer* by rendering the matrix-valued functions $J_\alpha(U)$, defined by (5.1.4), symmetric. For that purpose, one may employ symmetrizers $A(U)$ that are not necessarily Hessians of convex entropies, as it was assumed above. This extra flexibility has been exploited for constructing solutions to the Euler equations with regions of vacuum, where hyperbolicity breaks down. The method extends to the broader class of systems endowed with so-called symbolic symmetrizers. This includes all systems in which the multiplicity of each characteristic speed $\lambda_i(\nu;U)$ does not vary with ν or U.

5.2 The Role of Damping and Relaxation

In this section we consider the Cauchy problem

$$\partial_t U(x,t) + \operatorname{div} G(U(x,t)) + P(U(x,t)) = 0, \qquad x \in \mathbb{R}^m, \ t > 0, \tag{5.2.1}$$

$$U(x,0) = U_0(x), \qquad x \in \mathbb{R}^m, \tag{5.2.2}$$

for a homogeneous hyperbolic system of balance laws in canonical form, where $G(U)$ and $P(U)$ are smooth functions defined on $\mathscr{O}$. We assume that $P(\bar{U}) = 0$, for some $\bar{U} \in \mathscr{O}$, so that $U \equiv \bar{U}$ is a constant equilibrium solution of (5.2.1).

Suppose (5.2.1) is endowed with a C^3 entropy-entropy flux pair (η, Q), where $\eta(U)$ is locally uniformly convex, so that any classical solution satisfies the additional balance law

$$\partial_t \eta(U(x,t)) + \operatorname{div} Q(U(x,t)) + \mathrm{D}\eta(U(x,t))P(U(x,t)) = 0. \tag{5.2.3}$$

Without loss of generality, we may assume $\eta(\bar{U}) = 0$, $\mathrm{D}\eta(\bar{U}) = 0$, $Q(\bar{U}) = 0$, $\mathrm{D}Q_\alpha(\bar{U}) = 0$, $\alpha = 1, \dots, m$, since otherwise we simply replace (η, Q) with the pair $(\bar{\eta}, \bar{Q})$ defined by (4.1.6), (4.1.7).

For initial data U_0 with $\nabla U_0 \in H^\ell$, $\ell > m/2$, a straightforward extension of Theorem 5.1.1 yields the existence of a classical solution to (5.2.1), (5.2.2) on a maximal time interval $[0, T_\infty)$. The aim is to investigate whether the mechanism that causes the breaking of waves may be offset by a dissipative source term that keeps $\|\nabla U(\cdot,t)\|_{L^\infty}$ bounded for all $t > 0$. Our experience with Equation (4.2.2), in Section 4.2, indicates that dissipation is likely to prevail near equilibrium.

Damping manifests itself in that the entropy production is nonnegative on some open neighborhood $\mathscr{B} \subset \mathscr{O}$ of $\bar{U}$:

$$\mathrm{D}\eta(U)P(U) \geq 0, \qquad U \in \mathscr{B}. \tag{5.2.4}$$

Under this assumption, for as long as U takes values in $\mathscr{B}$,

$$\|U(\cdot,t) - \bar{U}\|_{L^2} \leq a\|U_0(\cdot) - \bar{U}\|_{L^2}\,, \tag{5.2.5}$$

which is obtained by integrating (5.2.3) over $\mathbb{R}^m \times (0,t)$. This, combined with the "interpolation" estimate

$$\|U(\cdot,t)-\bar{U}\|_{L^\infty} \le b\|\nabla U(\cdot,t)\|_{L^\infty}^{\rho}\|U(\cdot,t)-\bar{U}\|_{L^2}^{1-\rho}, \tag{5.2.6}$$

where $\rho = \frac{1}{2}m(\ell+1)$, in turn implies that $U(\cdot,t)$ will lie in $\mathscr{B}$ for as long as $\|\nabla U(\cdot,t)\|_{L^\infty}$ stays sufficiently small.

As in Section 5.1, we fix any multi-index r of order $1 \le |r| \le \ell+1$, then set $\partial^r U = U_r$, and apply ∂^r to the equation (5.2.1) to get

$$\partial_t U_r + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(U)\partial_\alpha U_r + \mathrm{D}P(U)U_r \tag{5.2.7}$$

$$= \sum_{\alpha=1}^{m}\{\mathrm{D}G_\alpha(U)\partial^r\partial_\alpha U - \partial^r[\mathrm{D}G_\alpha(U)\partial_\alpha U]\} + \{\mathrm{D}P(U)\partial^s\partial_\beta U - \partial^s[\mathrm{D}P(U)\partial_\beta U]\},$$

where β is any fixed index in $\{1,\dots,m\}$ with $r_\beta \ge 1$, and s is the multi-index with $s_\gamma = r_\gamma$, for $\gamma \ne \beta$, and $s_\beta = r_\beta - 1$. We recall (5.1.34) and note its analog

$$\|\mathrm{D}P(U)\partial^s\partial_\beta U - \partial^s[\mathrm{D}P(U)\partial_\beta U]\|_{L^2} \le c\|\nabla U\|_{L^\infty}\|\nabla U\|_\ell. \tag{5.2.8}$$

Here and below c stands for a generic constant depending solely on the maximum on $\overline{\mathscr{B}}$ of U, all derivatives $|\mathrm{D}^k G(U)|$ up to order $k=\ell+2$, and all derivatives $|\mathrm{D}^k P(U)|$ up to order $k=\ell+1$.

When (5.2.4) holds, the matrix $A(\bar{U})\mathrm{D}P(\bar{U})$, with A defined by (5.1.3), is at least positive semidefinite. In particular, P is strongly dissipative at $\bar{U}$ if

$$W^\top A(\bar{U})\mathrm{D}P(\bar{U})W \ge \mu > 0, \qquad W \in S^{n-1}. \tag{5.2.9}$$

In that case, multiplying (5.2.7), from the left, by $2U_r^\top A(\bar{U})$, summing over all multi-indices r with $1 \le |r| \le \ell+1$, and integrating the resulting equation over $\mathbb{R}^m \times (0,t)$, we arrive at an estimate of the form

$$\|\nabla U(\cdot,t)\|_\ell^2 + 2\mu\int_0^t \|\nabla U(\cdot,\tau)\|_\ell^2 d\tau \tag{5.2.10}$$

$$\le c\|\nabla U_0(\cdot)\|_\ell^2 + c\int_0^t \{\|\nabla U(\cdot,\tau)\|_{L^\infty} + \|U(\cdot,\tau)-\bar{U}\|_{L^\infty}\}\|\nabla U(\cdot,\tau)\|_\ell^2 d\tau.$$

So long as $\|\nabla U(\cdot,\tau)\|$ stays small, the integral on the left-hand side of (5.2.10) dominates the integral on the right-hand side and induces $\|\nabla U\|_\ell^2 \le c\|\nabla U_0\|_\ell^2$. Since $\|\nabla U\|_{L^\infty} \le \kappa\|\nabla U\|_\ell$, we conclude that if $\|\nabla U_0\|_\ell$ is sufficiently small, then $\|\nabla U\|_\ell$, and thereby $\|\nabla U\|_{L^\infty}$, stay small throughout the life span of the solution and thus the life span cannot be finite.

Unfortunately, assumption (5.2.9) is too stringent, as it generally rules out the type of source term associated with the dissipative mechanisms encountered in continuum physics. A typical example is the system that governs isentropic gas flow through a porous medium, namely (3.3.36) with body force $-v$:

$$\text{(5.2.11)} \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\ \partial_t(\rho v) + \operatorname{div}(\rho v v^\top) + \operatorname{grad} p(\rho) + \rho v = 0. \end{cases}$$

This difficulty is also encountered in systems with source terms induced by relaxation effects, for instance (3.3.25). Typically, in the applications, $A(\bar{U})\mathrm{D}P(\bar{U})$ is merely positive semidefinite. In this situation, the damping fails to be effective, unless the source term satisfies an additional condition which ensures that waves of any characteristic family, propagating in any direction $\nu \in S^{m-1}$, are properly damped. The appropriate assumption, similar to the Kawashima condition (4.6.4), reads

$$\text{(5.2.12)} \qquad \mathrm{D}P(\bar{U})R_i(\nu;\bar{U}) \neq 0, \qquad \nu \in S^{m-1},\ \ i = 1,\dots,n,$$

where $R_i(\nu;U)$ is any eigenvector of the matrix $\Lambda(\nu;U)$, in (4.1.2), associated with the eigenvalue $\lambda_i(\nu;U)$. To see the implications of (5.2.12), linearize (5.2.1) about $\bar{U}$:

$$\text{(5.2.13)} \qquad \partial_t V(x,t) + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(\bar{U})\partial_\alpha V(x,t) + \mathrm{D}P(\bar{U})V(x,t) = F(x,t).$$

Notice that when (5.2.12) is violated for some i and ν, then (5.2.13), with $F \equiv 0$, admits traveling wave solutions (4.6.5), which are not attenuated by the damping. On the other hand, it can be shown that when (5.2.12) holds, then, for any $\nu \in S^{m-1}$, there exists a skew symmetric $n \times n$ matrix $K(\nu)$ such that the matrix

$$\text{(5.2.14)} \qquad K(\nu)\Lambda(\nu;\bar{U}) + A(\bar{U})\mathrm{D}P(\bar{U})$$

is positive definite. This in turn implies that solutions of (5.2.13) satisfy an estimate

$$\text{(5.2.15)} \qquad \int_0^t \int_{\mathbb{R}^m} |V|^2(x,\tau)\,dx\,d\tau \leq \kappa \int_0^t \int_{\mathbb{R}^m} V^\top(x,\tau)A(\bar{U})\mathrm{D}P(\bar{U})V(x,\tau)\,dx\,d\tau$$
$$+\kappa \int_{\mathbb{R}^m} [|V|^2(x,t) + |V|^2(x,0)]\,dx + \kappa \int_0^t \int_{\mathbb{R}^m} |F|^2(x,\tau)\,dx\,d\tau.$$

As before, we multiply (5.2.7), from the left, by $2U_r^\top A(\bar{U})$, we sum over all multi-indices r with $1 \leq |r| \leq \ell + 1$, and integrate over $\mathbb{R}^m \times (0,t)$. Upon combining the resulting equation with the estimate (5.2.15), one reestablishes (5.2.10), for some $\mu > 0$, thus proving the following

5.2.1 Theorem. *Consider the hyperbolic system of balance laws* (5.2.1)*, with G in $C^{\ell+2}$ and P in $C^{\ell+1}$, for some $\ell > m/2$. Assume $P(\bar{U}) = 0$ and $\mathrm{D}P(\bar{U})$ satisfies* (5.2.12). *Furthermore, let η be a C^3 entropy for* (5.2.1) *such that $\mathrm{D}^2\eta(\bar{U})$ is positive definite and* (5.2.4) *holds on some neighborhood $\mathscr{B}$ of $\bar{U}$. When $U_0 - \bar{U} \in L^2(\mathbb{R}^m)$,*

$\nabla U_0 \in H^\ell$ and $\|\nabla U_0\|_\ell$ is sufficiently small, then the Cauchy problem (5.2.1)*,* (5.2.2) *admits a unique classical solution U on the upper half-space, such that*

$$\nabla U(\cdot,t) \in C^0([0,\infty);H^\ell) \cap L^2([0,\infty);H^\ell). \tag{5.2.16}$$

When $\bar{U}$ is a strict minimum of $\mathrm{D}\eta(U)P(U)$, it is expected that dissipation will drive the solution obtained in the above theorem to this isolated equilibrium point, as $t \to \infty$. Of far greater interest is the long time behavior of solutions of (5.2.1), (5.2.2) when the source vanishes on a manifold in state space. This is typically the case with systems governing relaxation phenomena.

Upon rescaling the coordinates by $(x,t) \mapsto (\mu x, \mu t)$, where $\mu > 0$ is the so-called *relaxation parameter*, we recast (5.2.1) in the form

$$\partial_t U(x,t) + \operatorname{div} G(U(x,t)) + \frac{1}{\mu} P(U(x,t)) = 0, \qquad x \in \mathbb{R}^m,\ t > 0. \tag{5.2.17}$$

Thus, the asymptotic behavior of solutions of (5.2.1), as $t \uparrow \infty$, will be derived from the asymptotic behavior of solutions of (5.2.17), as $\mu \downarrow 0$.

The following assumptions on P embody the structure typically encountered in systems governing relaxation phenomena:

(a) For some $k < n$, there is a constant $k \times n$ matrix K such that $KP(U) = 0$, for all $U \in \mathscr{O}$.
(b) There is a k-dimensional *local equilibrium manifold,* embedded in $\mathscr{O}$, which is defined by a smooth function $U = E(V)$, $V \in \mathscr{V} \subset \mathbb{R}^k$, such that $P(E(V)) = 0$ and $KE(V) = V$, for all $V \in \mathscr{V}$.

As a representative example, consider the system

$$\begin{cases} \partial_t u(x,t) + \partial_x v(x,t) = 0 \\ \partial_t v(x,t) + \partial_x p(u(x,t)) + \frac{1}{\mu}[v(x,t) - f(u(x,t))] = 0 \end{cases} \tag{5.2.18}$$

of two balance laws in one spatial variable, where $p'(u) = a^2(u)$, $a(u) > 0$. Here $K = (1,0)$, $V = u$, and $E(u) = (u, f(u))^\top$.

The expectation is that, as $\mu \downarrow 0$, the stiff source will induce U to relax on its local equilibrium manifold $U = E(V)$, with V satisfying the *relaxed system* of conservation laws

$$\partial_t V(x,t) + \operatorname{div} \hat{G}(V(x,t)) = 0, \qquad x \in \mathbb{R}^m,\ \ t > 0, \tag{5.2.19}$$

where

$$\hat{G}(V) = KG(E(V)), \qquad V \in \mathscr{V}. \tag{5.2.20}$$

For the system (5.2.18), (5.2.19) reduces to the scalar conservation law

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0. \tag{5.2.21}$$

We now explore the implications of the dissipativeness of the source P as encoded in the existence of an entropy-entropy flux pair (η, Q) for (5.2.17) which satisfies (5.2.4), for all $U \in \mathscr{O}$. In particular, $\mathrm{D}\eta(U)P(U)$ is minimized on the local equilibrium manifold, and so

$$\mathrm{D}\eta(E(V))\mathrm{D}P(E(V)) = 0, \qquad V \in \mathscr{V}. \tag{5.2.22}$$

We also have $K\mathrm{D}P(U) = 0$, $U \in \mathscr{O}$. Hence, assuming that the rank of $\mathrm{D}P(E(V))$ is $n-k$, for any $V \in \mathscr{V}$, we conclude

$$\mathrm{D}\eta(E(V)) = M(V)K, \qquad V \in \mathscr{O}, \tag{5.2.23}$$

for some k-row vector-valued function M on $\mathscr{O}$.

We now set

$$\hat{\eta}(V) = \eta(E(V)), \qquad \hat{Q}(V) = Q(E(V)), \tag{5.2.24}$$

and show that $(\hat{\eta}, \hat{Q})$ is an entropy-entropy flux pair for the relaxed system (5.2.19). Indeed, recalling (4.1.4), (5.2.20), (5.2.23) and noting that $KE(V) = V$ implies $K\mathrm{D}_V E = I$, we deduce, by the chain rule,

$$\begin{aligned} \mathrm{D}_V\hat{\eta}\,\mathrm{D}_V\hat{G}_\alpha &= \mathrm{D}_U\eta\,\mathrm{D}_V E\,K\,\mathrm{D}_U G_\alpha\,\mathrm{D}_V E = MK\,\mathrm{D}_V E\,K\,\mathrm{D}_U G_\alpha\,\mathrm{D}_V E \\ &= MK\,\mathrm{D}_U G_\alpha\,\mathrm{D}_V E = \mathrm{D}_U\eta\,\mathrm{D}_U G_\alpha\,\mathrm{D}_V E = \mathrm{D}_U Q_\alpha\,\mathrm{D}_V E = \mathrm{D}_V\hat{Q}_\alpha\,. \end{aligned} \tag{5.2.25}$$

It has also been shown (references in Section 5.7) that if $\mathrm{D}(U)P(U)$ is strictly positive away from the local equilibrium manifold and $\mathrm{D}^2\eta(U)$ is positive definite on $\mathscr{O}$, then $\mathrm{D}^2\hat{\eta}(V)$ is positive definite on $\mathscr{V}$, in which case the relaxed system (5.2.19) is hyperbolic. Moreover, all characteristic speeds of (5.2.19), in any direction $\nu \in S^{m-1}$ and state $V \in \mathscr{V}$, are confined between the minimum and the maximum characteristic speed of (5.2.17), in the direction ν and state $U = E(V)$. This last property expresses the *subcharacteristic condition* which has important implications for stability.

As noted above, the objective is to demonstrate that, as $\mu \downarrow 0$, the solution U_μ of (5.2.17), (5.2.2) converges to $E(V)$, where V is the solution of the relaxed system (5.2.19) with initial value $V_0 = KU_0$. When the initial data U_0 do not lie on the local equilibrium manifold, i.e., $U_0 \neq E(V_0)$, then as $\mu \downarrow 0$, U_μ will develop a boundary layer across $t = 0$, connecting U_0 to $E(V_0)$.

The asymptotic behavior of U_μ, as $\mu \downarrow 0$, has been analyzed within the context of classical solutions, for quite general systems. The reader should consult the relevant references cited in Section 5.7. Additional information can be found in Sections 6.6 and 16.5.

An intimate relation exists between dissipation induced by relaxation and dissipation induced by viscosity. The reader may catch a first glimpse through the following formal calculation for the simple system (5.2.18).

We set

$$v = f(u) + \mu w \tag{5.2.26}$$

and substitute into (5.2.18). Dropping, formally, all terms of order μ and then eliminating $\partial_t u$ between the two equations of the system yields

$$w = [f'(u)^2 - a(u)^2]\partial_x u. \tag{5.2.27}$$

Upon combining $(5.2.18)_1$ with (5.2.26) and (5.2.27), we deduce that, formally, to leading order, u satisfies the equation

$$\partial_t u + \partial_x f(u) = \mu \partial_x \{[a^2(u) - f'(u)^2]\partial_x u\}. \tag{5.2.28}$$

For well-posedness we need

$$-a(u) < f'(u) < a(u). \tag{5.2.29}$$

Since $\pm a(u)$ are the characteristic speeds of (5.2.18) and $f'(u)$ is the characteristic speed of (5.2.21), (5.2.29) expresses the subcharacteristic condition encountered above.

An analogous calculation, with analogous conclusions, applies to the general system (5.2.17) as well. In fact the Kawashima-type conditions (4.6.4) and (5.2.12) are intimately related. The reader can find details in the literature cited in Section 5.7.

In continuum physics, one encounters a host of evolutionary systems with the feature that wave amplification induced by nonlinear advection cohabits and competes with some kind of dissipation; and the former is in control far from equilibrium, while the latter prevails in the vicinity of equilibrium, securing the existence of smooth solutions in the large. Such systems are generally treated by methods akin to those employed in this section, namely "energy" type estimates that bring out the balance between amplification and damping. This subject, which already commands a large body of literature, lies beyond the scope of the present book. Nevertheless, in order to give a taste of the wide diversity of systems with such features, a few representative examples will be recorded below, and a small sample of relevant references will be listed in Section 5.7.

We begin with the so-called Euler-Poisson system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\ \partial_t(\rho v) + \operatorname{div}(\rho v v^\top) + \operatorname{grad} p(\rho) = a\rho \operatorname{grad}\psi \\ \Delta\psi = b(\rho - \bar\rho), \end{cases} \tag{5.2.30}$$

which models the movement of electrons in a plasma. In that connection, the aggregate of the electrons is regarded as an elastic fluid with density ρ and pressure $p(\rho)$, flowing with velocity v; while the much heavier ions are assumed stationary, merely providing a uniform background of positive charge, proportional to $\bar\rho$. The combined charge of electrons and ions, which is proportional to $\rho - \bar\rho$, generates the electrostatic potential ψ, and thereby the electric field $\operatorname{grad}\psi$ that sets the electrons in motion. The constants a and b are positive. As in (5.2.11), we are dealing here with the hyperbolic system of the Euler equations, with a source induced by some feedback mechanism, which derives from the Poisson equation $(5.2.30)_3$ and is dissipative at least when the flow of electrons is irrotational, $\operatorname{curl} v = 0$. Recall from Section 3.3.6 that flows starting out irrotational stay irrotational for as long as they are smooth. It has been shown that sufficiently smooth, irrotational Cauchy data, close to equilibrium $\rho = \bar{\rho}, v = 0, \psi = 0$, generate globally defined smooth solutions. On the other hand, solutions starting out far from equilibrium generally develop singularities in a finite time.

The situation is similar for the system

$$
\begin{cases} \partial_t \rho + \operatorname{div}(\rho v^\top) = 0 \\ \partial_t(\rho v) + \operatorname{div}(\rho v v^\top) + \operatorname{grad} p(\rho) + \mu^{-1} \rho v = a\rho \operatorname{grad} \psi \\ \Delta \psi = b(\rho - \bar{\rho}), \end{cases} \tag{5.2.31}
$$

associated with the hydrodynamic model of semiconductors. Here $\mu > 0$ is a relaxation parameter. Notice that (5.2.31) combines the dissipative mechanisms encountered in (5.2.11) and (5.2.30).

The balance laws for continuous media with internal friction, such as viscosity and heat conductivity, yield systems exhibiting similar behavior. The reason is that one may trace the lineage of these media back to elasticity, and hence, even though the resulting systems are not hyperbolic, they inherit features of hyperbolicity, giving rise to a destabilizing wave amplification mechanism that competes with the damping induced by the internal friction.

A first example is the system (4.6.2), which governs the flow of heat conducting thermoviscoelastic fluids with Newtonian viscosity. Internal friction manifests itself on the right-hand side of the second and the third equation, while the first equation retains its hyperbolic character.

Still another example with similar features is the system

$$
\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v - \partial_x \sigma(u,\theta) = 0 \\ \partial_t [\varepsilon(u,\theta) + \tfrac{1}{2} v^2] - \partial_x [\sigma(u,\theta) v] = \partial_x q(u,\theta,\partial_x \theta), \end{cases} \tag{5.2.32}
$$

which governs rectilinear motion, in Lagrangian coordinates, of a heat-conducting thermoelastic medium. Here u is the strain (deformation gradient), v is the velocity, θ is the (absolute) temperature, σ is the stress, ε is the internal energy, q is the heat flux, and the reference density is taken to be one. For compliance with (2.5.28) and (2.5.29), the material response functions ε, σ and q must satisfy $\varepsilon_u = \sigma - \theta \sigma_\theta$ and $q(u,\theta,g)g \geq 0$. These should be supplemented with the natural assumptions $\sigma_u > 0$, $\varepsilon_\theta > 0$ and $q_g > 0$. Here internal friction is provided by thermal diffusion.

Internal friction of yet another nature, but with similar effects, is induced by fading memory, encountered in viscoelastic continuous media in which the stress σ

at the particle x and time t is no longer solely determined, as in elastic materials, by the deformation gradient at (x,t), but also depends on the past history of the deformation gradient at x. The balance laws are then expressed by functional-partial differential equations. A simple, one-dimensional model system that captures the damping effect of memory reads

$$\begin{cases} \partial_t u(x,t) + \partial_x v(x,t) = 0 \\ \partial_t v(x,t) + \partial_x p(u(x,t)) + \int_{-\infty}^{t} k'(t-\tau)\partial_x q(u(x,\tau))d\tau = 0, \end{cases} \tag{5.2.33}$$

where k is a smooth integrable relaxation kernel on $[0,\infty)$, with $k(\tau) > 0$, $k'(\tau) < 0$ and $k''(\tau) \geq 0$, for $\tau \in [0,\infty)$, and $p'(u) > k(0)q'(u) > 0$. Notice that (5.2.33) is intimately related to (5.2.18), as the latter system, for $f \equiv 0$, may be rewritten in the form

$$\begin{cases} \partial_t u(x,t) + \partial_x v(x,t) = 0 \\ \partial_t v(x,t) + \partial_x p(u(x,t)) + \int_{-\infty}^{t} [\exp(-\frac{t-\tau}{\mu})]'\partial_x p(u(x,\tau))d\tau = 0. \end{cases} \tag{5.2.34}$$

The above systems, (5.2.32) and (5.2.33), share the property that smooth initial data near equilibrium generate globally defined smooth solutions, while smooth solutions starting out from "large" initial values generally blow up in finite time. See the relevant references in Section 5.7.

An alternative decay mechanism acting on the systems of balance laws of continuum physics is dispersion. It is particularly effective when the dimension of the space is large and solutions stay close to equilibrium. In systems that are fully nonlinear, such as the Euler equations, dispersion may delay but not prevent the breaking of waves. However, in systems with gentler nonlinearity, satisfying the so-called *null condition*, dispersion renders the existence of globally defined smooth solutions to the Cauchy problem, with initial data close to equilibrium. As a typical example, consider the system (3.3.19) of equations of isentropic elastodynamics. For convenience, assume that the reference space coincides with the physical space, and that the reference configuration, with $F \equiv I$, is an isotropic equilibrium state, so that the internal energy $\varepsilon(F)$ is a function (2.5.21) of the principal invariants (J_1, J_2, J_3) of the right stretch tensor (2.1.7). Assume, further, that $\varepsilon(F)$ is rank-one convex and satisfies the null condition

$$\sum_{i,j,k=1}^{3} \sum_{\alpha,\beta,\gamma=1}^{3} \frac{\partial^3 \varepsilon(F)}{\partial F_{i\alpha}\partial F_{j\beta}\partial F_{k\gamma}} \nu_\alpha \nu_\beta \nu_\gamma \nu_i \nu_j \nu_k = 0, \tag{5.2.35}$$

at $F = I$, for any vector $\nu \in \mathbb{R}^3$. Then the Cauchy problem with initial data (F_0, v_0) close to $(I,0)$, in an appropriate Sobolev space, admits a unique, globally defined classical solution. For isotropic incompressible elastic media, the relevant null condition is automatically satisfied. There is voluminous literature on these issues, a sample of which is cited in Section 5.7.

5.3 Convex Entropy and the Stability of Classical Solutions

The aim here is to show that the presence of a convex entropy guarantees that classical solutions of the initial value problem depend continuously on the initial data, even within the broader class of admissible bounded weak solutions.

5.3.1 Theorem. *Assume the system of conservation laws* (5.1.1) *is endowed with an entropy-entropy flux pair* (η, Q), *where* $\mathrm{D}^2\eta(U)$ *is positive definite on* $\mathscr{O}$. *Suppose* $\bar{U}$ *is a classical solution of* (5.1.1) *on* $[0,T)$, *taking values in a convex compact subset* $\mathscr{D}$ *of* $\mathscr{O}$, *with initial data* $\bar{U}_0$. *Let* U *be any weak solution of* (5.1.1) *on* $[0,T)$, *taking values in* $\mathscr{D}$, *which satisfies the entropy admissibility condition* (4.5.3), *and has initial data* U_0 . *Then*

$$\int_{|x|<r} |U(x,t) - \bar{U}(x,t)|^2 dx \le ae^{bt} \int_{|x|<r+st} |U_0(x) - \bar{U}_0(x)|^2 dx \tag{5.3.1}$$

holds for any $r > 0$ *and* $t \in [0,T)$, *with positive constants* s, a, *depending solely on* $\mathscr{D}$, *and* b *that also depends on the Lipschitz constant of* $\bar{U}$. *In particular,* $\bar{U}$ *is the unique admissible weak solution of* (5.1.1) *with initial data* $\bar{U}_0$ *and values in* $\mathscr{D}$.

Proof. On $\mathscr{D} \times \mathscr{D}$ we define the functions

$$h(U,\bar{U}) = \eta(U) - \eta(\bar{U}) - \mathrm{D}\eta(\bar{U})[U - \bar{U}], \tag{5.3.2}$$

$$Y_\alpha(U,\bar{U}) = Q_\alpha(U) - Q_\alpha(\bar{U}) - \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})], \tag{5.3.3}$$

$$Z_\alpha(U,\bar{U}) = A(\bar{U})\{G_\alpha(U) - G_\alpha(\bar{U}) - \mathrm{D}G_\alpha(\bar{U})[U - \bar{U}]\}, \tag{5.3.4}$$

all of quadratic order in $U - \bar{U}$ (recall (4.1.4) and (5.1.3)). Consequently, since $\mathrm{D}^2\eta(U)$ is positive definite, uniformly on $\mathscr{D}$, there is a positive constant s such that

$$|Y(U,\bar{U})| \le sh(U,\bar{U}). \tag{5.3.5}$$

Let us fix any nonnegative, Lipschitz continuous test function ψ with compact support on $\mathbb{R}^m \times [0,T)$ and evaluate h, Y and Z along the two solutions $U(x,t), \bar{U}(x,t)$. Since U satisfies the inequality (4.5.3), while $\bar{U}$, being a classical solution, satisfies identically (4.5.3) as an equality, we deduce

(5.3.6)

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^m} [\partial_t \psi\, h(U,\bar{U}) + \sum_{\alpha=1}^m \partial_\alpha \psi\, Y_\alpha(U,\bar{U})] dxdt + \int_{\mathbb{R}^m} \psi(x,0)\, h(U_0(x), \bar{U}_0(x))\, dx \\
&\ge -\int_0^T \int_{\mathbb{R}^m} \{\partial_t \psi \mathrm{D}\eta(\bar{U})[U - \bar{U}] + \sum_{\alpha=1}^m \partial_\alpha \psi \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})]\} dxdt \\
&\qquad - \int_{\mathbb{R}^m} \psi(x,0)\, \mathrm{D}\eta(\bar{U}_0(x))[U_0(x) - \bar{U}_0(x)] dx.
\end{aligned}$$

Next we write (4.3.2) for both solutions U and $\bar{U}$, using the Lipschitz continuous vector field $\psi \mathrm{D}\eta(\bar{U})$ as test function Φ, to get

(5.3.7) $$\int_0^T \int_{\mathbb{R}^m} \{\partial_t[\psi \mathrm{D}\eta(\bar{U})][U-\bar{U}] + \sum_{\alpha=1}^m \partial_\alpha[\psi \mathrm{D}\eta(\bar{U})][G_\alpha(U) - G_\alpha(\bar{U})]\} dxdt$$
$$+ \int_{\mathbb{R}^m} \psi(x,0) \mathrm{D}\eta(\bar{U}_0(x))[U_0(x) - \bar{U}_0(x)]dx = 0.$$

Since $\bar{U}$ is a classical solution of (5.1.1), and by virtue of (5.1.3), (5.1.4),

(5.3.8)
$$\partial_t \mathrm{D}^2\eta(\bar{U}) = \partial_t \bar{U}^\top A(\bar{U}) = -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top J_\alpha(\bar{U})^\top = -\sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top A(\bar{U}) \mathrm{D}G_\alpha(\bar{U})$$

so that, recalling (5.3.4),

(5.3.9) $$\partial_t \mathrm{D}\eta(\bar{U})[U-\bar{U}] + \sum_{\alpha=1}^m \partial_\alpha \mathrm{D}\eta(\bar{U})[G_\alpha(U) - G_\alpha(\bar{U})] = \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U,\bar{U}).$$

Combining (5.3.6), (5.3.7) and (5.3.9) yields

(5.3.10)
$$\int_0^T \int_{\mathbb{R}^m} [\partial_t \psi h(U,\bar{U}) + \sum_{\alpha=1}^m \partial_\alpha \psi Y_\alpha(U,\bar{U})] dxdt + \int_{\mathbb{R}^m} \psi(x,0) h(U_0(x), \bar{U}_0(x)) dx$$
$$\geq \int_0^T \int_{\mathbb{R}^m} \psi \sum_{\alpha=1}^m \partial_\alpha \bar{U}^\top Z_\alpha(U,\bar{U}) dxdt.$$

We now fix $t \in (0,T)$ and $r > 0$. For any $\sigma \in (0,t]$ and ε positive small, write (5.3.10) for the test function $\psi(x,\tau) = \chi(x,\tau)\omega(\tau)$, with

(5.3.11) $$\omega(\tau) = \begin{cases} 1 & 0 \leq \tau < \sigma \\ \varepsilon^{-1}(\sigma - \tau) + 1 & \sigma \leq \tau < \sigma + \varepsilon \\ 0 & \sigma + \varepsilon \leq \tau < \infty \end{cases}$$

(5.3.12) $$\chi(x,\tau) = \begin{cases} 1 & |x| - r - s(\sigma - \tau) < 0 \\ \varepsilon^{-1}[r + s(t-\tau) - |x|] + 1 & 0 \leq |x| - r - s(t-\tau) < \varepsilon \\ 0 & |x| - r - s(t-\tau) \geq \varepsilon \end{cases}$$

where s is the constant appearing in (5.3.5). The calculation gives

(5.3.13)

$$\frac{1}{\varepsilon}\int_{\sigma}^{\sigma+\varepsilon}\int_{|x|<r+s(t-\sigma)} h(U(x,\tau),\bar{U}(x,\tau))dxd\tau \le \int_{|x|<r+st} h(U_0(x),\bar{U}_0(x))dx$$
$$-\frac{1}{\varepsilon}\int_0^{\sigma}\int_{r+s(t-\tau)<|x|<r+s(t-\tau)+\varepsilon}\left[sh(U,\bar{U})+\frac{Y(U,\bar{U})x}{|x|}\right]dxd\tau$$
$$-\int_0^{\sigma}\int_{|x|<r+s(t-\tau)}\sum_{\alpha=1}^{m}\partial_\alpha\bar{U}^\top Z_\alpha(U,\bar{U})dxd\tau+O(\varepsilon).$$

We let $\varepsilon\downarrow 0$. The second integral on the right-hand side of (5.3.13) is nonnegative on account of (5.3.5). Hence,

$$\int_{|x|<r+s(t-\sigma)} h(U(x,\sigma),\bar{U}(x,\sigma))dx \le \int_{|x|<r+st} h(U_0(x),\bar{U}_0(x))dx \tag{5.3.14}$$
$$-\int_0^{\sigma}\int_{|x|<r+s(t-\tau)}\sum_{\alpha=1}^{m}\partial_\alpha\bar{U}^\top Z_\alpha(U,\bar{U})dxd\tau,$$

for all points σ of L^∞ weak* continuity of $\eta(U(\cdot,\tau))$ in $(0,t)$. As noted above, $h(U,\bar{U})$ and the $Z_\alpha(U,\bar{U})$ are of quadratic order in $U-\bar{U}$ and, in addition, $h(U,\bar{U})$ is positive definite, due to the convexity of η. Thus, upon setting

$$u(\tau)=\int_{|x|<r+s(t-\tau)}|U(x,\tau)-\bar{U}(x,\tau)|^2dx\,, \tag{5.3.15}$$

(5.3.14) implies

$$u(\sigma)\le au(0)+b\int_0^{\sigma}u(\tau)d\tau\,, \tag{5.3.16}$$

for almost all $\sigma\in(0,t)$. Since $u(\cdot)$ is weakly lower semicontinuous, (5.3.16) holds for all $\sigma\in[0,t]$. Then Gronwall's inequality yields $u(t)\le au(0)e^{bt}$, which is (5.3.1). Notice that a and s depend solely on $\mathscr{D}$ while b also depends on the Lipschitz constant of $\bar{U}$. This completes the proof.

It is remarkable that a single entropy inequality, with convex entropy, manages to weed out all but one solution of the initial value problem, so long as a classical solution exists. As we shall see, however, when no classical solution exists, just one entropy inequality is no longer generally sufficient to single out any particular weak solution. In particular, it has been shown recently (references in Section 4.8) that the Cauchy problem for the Euler equations (3.3.36), under specially constructed initial data, admits infinitely many weak solutions satisfying the entropy admissibility condition (4.5.3), relative to the entropy $\rho\varepsilon(\rho)+\frac{1}{2}\rho|v|^2$, as an equality. The issue of uniqueness of weak solutions is knotty and will be a major topic of discussion in subsequent chapters.

The functions $h(U,\bar{U})$ and $Y(U,\bar{U})$, defined by (5.3.2) and (5.3.3), are commonly called the *relative entropy* and associated *relative entropy flux*, with respect to the state $\bar{U}$.

5.3.2 Remark. In the proof of Theorem 5.3.1 one needs only that $h(U,\bar{U})$ be positive definite for all $\bar{U}$ in the range of the classical solution. This may well hold, even for η that fails to be convex, when the classical solution is special, e.g., it is a constant state $\bar{U}$ which is a strong minimum of η.

5.4 Involutions

The previous three sections have illustrated the beneficent role of convex entropies. Nevertheless, the entropy associated with systems of balance laws in continuum physics is not always convex. Indeed, we have already encountered, in Chapter III, the cases of isentropic elastodynamics (Section 3.3.3) and electrodynamics (Section 3.3.8), in which invariance, dictated by physics, is incompatible with global convexity of the entropy. The objective in this, and the following sections is to identify special structure in such systems that may compensate for the failure of convexity in the entropy.

Recall that solutions of the system (3.3.19) with relevance to elastodynamics should also satisfy the equations (3.3.10). Notice that (3.3.10) is not independent of (3.3.19). Indeed, in a Cauchy problem, $(3.3.19)_1$ implies that when (3.3.10) is satisfied by the initial data, then it will hold for all $t > 0$.

The equations of electrodynamics exhibit similar behavior: in addition to the hyperbolic system (3.3.66), the magnetic induction and the electric displacement must also satisfy (3.3.67). However, in a Cauchy problem, by virtue of (3.3.66) and (3.3.68), (3.3.67) will hold automatically for all $t > 0$, so long as they are satisfied by the initial data.

One recognizes a similar structure in many other systems arising in continuum physics, and so an examination of its implications in a general framework is warranted:

5.4.1 Definition. The first order system

$$\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha U = 0 \tag{5.4.1}$$

of differential equations, with M_α $k \times n$ matrices, $\alpha = 1, \dots, m$, is called an *involution* of the system (5.1.1) of conservation laws if any (generally weak) solution of the Cauchy problem (5.1.1),(5.1.2) satisfies (5.4.1) identically, whenever the initial data do so.

Thus (3.3.10) is an involution of (3.3.19) and (3.3.67) is an involution of (3.3.66).

A sufficient condition for (5.4.1) to be an involution of (5.1.1) is

$$M_\alpha G_\beta(U) + M_\beta G_\alpha(U) = 0, \qquad \alpha, \beta = 1, \cdots, m, \tag{5.4.2}$$

for any $U \in \mathscr{O}$. We shall focus our investigation on this special case which covers, in particular, the prototypical examples (3.3.10) and (3.3.67). The aim is to demonstrate that, in the presence of involutions, one may establish existence and stability of classical solutions under the weaker hypothesis that the entropy is convex just in the direction of a certain cone in state space, which is constructed by the following procedure:

With any $\nu \in S^{m-1}$, we associate the $k \times n$ matrix

$$N(\nu) = \sum_{\alpha=1}^{m} \nu_\alpha M_\alpha \,. \tag{5.4.3}$$

5.4.2 Definition. The *involution cone* in $\mathbb{R}^n$ of the involution (5.4.1) is

$$\mathscr{C} = \bigcup_{\nu \in S^{m-1}} \ker N(\nu). \tag{5.4.4}$$

In light of the notation (4.1.2), (5.4.2) implies

$$N(\nu)\Lambda(\nu;U) = 0. \tag{5.4.5}$$

Thus involutions are related to stationary fronts and the rows of $N(\nu)$ are left eigenvectors of $\Lambda(\nu;U)$, with zero characteristic speed, $\lambda(\upsilon;U) = 0$. To simplify the presentation, we shall proceed under the assumption that the rows of $N(\nu)$ span the left eigenspace of $\Lambda(\nu,U)$ associated with the zero eigenvalue, i.e., for any $\nu \in S^{m-1}$, the rank of $N(\nu)$ equals the dimension of the kernel of $\Lambda(\nu;U)$. This condition is indeed satisfied in the prototypical systems (3.3.19), of isentropic elastodynamics, and (3.3.66), of electrodynamics, but it is not universally valid. For example, it fails in the system (3.3.4) of nonisentropic elastodynamics in which the zero characteristic speed has multiplicity seven whereas the rank of $N(\nu)$ induced by the involutions (3.3.10) is only six. Nevertheless, even (3.3.4) can be treated by the methods expounded below, albeit at the expense of complicating the formalism, which should be avoided here.

The implications of our assumptions become clear if one considers shock waves for systems (5.1.1) endowed with involutions (5.4.1). The Rankine-Hugoniot jump condition (3.1.3) reads

$$\sum_{\alpha=1}^{m} \nu_\alpha [G_\alpha(U_+) - G_\alpha(U_-)] = s[U_+ - U_-]. \tag{5.4.6}$$

The shock will also satisfy the involution (5.4.1) if

$$N(\nu)[U_+ - U_-] = 0. \tag{5.4.7}$$

Thus the amplitude $U_+ - U_-$ of any shock satisfying the involution lies on the involution cone $\mathscr{C}$.

Notice that (5.4.6) and (5.4.2) imply

$$sN(\nu)[U_+ - U_-] = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m} \nu_\alpha \nu_\beta M_\alpha [G_\beta(U_+) - G_\beta(U_-)] = 0. \tag{5.4.8}$$

Hence any shock with nonzero speed automatically satisfies the involution (5.4.7). By (5.4.6) and (4.1.2), shocks with speed $s = 0$ must satisfy $\bar{\Lambda}[U_+ - U_-] = 0$, where

$$\bar{\Lambda} = \int_0^1 \Lambda(\nu;(1-\tau)U_- + \tau U_+)d\tau. \tag{5.4.9}$$

In particular, any right eigenvector of $\Lambda(\nu;U)$ with zero characteristic speed can be the amplitude of a stationary shock, but none of these shocks is compatible with the involution (5.4.7), under the assumption that the rows of $N(\nu)$ span the left eigenspace associated with the eigenvalue $\lambda(\nu;U) = 0$. Thus the involution rules out all stationary shocks of (at least) small strength.

For the system (3.3.19) of isentropic elastodynamics, with involutions (3.3.10), the involution cone $\mathscr{C}$ consists of the states (F,v), with velocity $v \in \mathbb{R}^3$ and deformation gradient F of rank one, $F = uw^\top, u \in \mathbb{R}^3, w \in \mathbb{R}^3$. Notice that the entropy $\varepsilon(F) + \frac{1}{2}|v|^2$ may not be globally convex but it is convex at least in the direction of $\mathscr{C}$. This follows from the assumption that the internal energy is rank-one convex, (3.3.7), so that the system is hyperbolic. Here the project is to investigate whether, in the presence of involutions, mere convexity of the entropy in the direction of the involution cone would suffice for existence and stability of solutions to the Cauchy problem. This will be attained with the help of the following

5.4.3 Lemma. *Assume the system of conservation laws* (5.1.1) *is endowed with an involution* (5.4.1) *with involution cone $\mathscr{C}$. Suppose P is a symmetric $n \times n$ matrix-valued L^∞ function on $\mathbb{R}^m$ which is uniformly positive definite in the direction of $\mathscr{C}$, i.e.*,

$$X^\top P(x) X \geq \mu |X|^2, \qquad X \in \mathscr{C},\ x \in \mathbb{R}^m, \tag{5.4.10}$$

for some $\mu > 0$. Furthermore, there is a covering of $\mathbb{R}^m$ by the union of open sets $\Omega_0, \Omega_1, \cdots, \Omega_K$, such that, for $J = 0, 1, \cdots, K$,

$$|P(y) - P(x)| < \mu - 2\delta\ , \qquad x, y \in \Omega_J\ , \tag{5.4.11}$$

with $\delta > 0$. Then there is b, depending solely on the covering, such that

$$\int_{\mathbb{R}^m} V^\top(x)P(x)V(x)dx \geq \delta \|V\|^2_{L^2} - b\|V\|^2_{W^{-1,2}} \tag{5.4.12}$$

holds for any $V \in L^2(\mathbb{R}^m;\mathbb{R}^n)$ that satisfies the involution

$$\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha V = 0, \tag{5.4.13}$$

in the sense of distributions on $\mathbb{R}^m$.

Proof. Fix $\tilde{U} \in \mathscr{O}$ and consider the differential operator

(5.4.14) $$\mathscr{L} = \sum_{\beta=1}^{m} \mathrm{D}G_\beta(\tilde{U})\partial_\beta\,.$$

We construct $\Phi \in W^{1,2}(\mathbb{R}^m,\mathbb{R}^n)$ such that

(5.4.15) $$\mathscr{L}\Phi + \Phi = V,$$

(5.4.16) $$\sum_{\alpha=1}^{m} M_\alpha \partial_\alpha \Phi = 0.$$

We solve the above equations by using Fourier transform. Recalling (4.1.2) and (5.4.3), (5.4.15) and (5.4.16) imply

(5.4.17) $$\{i|\xi|\Lambda(|\xi|^{-1}\xi;\tilde{U}) + I\}\widehat{\Phi}(\xi) = \widehat{V}(\xi),$$

(5.4.18) $$N(|\xi|^{-1}\xi)\widehat{\Phi}(\xi) = 0.$$

The above linear system is solvable because, by virtue of (5.4.13),

(5.4.19) $$N(|\xi|^{-1}\xi)\widehat{V}(\xi) = 0.$$

Furthermore, we have

(5.4.20) $$|\widehat{\Phi}(\xi)|^2 \le a^2(1+|\xi|^2)^{-1}|\widehat{V}(\xi)|^2,$$

which implies

(5.4.21) $$\|\Phi\|_{L^2} \le a\|V\|_{W^{-1,2}}\,.$$

Next we fix a partition of unity $\phi_0,\phi_1,\cdots,\phi_K$ subordinate to the covering $\Omega_0,\Omega_1,\cdots,\Omega_K$, i.e., for $J=0,\cdots,K$, $\phi_J \in C^\infty(\mathbb{R}^m)$, $\operatorname{spt}\phi_J \subset \Omega_J$ and

(5.4.22) $$\sum_{J=0}^{K} \phi_J^2(x) = 1, \qquad x \in \mathbb{R}^m.$$

We also fix any $y_J \in \Omega_J,\quad J=0,\cdots,K,$ and write

(5.4.23) $$\int_{\mathbb{R}^m} V^\top(x)P(x)V(x)dx = \sum_{J=0}^{K}\int_{\mathbb{R}^m} \phi_J^2(x)V^\top(x)P(x)V(x)dx$$

$$= \sum_{J=0}^{K}\int_{\mathbb{R}^m} \phi_J^2(x)V^\top(x)P(y_J)V(x)dx + \sum_{J=1}^{K}\int_{\mathbb{R}^m} \phi_J^2(x)V^\top(x)[P(x)-P(y_J)]V(x)dx.$$

By virtue of (5.4.11),

$$\sum_{J=0}^{K}\int_{\mathbb{R}^m}\phi_J^2(x)V^\top(x)[P(x)-P(y_J)]V(x)dx \geq -(\mu-2\delta)\|V\|_{L^2}^2\,. \tag{5.4.24}$$

For each $J=0,\cdots,K$, we split $\phi_J V$ into

$$\phi_J V = S_J + T_J\,, \tag{5.4.25}$$

where

$$S_J = \mathscr{L}(\phi_J\Phi), \tag{5.4.26}$$

$$T_J = [\phi_J I - \sum_{\beta=1}^{m}\partial_\beta\phi_J \mathrm{D}G_\beta(\tilde{U})]\Phi. \tag{5.4.27}$$

Notice that

$$N(|\xi|^{-1}\xi)\widehat{S}_J(\xi) = i|\xi|N(|\xi|^{-1}\xi)\Lambda(|\xi|^{-1}\xi;\tilde{U})(\widehat{\phi_J\Phi}) = 0, \tag{5.4.28}$$

so that both the real and imaginary part of $\widehat{S}_J(\xi)$ are in $\mathscr{C}$, for any $\xi\in\mathbb{R}^m$ and for $J=0,\dots,K$. Thus, applying Parseval's relation and using (5.4.10) results in

$$\int_{\mathbb{R}^m} S_J^\top(x)P(y_J)S_J(x)dx = \int_{\mathbb{R}^m}\widehat{S}_J^*(\xi)P(y_J)\widehat{S}_J(\xi)d\xi \geq \mu\int_{\mathbb{R}^m}|S_J^*(x)|^2dx. \tag{5.4.29}$$

Moreover, from (5.4.27) and (5.4.22) we infer, for $J=1,\cdots,K$,

$$\int_{\mathbb{R}^m}|T_J(x)|^2dx \leq c\|V\|_{W^{-1,2}}^2\,. \tag{5.4.30}$$

We now return to (5.4.23). From (5.4.25), (5.4.29) and (5.4.30) it follows that

$$\begin{aligned}\int_{\mathbb{R}^m}&\phi_J^2(x)V^\top(x)P(y_J)V(x)dx\\ &\geq\left(1-\frac{\delta}{2\mu}\right)\int_{\mathbb{R}^m}S_J^\top(x)P(y_J)S_J(x)dx-\frac{2\mu}{\delta}\int_{\mathbb{R}^m}T_J^\top(x)P(y_J)T_J(x)dx\\ &\geq\left(\mu-\frac{\delta}{2}\right)\int_{\mathbb{R}^m}|S_J(x)|^2dx-c\|V\|_{W^{-1,2}}^2\,.\end{aligned} \tag{5.4.31}$$

Again by (5.4.25) and (5.4.30),

$$\int_{\mathbb{R}^m}|S_J(x)|^2dx \geq \left(1-\frac{\delta}{2\mu}\right)\int_{\mathbb{R}^m}\phi_J^2(x)|V(x)|^2dx - c\|V\|_{W^{-1,2}}^2. \tag{5.4.32}$$

Combining (5.4.23), (5.4.22), (5.4.31) and (5.4.32), we arrive at (5.4.12). The proof is complete.

5.4.4 Theorem. *Assume the system of conservation laws* (5.1.1) *is endowed with an involution* (5.4.1) *and is equipped with a* C^3 *entropy* η, *with* $\mathrm{D}^2\eta(U)$ *positive definite in the direction of the involution cone* $\mathscr{C}$. *Suppose the initial data* U_0 *take values in a compact subset of* $\mathscr{O}$, *satisfy the involution* (5.4.1) *and also* $U_0 - \bar{U} \in L^2(\mathbb{R}^m)$, *for some constant state* $\bar{U} \in \mathscr{O}$, *and* $\nabla U_0 \in H^\ell$, *for some* $\ell > m/2$. *Moreover, let* $G \in C^{\ell+2}$. *Then there exists* $T_\infty \leq \infty$ *and a unique classical solution* U *of* (5.1.1), (5.1.2) *on* $\mathbb{R}^m \times [0,T_\infty)$, *taking values in* $\mathscr{O}$, *such that*

$$\nabla U(\cdot,t) \in \bigcap_{k=0}^{\ell} W^{k,\infty}_{\mathrm{loc}}([0,T_\infty);H^{\ell-k}). \tag{5.4.33}$$

The interval $[0,T_\infty)$ *is maximal in that if* $T_\infty < \infty$ *then*

$$\limsup_{t\uparrow T_\infty} \|\nabla U(\cdot,t)\|_\ell = \infty \tag{5.4.34}$$

and/or the range of $U(\cdot,t)$ *escapes from every compact subset of* $\mathscr{O}$, *as* $t \uparrow T_\infty$.

Proof. We apply the method used for proving Theorem 5.1.1, by constructing the solution to (5.1.1), (5.1.2) as the $\varepsilon \downarrow 0$ limit of solutions to (5.1.8), (5.1.2).

Lemma 5.1.2, which establishes the existence of a solution U to (5.1.8), (5.1.2) on a time interval $[0,T_\varepsilon)$, carries over here, without any modification, as it does not rely on any special properties of the flux G. By virtue of (5.4.2), the function $Z = \sum M_\alpha \partial_\alpha U$ satisfies the parabolic system $\partial_t Z = \varepsilon \Delta Z$. Since $Z(\cdot,0) = 0$, Z must vanish identically on $\mathbb{R}^m \times [0,T_\varepsilon)$, and thus U will satisfy (5.4.1) on $\mathbb{R}^m \times [0,T_\varepsilon)$. Furthermore, the matrix $A(U(x,t))$, defined by (5.1.3), will be positive definite in the direction of the involution cone, i.e.,

$$X^\top A(U(x,t))X \geq \mu |X|^2, \qquad X \in \mathscr{C},\ x \in \mathbb{R}^m,\ t \in [0,T_\varepsilon), \tag{5.4.35}$$

for some $\mu > 0$. Using that $U_0(\cdot) - \bar{U} \in L^2(\mathbb{R}^m)$ and $\|\nabla U\|_\ell < \omega$, one easily infers from (5.1.8) that

$$|A(U(x,t)) - A(\bar{U})| < \frac{1}{4}\mu, \qquad x \in \Omega_0,\ t \in [0,T], \tag{5.4.36}$$

where Ω_0 is the complement of a closed ball $\overline{\mathscr{B}}_\rho$, of sufficiently large radius ρ, and $T \in (0,T_\varepsilon)$ is sufficiently small. Both ρ and T are selected independently of the parameter $\varepsilon > 0$. Next we notice that, since $|\nabla U| < c\omega$, one may cover $\overline{\mathscr{B}}_\rho$ by the union of balls $\Omega_1, \cdots, \Omega_K$, of sufficiently small radii so that, for $J = 1, \cdots, K$,

$$|A(U(y,t)) = A(U(x,t))| < \frac{1}{2}\mu\ , \quad x,y \in \Omega_J,\ t \in [0,T]. \tag{5.4.37}$$

After this preparation, we move on to Lemma 5.1.3 and retrace the steps in its proof, without any modification, up until (5.1.30) is derived. However, we no longer have (5.1.31), as $A(U)$ is not necessarily positive definite. In its place, we integrate

(5.1.30) over $\mathbb{R}^m$ and employ Lemma 5.4.3, together with the Cauchy inequality, to derive an estimate of the form

$$-\int_{\mathbb{R}^m} U_r^\top A(U)\Delta U_r dx \geq \frac{1}{4}\mu \int_{\mathbb{R}^m} |\nabla U_r|^2 dx - c\int_{\mathbb{R}^m} |\nabla U|^2 |U_r|^2 dx, \tag{5.4.38}$$

with a constant c that does not depend on $t \in [0,T]$ or on $\varepsilon > 0$. With the help of (5.4.38) and (5.1.32), we thus reestablish (5.1.33).

We must use Lemma 5.4.3 in order to estimate the left-hand side of (5.1.33). For $|r| = 2, \cdots, \ell+1$,

$$\int_{\mathbb{R}^m} U_r^\top(x,t) A(U(x,t)) U_r(x,t) dx \geq \frac{1}{4}\mu \int_{\mathbb{R}^m} |U_r(x,t)|^2 dx - c\|\nabla U(\cdot,t)\|_{r-2}^2 \tag{5.4.39}$$

while for $|r| = 1$

$$\int_{\mathbb{R}^m} U_r^\top(x,t) A(U(x,t)) U_r(x,t) dx \geq \frac{1}{4}\mu \int_{\mathbb{R}^m} |U_r(x,t)|^2 dx - c\int_{\mathbb{R}^m} |U(x,t) - \bar{U}|^2 dx. \tag{5.4.40}$$

The last term is estimated with the help of

$$\int_{\mathbb{R}^m} |U(x,t) - \bar{U}|^2 dx \leq 2\int_{\mathbb{R}^m} |U_0(x) - \bar{U}|^2 dx + ct\int_0^t \int_{\mathbb{R}^m} |\nabla U(x,t)|^2 dx d\tau, \tag{5.4.41}$$

which is easily derived from (5.1.8). Therefore, upon summing (5.1.33) over all multi-indices r of order $1 \leq |r| \leq \ell+1$ and using (5.1.34), (5.4.39), (5.4.40) and (5.4.41), one ends up with an estimate of the form

$$\|\nabla U(\cdot,t)\|_\ell^2 \leq c\|\nabla U_0(\cdot)\|_\ell^2 + c\|U_0(\cdot) - \bar{U}\|_{L^2} + \int_0^t [ct + g(\tau)]\|\nabla U(\cdot,\tau)\|_\ell^2 d\tau, \tag{5.4.42}$$

where g is bounded as in (5.1.36).

By retracing the remaining steps in the proof of Lemma 5.1.3, which require virtually no modification, we establish the existence of a classical solution U to (5.1.1), (5.1.2), on a time interval $[0,T]$, with $\nabla U(\cdot,t) \in L^\infty([0,T];H^\ell)$. In particular, $\nabla U(\cdot,T) \in H^\ell$. Furthermore, by virtue of (5.4.41), $U(\cdot,T) - \bar{U} \in L^2(\mathbb{R}^m)$. We may thus repeat the construction and extend U to a longer interval $[0,T']$. By continuing the process, we end up with a solution on a maximal interval $[0,T_\infty)$, and if $T_\infty < \infty$ then (5.4.34) holds and/or the range of $U(\cdot,t)$ escapes from every compact subset of $\mathscr{O}$, as $t \uparrow T_\infty$. From $\nabla U(\cdot,t) \in L^\infty_{\mathrm{loc}}([0,T_\infty);H^\ell)$ and (5.1.1) one shows inductively that $\partial_t^k \nabla U(\cdot,t) \in L^\infty_{\mathrm{loc}}([0,T_\infty);H^{\ell-k})$ thus establishing (5.4.33). The proof is complete.

The next proposition, akin to Theorem 5.3.1, establishes the uniqueness of classical solutions within the broader class of admissible weak solutions of sufficiently small local oscillation.

5.4.5 Theorem. *Assume the system of conservation laws* (5.1.1) *is endowed with an involution* (5.4.1), *and is equipped with an entropy-entropy flux pair* (η, Q), *where*

$\mathrm{D}^2\eta(U)$ is positive definite in the direction of the involution cone $\mathscr{C}$. Let $\bar{U}$ be a classical solution of (5.1.1), *on a time interval* $[0,T]$, *which takes values in a convex, compact subset $\mathscr{D}$ of $\mathscr{O}$ and has initial values $\bar{U}_0$ satisfying the involution. Suppose U is any weak solution of* (5.1.1) *which also takes values in $\mathscr{D}$, satisfies the entropy admissibility condition* (4.5.3), *and has the same initial values $\bar{U}_0$ as $\bar{U}$. Moreover, assume that $U(x,t)-\bar{U}(x,t)\to 0$, as $|x|\to\infty$, uniformly in $t\in[0,T]$, and*

$$\limsup_{x\to y\ t\to\tau} |U(x,t)-U(y,\tau)| < \kappa\,, \qquad y\in\mathbb{R}^m,\ \tau\in[0,T]. \tag{5.4.43}$$

Then, if κ is sufficiently small, U coincides with $\bar{U}$.

Proof. Retracing the steps in the proof of Theorem 5.3.1, we write (5.3.14) for $r=\infty$ and $U_0\equiv\bar{U}_0$:

$$\int_{\mathbb{R}^m} h(U(x,\sigma),\bar{U}(x,\sigma))dx \le -\int_0^\sigma\int_{\mathbb{R}^m}\sum_{\alpha=1}^m \partial_\alpha\bar{U}^\top Z_\alpha(U,\bar{U})dxd\tau, \tag{5.4.44}$$

for any point σ of L^∞ weak* continuity of $\eta(U(\cdot,\tau))$ in $(0,T)$. From (5.3.2),

$$h(U,\bar{U}) = (U-\bar{U})^\top\bar{A}(U,\bar{U})(U-\bar{U}), \tag{5.4.45}$$

where

$$\bar{A}(U,\bar{U}) = \int_0^1\int_0^\lambda A(\rho U+(1-\rho)\bar{U})d\rho d\lambda. \tag{5.4.46}$$

In particular,

$$X^\top\bar{A}(U,\bar{U})X \ge \mu|X|^2, \qquad X\in\mathscr{C}. \tag{5.4.47}$$

We fix r sufficiently large so that, setting $\Omega_0=\overline{B}_r^c$,

$$|U(x,t)-\bar{U}(x,t)| < \kappa, \quad x\in\Omega_0,\ t\in[0,T]. \tag{5.4.48}$$

Next we note that since $\bar{U}$ is continuous and U satisfies (5.4.43), there is $\delta>0$ such that, for any $t\in[0,T]$ and $x,y\in\overline{\mathscr{B}_r}$, with $|y-x|<\delta$,

$$|\bar{U}(x,t)-\bar{U}(y,t)| < \kappa, \qquad |U(x,t)-U(y,t)| < 2\kappa. \tag{5.4.49}$$

Hence, if we cover $\overline{\mathscr{B}_r}$ by the union of balls $\Omega_1,\cdots,\Omega_K$ of radius $\frac{1}{2}\delta$, and select κ sufficiently small, then, for $J=0,\cdots,K$,

$$|\bar{A}(U(y,t),\bar{U}(y,t))-\bar{A}(U(x,t),\bar{U}(x,t))| < \frac{1}{2}\mu, \tag{5.4.50}$$

for any $t\in[0,T]$ and any $x,y\in\Omega_J$.

We now return to (5.4.4) and apply Lemma 5.4.3 to get

$$\int_{\mathbb{R}^m} h(U(x,\sigma),\bar{U}(x,\sigma))dx \ge \frac{1}{4}\mu\|U(\cdot,\sigma)-\bar{U}(\cdot,\sigma)\|_{L^2}^2 - b\|U(\cdot,\sigma)-\bar{U}(\cdot,\sigma)\|_{W^{-1,2}}^2 \tag{5.4.51}$$

Upon using that both U and $\bar{U}$ satisfy (5.1.1),

$$
\begin{aligned}
(5.4.52)\qquad \|U(\cdot,\sigma)-\bar{U}(\cdot,\sigma)\|_{W^{-1,2}} &\le \int_0^\sigma \|\partial_t[U(\cdot,\tau)-\bar{U}(\cdot,\tau)]\|_{W^{-1,2}}d\tau \\
&\le \int_0^\sigma \sum_{\alpha=1}^m \|G_\alpha(U(\cdot,\tau))-G_\alpha(\bar{U}(\cdot,\tau))\|_{L^2}d\tau \\
&\le c\int_0^\sigma \|U(\cdot,\tau)-\bar{U}(\cdot,\tau)\|_{L^2}d\tau.
\end{aligned}
$$

Therefore, combining (5.4.44), (5.3.4), (5.4.51) and (5.4.52), we infer

$$
(5.4.53)\qquad \|U(\cdot,\sigma)-\bar{U}(\cdot,\sigma)\|_{L^2}^2 \le c(1+T)\int_0^\sigma \|U(\cdot,\tau)-\bar{U}(\cdot,\tau)\|_{L^2}^2 d\tau,
$$

for almost all $\sigma\in(0,T)$, whence $U\equiv\bar{U}$. The proof is complete.

In particular, Theorems 5.4.4 and 5.4.5 establish the existence and uniqueness of classical solutions to the Cauchy problem for the system (3.3.19) of isentropic elastodynamics, when the internal energy is not globally convex, but it merely satisfies the rank-one convexity condition (3.3.7). By contrast, the above theorems provide no relief for the system (3.3.66) of electrodynamics, with no convex electromagnetic field energy, because in that case the involution cone $\mathscr{C}$ coincides with the whole state space $\mathbb{R}^6$.

The reader must have noticed that Theorem 5.1.1 ascribes to solutions a higher level of regularity than does Theorem 5.4.4. The reason is that in the latter case one cannot depend on Lemma 5.1.5, which requires weak lower semicontinuity of the entropy, induced by the convexity. It should be noted, however, that convexity is merely a sufficient condition for weak lower semicontinuity. The following, more general, condition is necessary as well as sufficient.

5.4.6 Definition. An entropy η for the system of conservation laws (5.1.1), endowed with an involution (5.4.1), is called *quasiconvex* if for any $U\in L^\infty(\mathbb{R}^m;\mathscr{O})$, which is periodic in x_α, $\alpha=1,\dots,m$, with period one, satisfies (5.4.1) and has mean

$$
(5.4.54)\qquad \hat{U}=\int_{\mathscr{K}} U(y)dy
$$

over the standard hypercube $\mathscr{K}$ in $\mathbb{R}^m$ with edge length one, it is

$$
(5.4.55)\qquad \eta(\hat{U})\le\int_{\mathscr{K}}\eta(U(y))dy.
$$

Roughly, quasiconvexity stipulates that the uniform state minimizes the total entropy, among all states that are compatible with the involution and have the same "mass". This is in the spirit of the fundamental law of classical thermostatics, which affirms that the physical entropy is maximized at the equilibrium state.

The relevance of quasiconvexity is demonstrated by the following proposition, whose proof may be found in the references cited in Section 5.7:

5.4.7 Theorem. *Assume the system of conservation laws* (5.1.1) *is endowed with an entropy* η *and an involution* (5.4.1)*, such that the rank of* $N(\nu)$ *is constant, for any* $\nu \in S^{m-1}$*, and equal to the dimension of the kernel of* $\Lambda(\nu;U)$*. Then* $\int\limits_{|x|<r} \eta(U)dx$ *is weak* lower semicontinuous on the space of* L^∞ *vector fields* U *that satisfy* (5.4.1)*, if and only if* η *is quasiconvex. Furthermore, any quasiconvex* η *is necessarily convex in the direction of the involution cone* $\mathscr{C}$*.*

Because of the above proposition, the notion of quasiconvexity plays a fundamental role in the calculus of variations. Unfortunately, Definition 5.4.6 does not provide any clue as to how to test whether a given entropy is quasiconvex. The conjecture that convexity in the direction of the involution cone is also sufficient for quasiconvexity is valid when the entropy is quadratic: $\eta = U^\top AU$. In general, however, quasiconvexity is a more stringent condition than mere convexity in the direction of the involution cone.

The above may be illustrated in the context of our prototypical example, namely the system (3.3.19) of isentropic elastodynamics, with involution (3.3.10) and entropy $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$. In that case, η is quasiconvex when $\varepsilon(F)$ is quasiconvex in the sense of Morrey: For any constant deformation gradient $\hat{F}$ and any Lipschitz function χ from $\mathscr{K}$ to $\mathbb{R}^3$, with compact support in $\mathscr{K}$,

$$\varepsilon(\hat{F}) \leq \int\limits_{\mathscr{K}} \varepsilon(\hat{F} + \nabla\chi)dy. \tag{5.4.56}$$

In other words, a homogeneous deformation of $\mathscr{K}$ minimizes the total internal energy among all placements of $\mathscr{K}$ with the same boundary values. Any quasiconvex internal energy is rank-one convex (3.3.7). On the other hand, there exist rank-one convex functions $\varepsilon(F)$ that fail to be quasiconvex.

A placement of an elastic body is in (isentropic) equilibrium when its total internal energy $\int \varepsilon(F)dx$ is minimum over all placements with the same boundary conditions. Thus, quasiconvexity is necessary and sufficient for attaining equilibria by minimizing sequences of placements that are merely bounded in $W^{1,\infty}$.

As null Lagrangians, $\pm \det F$ and the entries of $\pm F^*$ are all quasiconvex. Therefore, $\det F$ and F^* are L^∞ weak* continuous functions on the space of F that satisfy the involution (3.3.10).

To conclude this section, it should be noted that there are systems of conservation laws equipped with nonlinear involutions, that is nonlinear functions of the state vector and its spatial derivatives that are preserved by classical or weak solutions to the Cauchy problem, once introduced by the initial data. For example, (2.2.12) and (2.2.13) may be regarded as nonlinear involutions of the system (3.3.19) of isentropic elastodynamics. As another example, notice that if one writes the system (3.3.19) in Eulerian coordinates then the linear involutions (3.3.10) become nonlinear: $F_{j\beta}\partial_j F_{i\alpha} - F_{j\alpha}\partial_j F_{i\beta} = 0$ (with summation convention). At the same time, the reader should be warned that there may be linear combinations of derivatives that are conserved by smooth solutions, without being involutions. A case in point is the irrotationality condition $\omega = \operatorname{curl} v = 0$ which is preserved by smooth solutions of the Euler equation (3.3.36), but it breaks down after discontinuities develop.

5.5 Contingent Entropies and Polyconvexity

The attempt in the previous section to compensate for the breakdown of convexity in the entropy by employing involutions was met with partial success in the case of elastodynamics and with failure in the case of electrodynamics. Here, we shall address these difficulties with the help of the special structure manifested in the presence of equations like (3.3.11), (3.3.12), for elastodynamics, and (3.3.74), for electrodynamics. At first glance, these equations seem to identify additional entropy-entropy flux pairs for the systems (3.3.19) and (3.3.66). However, this is not accurate, because, as it was already pointed out in Section 3.3, (3.3.11) and (3.3.12) hold only for solutions of (3.3.19) that satisfy the involutions (3.3.10); and similarly (3.3.74) holds only for solutions of (3.3.66) that satisfy the involutions (3.3.67). It is thus expedient to introduce extended entropy balance laws that hold only contingent on the involutions.

5.5.1 Definition. In a system of conservation laws (5.1.1), endowed with the involution (5.4.1), a smooth, scalar-valued function $\eta(U)$ on $\mathscr{O}$ is a *contingent entropy*, associated with the $1 \times k$ matrix-valued *contingent entropy flux* $Q(U)$, if there is a k-vector-valued function $\Xi(U)$ on $\mathscr{O}$ such that

$$\mathrm{D}Q_\alpha(U) = \mathrm{D}\eta(U)\mathrm{D}G_\alpha(U) + \Xi(U)^\top M_\alpha, \qquad \alpha = 1, \cdots, m. \tag{5.5.1}$$

In particular, any entropy is a contingent entropy, with $\Xi(U)$ zero. Clearly, (5.5.1) implies that

$$\partial_t \eta(U(x,t)) + \sum_{\alpha=1}^{m} \partial_\alpha Q_\alpha(U(x,t)) = 0 \tag{5.5.2}$$

holds for any classical solution U of (5.1.1) that satisfies the involution (5.4.1). Thus, $\Xi(U)$ plays the role of a Lagrange multiplier.

In our prototypical examples, $\det F$ and the nine entries of F^* are contingent entropies for the system (3.3.19) of elastodynamics, and the three components of $B \wedge D$ are contingent entropies for the system (3.3.66) of electrodynamics.

The running assumption throughout this section will be that (5.1.1) is a system of n conservation laws that is endowed with k involutions (5.4.1) and is equipped with a principal contingent entropy-entropy flux pair (η, Q), as well as ℓ supplemental linearly independent contingent entropy-entropy flux pairs. Admissibility of weak solutions will be dictated by the inequality (4.5.3) for the principal contingent entropy-entropy flux pair.

We set $N = n + \ell$ and compose the N-vector-valued function $W(U)$ whose first n components are the components $U^1, \cdots, U^n$ of the state vector U while the remaining ℓ components are the supplemental contingent entropies. Thus

$$\partial_t W(U(x,t)) + \sum_{\alpha=1}^{m} \partial_\alpha X_\alpha(U(x,t)) = 0, \tag{5.5.3}$$

holds for any classical solution U of (5.1.1) that satisfies the involution (5.4.1), where X_α is the α-th column vector of the $N \times m$ matrix-valued function X whose I-th row X^I is the contingent entropy flux associated with the contingent entropy W^I. In particular, for $I = 1, \cdots, n$, $X_\alpha^I = G_\alpha^I$.

The principal contingent entropy-entropy flux pair (η, Q) satisfies (5.5.1) for some Lagrange multiplier Ξ. Similarly,

$$\mathrm{D}X_\alpha(U) = \mathrm{D}W(U)\mathrm{D}G_\alpha(U) + \Omega(U)^\top M_\alpha\,, \qquad \alpha = 1, \cdots, m, \tag{5.5.4}$$

where Ω is the $k \times N$-matrix-valued function whose I-th column vector Ω^I is the Lagrange multiplier associated with the contingent entropy-entropy flux pair (W^I, X^I). In particular, for $I = 1, \cdots, n$, $\Omega^I = 0$.

The objective of this section is to demonstrate that in the above setting the requirement of convexity on the principal entropy may be relaxed into the following weaker condition:

5.5.2 Definition. The principal contingent entropy η is called *polyconvex*, relative to the contingent entropies W, if it admits a representation

$$\eta(U) = \theta(W(U)), \qquad U \in \mathscr{O}, \tag{5.5.5}$$

where θ is a smooth function defined on an open neighborhood $\mathscr{F}$ of $W(\mathscr{O})$ in $\mathbb{R}^N$, whose Hessian matrix is positive definite at every $W \in \mathscr{F}$.

In the example of elastodynamics, $W = (F, v, F^*, \det F)$, arranged as a 22-vector. The principal entropy $\eta = \varepsilon(F) + \frac{1}{2}|v|^2$ will be polyconvex when the internal energy function $\varepsilon(F)$ admits a representation

$$\varepsilon(F) = \phi(F, F^*, \det F), \tag{5.5.6}$$

where $\phi(F, H, \delta)$ is a smooth function with positive definite Hessian on an open neighborhood of the manifold $\{(F, H, \delta) : \det F > 0,\, H = F^*,\, \delta = \det F\}$, embedded in $\mathbb{R}^{19}$. This is a physically reasonable assumption which has been discussed thoroughly in the literature, especially in the context of elastostatics. In particular, the (isentropic) internal energy for elastic fluids is polyconvex, as it is of the form $\varepsilon(F) = \phi(\det F)$, with ϕ convex on $(0, \infty)$. Similarly, in electrodynamics, where

$W = (B, D, B \wedge D)$, arranged as a 9-vector, polyconvexity is a natural assumption for the electromagnetic field energy η, which serves as the principal entropy. Indeed, in the Born-Infeld case (3.3.73), η is polyconvex.

We proceed to derive certain implications of polyconvexity. Notice first that (5.5.1) and (5.5.4) yield the symmetry relations

$$\mathrm{D}^2\eta(U)\mathrm{D}G_\alpha(U) + \mathrm{D}\Xi(U)^\top M_\alpha = \mathrm{D}G_\alpha(U)^\top \mathrm{D}^2\eta(U) + M_\alpha^\top \mathrm{D}\Xi(U), \tag{5.5.7}$$

$$\mathrm{D}^2W^I(U)\mathrm{D}G_\alpha(U) + \mathrm{D}\Omega^I(U)^\top M_\alpha = \mathrm{D}G_\alpha(U)^\top \mathrm{D}^2W^I(U) + M_\alpha^\top \mathrm{D}\Omega^I(U), \tag{5.5.8}$$

for $I = 1, \cdots, N$.

In the sequel, $\theta_W(W)$ will denote the differential of the function $\theta(W)$, regarded as a $1\times N$ matrix with entries $\theta_{W^I} = \partial\theta/\partial W^I$; and $\theta_{WW}(W)$ will stand for the $N\times N$ Hessian matrix of $\theta(W)$.

For $U \in \mathscr{O}$, we define the symmetric $n\times n$ matrix

$$A(U) = \mathrm{D}^2\eta(U) - \sum_{I=1}^{N} \theta_{W^I}(W(U))\mathrm{D}^2W^I(U). \tag{5.5.9}$$

Using (5.5.5),

$$A(U) = \mathrm{D}W(U)^\top \theta_{WW}(W(U))\mathrm{D}W(U), \tag{5.5.10}$$

so that $A(U)$ is positive definite when η is polyconvex. Furthermore, by virtue of (5.5.7) and (5.5.8), the $n\times n$ matrix-valued functions

$$J_\alpha(U) = A(U)\mathrm{D}G_\alpha(U) + \Gamma(U)^\top M_\alpha, \qquad \alpha = 1, \cdots, m, \tag{5.5.11}$$

where

$$\Gamma(U) = \mathrm{D}\Xi(U) - \sum_{I=1}^{N} \theta_{W^I}(W(U))\mathrm{D}\Omega^I(U), \tag{5.5.12}$$

are symmetric.

The following proposition establishes the existence of classical solutions to the Cauchy problem.

5.5.3 Theorem. *Assume the system of conservation laws* (5.1.1) *is endowed with the involution* (5.4.1) *and is equipped with a C^3 principal contingent entropy-entropy flux pair (η, Q), and with supplemental contingent entropy-entropy flux pairs that render $\eta(U)$ polyconvex, in the sense of Definition* 5.5.2. *Suppose the initial data U_0 are continuously differentiable on $\mathbb{R}^m$, take values in some compact subset of $\mathscr{O}$, satisfy the involution, and $\nabla U_0 \in H^\ell$, for some $\ell > m/2$. Moreover, let $G \in C^{\ell+2}$. Then there exists $T_\infty \le \infty$ and a unique continuously differentiable function U on $\mathbb{R}^m \times [0, T_\infty)$, taking values in $\mathscr{O}$, which is a classical solution of the Cauchy problem* (5.1.1), (5.1.2) *on $[0, T_\infty)$. Furthermore,*

$$\nabla U(\cdot,t) \in \bigcap_{k=0}^{\ell} C^k([0,T_\infty);H^{\ell-k}). \tag{5.5.13}$$

The interval $[0,T_\infty)$ is maximal in that if $T_\infty < \infty$ then

$$\int_0^{T_\infty} \|\nabla U(\cdot,t)\|_{L^\infty(\mathbb{R}^m)} dt = \infty \tag{5.5.14}$$

and/or the range of $U(\cdot,t)$ escapes from every compact subset of $\mathscr{O}$, as $t \uparrow T_\infty$.

Proof. We apply the method used for proving Theorem 5.1.1, by constructing the solution to (5.1.1), (5.1.2) as the $\varepsilon \downarrow 0$ limit of solutions to (5.1.8), (5.1.2).

We begin with Lemma 5.1.2, which establishes the existence of a solution U to (5.1.8), (5.1.2) on a time interval $[0,T_\varepsilon)$. This carries over here, without any change, as it does not rely on any particular properties of G. As in the proof of Theorem 5.4.4, it is important to remember that U satisfies the involution (5.4.1).

The next step is to construct a solution to (5.1.1), (5.1.2) on a time interval $[0,T]$ by retracing the steps in the proof of Lemma 5.1.3. However, in the place of A and J_α given by (5.1.3) and (5.1.4), we are here using A and J_α defined by (5.5.9) and (5.5.11). Since $A(U)$ is symmetric and positive definite, Equations (5.1.24)-(5.1.31) are still valid here. However, in the place of (5.1.32) we now have

(5.5.15)

$$\sum_{\alpha=1}^{m} 2U_r^\top A(U)\mathrm{D}G_\alpha(U)\partial_\alpha U_r = \sum_{\alpha=1}^{m} 2U_r^\top [A(U)\mathrm{D}G_\alpha(U) + \Gamma(U)^\top M_\alpha]\partial_\alpha U_r$$

$$= \sum_{\alpha=1}^{m} 2U_r^\top J_\alpha(U)\partial_\alpha U_r = \sum_{\alpha=1}^{m} \partial_\alpha[U_r^\top J_\alpha(U)U_r] - \sum_{\alpha=1}^{m} U_r^\top \partial_\alpha J_\alpha(U)U_r.$$

which follows from (5.5.11), (5.4.1) and the symmetry of $J_\alpha(U)$. Thus, the remaining estimates (5.1.33)-(5.1.39), and thereby the assertion of Lemma 5.1.3, apply to the present case as well.

The situation is similar with Lemmas 5.1.4 and 5.1.5. The estimates (5.1.40)-(5.1.53) carry over to the present setting, with A and J_α defined through (5.5.9) and (5.5.11). In particular, (5.1.43) and (5.1.44) now follow from (5.1.29) and (5.5.15).

Armed with Lemmas 5.1.2-5.1.5, one easily completes the proof of Theorem 5.5.3 by retracing the steps in the proof of Theorem 5.1.1.

We now turn to the question of uniqueness and stability of classical solutions within a class of weak solutions that will be dubbed mild.

5.5.4 Definition. A locally bounded measurable function U, defined on $\mathbb{R}^m \times [0,T)$ and taking values in $\mathscr{O}$, is a *mild solution* to (5.1.1), (5.1.2) if

(5.5.16)
$$\int_0^T \int_{\mathbb{R}^m} [\partial_t V^\top W(U) + \sum_{\alpha=1}^m \partial_\alpha V^\top X_\alpha(U)] dxdt + \int_{\mathbb{R}^m} V^\top(x,0) W(U_0(x)) dx = 0$$

holds for all Lipschitz N-vector-valued test functions V, with compact support in $\mathbb{R}^m \times [0,T)$.

Notice that (5.5.16) holds when U satisfies (5.5.3), in the sense of distributions, together with the initial condition $W(U(\cdot,t)) \to W(U_0(\cdot))$ in L^∞ weak*, as $t \to 0$. In particular, any mild solution of (5.1.1), (5.1.2) is a weak solution, as (5.1.1) is embedded in (5.5.3). Clearly, any classical solution of (5.1.1), (5.1.2) is a mild solution, because (5.5.3) and the initial conditions are automatically satisfied in that case. However, it comes as a surprise that in the applications one often encounters even discontinuous mild solutions. For example, any weak solution (F,v) of the system (3.3.19) of isentropic elastodynamics is mild. Indeed, as we saw in Section 2.3, (3.3.11) and (3.3.12) hold for any L^∞ fields that satisfy $(3.3.19)_1$ and the involution (3.3.10). Moreover, as stated in the previous section, F^* and $\det F$ are continuous functions in L^∞ weak*, and hence $F(\cdot,t) \to F_0(\cdot)$, as $t \to 0$, in L^∞ weak*, implies $F^*(\cdot,t) \to F_0^*(\cdot)$ and $\det F(\cdot,t) \to \det F_0(\cdot)$, as $t \to 0$, in L^∞ weak*. Similarly, *BV* weak solutions (B,D) of the system (3.3.66) of electrodynamics, with Born-Infeld constitutive relations (3.3.73) and involutions (3.3.67), are necessarily mild solutions, because all shocks satisfy (3.3.80). Thus (3.3.74) will hold for such solutions. Moreover, in the *BV* setting there is enough regularity so that $B(\cdot,t) \to B_0(\cdot)$ and $D(\cdot,t) \to D_0(\cdot)$, as $t \to 0$, implies $B(\cdot,t) \wedge D(\cdot,t) \to B_0(\cdot) \wedge D_0(\cdot)$, as $t \to 0$.

A mild solution U will be *admissible* if it is admissible as a weak solution, i.e., if (4.5.3) is satisfied for the principal contingent entropy-entropy flux pair. In particular, any *BV* solution of (3.3.66), under the Born-Infeld constitutive relation, is admissible, as shocks do not incur energy production. Of course, this is not the case with the system (3.3.19) of elastodynamics.

The following proposition should be compared with Theorem 5.3.1.

5.5.5 Theorem. *Assume the system of conservation laws* (5.1.1) *is endowed with the involution* (5.4.1) *and is equipped with a principal contingent entropy-entropy flux pair* (η, Q), *and also with supplemental contingent entropy-entropy flux pairs that render* $\eta(U)$ *polyconvex, in the sense of Definition* 5.5.2. *Let* $\mathscr{D}$ *be a compact subset of* $\mathscr{O}$ *such that* $W(\mathscr{D})$ *is contained in a convex subset of* $\mathscr{F}$. *Suppose* $\bar{U}$ *is a classical solution of* (5.1.1) *on* $[0,T)$, *taking values in* $\mathscr{D}$, *with initial data* $\bar{U}_0$ *satisfying the involution* (5.4.1). *Let* U *be any admissible mild solution of* (5.1.1) *on* $[0,T)$, *which also takes values in* $\mathscr{D}$ *and has initial data* U_0 *satisfying the involution* (5.4.1). *Then*

$$\int_{|x|<r} |U(x,t) - \bar{U}(x,t)|^2 dx \le a e^{bt} \int_{|x|<r+st} |U_0(x) - \bar{U}_0(x)|^2 dx \tag{5.5.17}$$

holds for any $r > 0$ *and* $t \in [0,T)$, *with positive constants* s, a, *depending solely on* $\mathscr{D}$, *and* b *that also depends on the Lipschitz constant of* $\bar{U}$. *In particular,* $\bar{U}$ *is the* *unique admissible mild solution of* (5.1.1) *with initial data* $\bar{U}_0$.

Proof. We retrace the steps in the proof of Theorem 5.3.1, with the needed modifications. On $\mathscr{D} \times \mathscr{D}$ we define

$$h(U,\bar{U}) = \eta(U) - \eta(\bar{U}) - \theta_W(W(\bar{U}))[W(U) - W(\bar{U})], \tag{5.5.18}$$

$$\begin{aligned} Y_\alpha(U,\bar{U}) = Q_\alpha(U) - Q_\alpha(\bar{U}) - \theta_W(W(\bar{U}))[X_\alpha(U) - X_\alpha(\bar{U})] \\ +[\theta_W(W(\bar{U}))\Omega(\bar{U})^\top - \Xi(\bar{U})^\top]M_\alpha[U - \bar{U}], \end{aligned} \tag{5.5.19}$$

$$\begin{aligned} Z_\alpha(U,\bar{U}) = -\mathrm{D}G_\alpha(\bar{U})^\top \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))[W(U) - W(\bar{U})] \\ +\mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))[X_\alpha(U) - X_\alpha(\bar{U})] \\ -\mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))\Omega(\bar{U})^\top M_\alpha[U - \bar{U}] \\ +\Gamma(\bar{U})^\top M_\alpha[U - \bar{U}], \end{aligned} \tag{5.5.20}$$

where Γ is given by (5.5.12).

Recalling Definition 5.5.2, we see that $h(U,\bar{U})$ is of quadratic order in $U - \bar{U}$ and positive definite. Upon using (5.5.1), (5.5.4) and (5.5.5), we deduce

$$\begin{aligned} \mathrm{D}Y_\alpha(U,\bar{U}) = [\theta_W(W(U)) - \theta_W(W(\bar{U}))]\mathrm{D}W(U)\mathrm{D}G_\alpha(U) \\ +[\Xi(U) - \Xi(\bar{U})]^\top M_\alpha - \theta_W(W(\bar{U}))[\Omega(U) - \Omega(\bar{U})]^\top M_\alpha\,, \end{aligned} \tag{5.5.21}$$

which vanishes at $U = \bar{U}$, so that $Y(U,\bar{U})$ is also of quadratic order in $U - \bar{U}$. In particular, for s large, (5.3.5) holds.

Turning to $Z(U,\bar{U})$, and by virtue of (5.5.4),

$$\begin{aligned} \mathrm{D}Z_\alpha(U,\bar{U}) = -\mathrm{D}G_\alpha(\bar{U})^\top \mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))\mathrm{D}W(U) \\ +\mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))\mathrm{D}W(U)\mathrm{D}G_\alpha(U) \\ +\mathrm{D}W(\bar{U})^\top \theta_{WW}(W(\bar{U}))[\Omega(U) - \Omega(\bar{U})]^\top M_\alpha \\ +\Gamma(\bar{U})^\top M_\alpha\,. \end{aligned} \tag{5.5.22}$$

Recalling (5.5.10), (5.5.11) and since J_α is symmetric, we conclude that

(5.5.23)
$$\mathrm{D}Z_\alpha(\bar{U},\bar{U}) = -\mathrm{D}G_\alpha(\bar{U})^\top A(\bar{U}) + A(\bar{U})\mathrm{D}G_\alpha(\bar{U}) + \Gamma(\bar{U})^\top M_\alpha = M_\alpha^\top \Gamma(\bar{U}).$$

As in the proof of Theorem 5.3.1, we fix a nonnegative, Lipschitz continuous test function ψ with compact support in $\mathbb{R}^m \times [0,T)$, and evaluate h, Y and Z along the two solutions $U(x,t)$ and $\bar{U}(x,t)$. As an admissible weak solution, U must satisfy the inequality (4.5.3), while $\bar{U}$ being a classical solution, will satisfy (4.5.3) as an equality. We thus deduce

(5.5.24)
$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^m} [\partial_t \psi\, h(U,\bar{U}) + \sum_{\alpha=1}^m \partial_\alpha \psi Y_\alpha(U,\bar{U})]dxdt + \int_{\mathbb{R}^m} \psi(x,0)\, h(U_0(x),\bar{U}_0(x))\, dx \\
&\qquad \geq -\int_0^T \int_{\mathbb{R}^m} \Big\{ \partial_t \psi\, \theta_W(W(\bar{U}))[W(U) - W(\bar{U})] \\
&\qquad\quad + \sum_{\alpha=1}^m \partial_\alpha \psi \{\theta_W(W(\bar{U}))[X_\alpha(U) - X_\alpha(\bar{U})] \\
&\qquad\quad - [\theta_W(W(\bar{U}))\Omega(\bar{U})^\top - \Xi(\bar{U})^\top] \times M_\alpha[U-\bar{U}]\} \Big\} dxdt \\
&\qquad\quad - \int_{\mathbb{R}^m} \psi(x,0)\theta_W(W(\bar{U}_0(x)))[W(U_0(x)) - W(\bar{U}_0(x))]dx.
\end{aligned}$$

Next we write (5.5.16) for both U and $\bar{U}$, with test function $V^T = \psi\theta_W(W(\bar{U}))$. This yields

$$\begin{aligned}
(5.5.25)\qquad &\int_0^T \int_{\mathbb{R}^m} \Big\{ \partial_t[\psi\theta_W(W(\bar{U}))][W(U) - W(\bar{U})] \\
&\quad + \sum_{\alpha=1}^m \partial_\alpha[\psi\theta_W(W(\bar{U}))][X_\alpha(U) - X_\alpha(\bar{U})] \Big\} dxdt \\
&\quad + \int_{\mathbb{R}^m} \psi(x,0)\theta_W(W(\bar{U}_0(x)))[W(U_0(x)) - W(\bar{U}_0(x))]dx = 0.
\end{aligned}$$

Furthermore, since both U and $\bar{U}$ satisfy the involution (5.4.1),

$$(5.5.26)\qquad \int_0^T \int_{\mathbb{R}^m} \sum_{\alpha=1}^m \partial_\alpha \Big\{ \psi[\theta_W(W(\bar{U}))\Omega(\bar{U})^\top - \Xi(\bar{U})^\top] \Big\} M_\alpha[U-\bar{U}]dxdt = 0.$$

By virtue of (5.5.4) and $\sum M_\alpha \partial_\alpha \bar{U} = 0$,

$$
\begin{aligned}
(5.5.27)\ \partial_t \theta_W(W(\bar U)) &= \partial_t W(\bar U)^\top \theta_{WW}(W(\bar U)) \\
&= \sum_{\alpha=1}^m \partial_\alpha X_\alpha(\bar U)^\top \theta_{WW}(W(\bar U)) \\
&= -\sum_{\alpha=1}^m \partial_\alpha \bar U^\top \mathrm{D}X_\alpha(\bar U)^\top \theta_{WW}(W(\bar U)) \\
&= -\sum_{\alpha=1}^m \partial_\alpha \bar U^\top [\mathrm{D}W(\bar U)\mathrm{D}G_\alpha(\bar U) + \Omega(\bar U)^\top M_\alpha]^\top \theta_{WW}(W(\bar U)) \\
&= -\sum_{\alpha=1}^m \partial_\alpha \bar U^\top \mathrm{D}G_\alpha(\bar U)^\top \mathrm{D}W(\bar U)^\top \theta_{WW}(W(\bar U)).
\end{aligned}
$$

Similarly,

$$
\partial_\alpha \theta_W(W(\bar U)) = \partial_\alpha \bar U^\top \mathrm{D}W(\bar U)^\top \theta_{WW}(W(\bar U)), \tag{5.5.28}
$$

(5.5.29)

$$
\begin{aligned}
\partial_\alpha[\theta_W(W(\bar U))\Omega(\bar U)^\top - \Xi(\bar U)^\top] = \partial_\alpha \bar U^\top \mathrm{D}W(\bar U)^\top \theta_{WW}(W(\bar U))\Omega(\bar U)^\top \\
-\partial_\alpha \bar U^\top \Gamma(\bar U)^\top.
\end{aligned}
$$

Therefore, recalling (5.5.20),

$$
\begin{aligned}
(5.5.30)\quad \partial_t \theta_W(W(\bar U))[W(U)-W(\bar U)] + \sum_{\alpha=1}^m \partial_\alpha \theta_W(W(\bar U))[X_\alpha(U) - X_\alpha(\bar U)] \\
-\sum_{\alpha=1}^m \partial_\alpha[\theta_W(W(\bar U))\Omega(\bar U)^\top - \Xi(\bar U)^\top] M_\alpha [U-\bar U] = \sum_{\alpha=1}^m \partial_\alpha \bar U^\top Z_\alpha(U,\bar U).
\end{aligned}
$$

On account of (5.5.23),

$$
\sum_{\alpha=1}^m \partial_\alpha \bar U^\top \mathrm{D}Z_\alpha(\bar U,\bar U) = \left[\sum_{\alpha=1}^m M_\alpha \partial_\alpha \bar U\right]^\top \Gamma(\bar U) = 0. \tag{5.5.31}
$$

consequently, the right-hand side of (5.5.30) is of quadratic order in $U-\bar U$.

By combining (5.5.24), (5.5.25), (5.5.26) and (5.5.30), we recover (5.3.10). The remainder of the proof follows along the lines of the proof of Theorem 5.3.1: departing from (5.3.10) and fixing any $t \in (0,T)$, we derive (5.3.13), for $\sigma \in (0,t)$, and then (5.3.14), for any σ of L^∞ weak* continuity of $\eta(U(\cdot,\tau))$. This in turn yields (5.3.16), for u defined by (5.3.15), and thereby (5.5.17). The proof is complete.

In particular, Theorems 5.5.3 and 5.5.5 apply to the class of systems of conservation laws that are endowed with an involution and are equipped with a convex contingent entropy $\eta(U)$ (just take $W(U) \equiv U$). One may attempt to reduce the more general class of systems endowed with an involution and equipped with a polyconvex contingent entropy to the above special class by means of the following procedure. Assume that the system (5.1.1) is endowed with the involution (5.4.1) and is equipped with a principal contingent entropy-entropy flux pair $(\eta(U), Q(U))$ which is polyconvex (5.5.5), relative to the contingent entropies W. We seek functions $S(\Psi)$ and $\Pi(\Psi)$, defined on $\mathbb{R}^N$ and taking values in $\mathbb{M}^{N\times m}$ and $\mathbb{M}^{1\times m}$, respectively, such that

$$S(W(U)) = W(U), \qquad \Pi(W(U)) = Q(U) \tag{5.5.32}$$

and, in addition, $(\theta(\Psi), \Pi(\Psi))$ is a (generally contingent) entropy-entropy flux pair for the *extended system*

$$\partial_t \Psi(x,t) + \operatorname{div} S(\Psi(x,t)) = 0. \tag{5.5.33}$$

When functions satisfying the above specifications can be found, one may construct solutions to the Cauchy problem (5.1.1), (5.1.2) by first solving (5.5.33) with initial conditions

$$\Psi(x,0) = W(U_0(x)), \tag{5.5.34}$$

and then getting U from the equation $W(U) = \Psi$. The merit of this approach lies in that (5.5.33) is now equipped with a convex (possibly contingent) entropy θ.

The above program has been implemented successfully for the systems of elastodynamics and electrodynamics.

In elastodynamics, $U = (F, v)^\top, \Psi = (F, v, \Theta, \omega)^\top$, $\sigma = \sigma(F, \Theta, \omega)$, the extended system reads

$$\begin{cases} \partial_t F_{i\alpha} - \partial_\alpha v_i = 0, & \alpha = 1,2,3; \quad i = 1,2,3 \\ \partial_t v_i - \partial_\alpha \left(\dfrac{\partial \sigma}{\partial F_{i\alpha}} + \dfrac{\partial \sigma}{\partial \Theta_{\beta j}} \dfrac{\partial F^*_{\beta j}}{\partial F_{i\alpha}} + \dfrac{\partial \sigma}{\partial \omega} \dfrac{\partial \det F}{\partial F_{i\alpha}} \right) = 0, & i = 1,2,3 \\ \partial_t \Theta_{\beta i} - \partial_\alpha \left(\dfrac{\partial F^*_{\beta i}}{\partial F_{j\alpha}} v_j \right) = 0, & \beta = 1,2,3; \quad i = 1,2,3 \\ \partial_t \omega - \partial_\alpha \left(\dfrac{\partial \det F}{\partial F_{j\alpha}} v_j \right) = 0, & \end{cases} \tag{5.5.35}$$

and the entropy-entropy flux pair is

$$\theta = \tfrac{1}{2}|v|^2 + \sigma(F, \Theta, \omega), \tag{5.5.36}$$

$$\Pi_\alpha = -\left(\frac{\partial \sigma}{\partial F_{i\alpha}} + \frac{\partial \sigma}{\partial \Theta_{\beta j}} \frac{\partial F^*_{\beta j}}{\partial F_{i\alpha}} + \frac{\partial \sigma}{\partial \omega} \frac{\partial \det F}{\partial F_{i\alpha}} \right) v_i\,. \tag{5.5.37}$$

On the "manifold" $\Psi = W(U) = (F, v, F^*, \det F)^\top$, (5.5.35) reduces to the system (3.3.19) (with $b = 0$) together with the kinematic conservation laws (3.3.11), (3.3.12), while (θ, Π) reduces to the classical entropy-entropy flux pair recorded in Section 3.3.3.

In electrodynamics, for the Born-Infeld constitutive relations, where $U = (B, D)^\top$, $\Psi = (B, D, P)^\top$, the extended system reads

$$\begin{cases} \partial_t B + \operatorname{curl}\big[\theta^{-1}(D + B \wedge P)\big] = 0 \\ \partial_t D - \operatorname{curl}\big[\theta^{-1}(B - D \wedge P)\big] = 0 \\ \partial_t P - \operatorname{div}\big[\theta^{-1}(I + BB^\top + DD^\top - PP^\top)\big] = 0, \end{cases} \tag{5.5.38}$$

and the entropy-entropy flux pair is

$$\theta = (1 + |B|^2 + |D|^2 + |P|^2)^{\frac{1}{2}}, \tag{5.5.39}$$

$$\Pi = P - \theta^{-2}[P - D\lambda B - (D \cdot P)D - (B \cdot P)B]. \tag{5.5.40}$$

Again, on the "manifold" $\Psi = W(U) = (B, D, D \wedge B)^\top$ (5.5.38) reduces to Maxwell's equations (3.3.66) (with $J = 0$), (3.3.73), together with the supplementary conservation law (3.3.74), while (θ, Π) reduces to the entropy-entropy flux pair (η, Q) recorded in (3.3.73).

5.6 Initial-Boundary Value Problems

The issue of properly formulating the initial-boundary value problem for systems of hyperbolic conservation laws and establishing local existence of classical solutions has been the object of intensive study in recent years. A fairly definitive, albeit highly technical and complicated, theory has emerged, which lies beyond the scope of this book. Fortunately, detailed expositions are now available, in books and survey articles, referenced in Section 5.7. In order to convey to the reader a taste of the current state of this theory, a representative result will be recorded here, along the lines of the formulation of initial-boundary value problems presented in Section 4.7.

We begin by fixing as domain the half-space

$$\mathscr{D} = \{x \in \mathbb{R}^m : \nu \cdot x < 0\}, \tag{5.6.1}$$

with outward unit normal $\nu \in S^{m-1}$. We seek solutions to the system

$$\partial_t U(x,t) + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U(x,t)) = 0, \qquad x \in \mathscr{D},\ t \in (0,T), \tag{5.6.2}$$

satisfying initial conditions

$$U(x,0) = U_0(x)\,, \qquad x \in \mathscr{D}\,, \tag{5.6.3}$$

and boundary conditions in the special form (4.7.1), (4.7.8), namely,

$$BG(U(x,t))\nu = 0, \qquad x \in \partial\mathscr{D},\ t \in (0,T)\,, \tag{5.6.4}$$

where B is a constant $n \times n$ matrix.

We make the following assumptions on the system (5.6.2). The flux $G(U)$ is a smooth $n \times m$ matrix-valued function defined on a convex domain $\mathscr{O} \subset \mathbb{R}^n$. For normalization, $0 \in \mathscr{O}$ and $G(0) = 0$. Furthermore, (5.6.2) is endowed with a smooth entropy $\eta(U)$ such that $\mathrm{D}^2\eta(U)$ is positive definite on $\mathscr{O}$. This implies, in particular, that (5.6.2) is hyperbolic, so that for any $U \in \mathscr{O}$ and $\xi \in S^{m-1}$, the matrix $\Lambda(\xi;U)$, defined by (4.1.2), possesses real eigenvalues (characteristic speeds) $\lambda_1(\xi;U) \leq \cdots \leq \lambda_n(\xi;U)$ and associated linearly independent eigenvectors $R_1(\xi;U), \cdots, R_n(\xi;U)$. We require that each eigenvalue has constant multiplicity on $S^{m-1} \times \mathscr{O}$.

Turning to the boundary conditions (5.6.4), we introduce the "manifold" of boundary data

$$\mathscr{M} = \{U \in \mathscr{O} : BG(U)\nu = 0\} \tag{5.6.5}$$

and assume that the boundary is *noncharacteristic*, in the sense that, for a certain $k = 0, \cdots, n$ and all $U \in \mathscr{M}$,

$$\lambda_k(\nu;U) < 0 < \lambda_{k+1}(\nu;U), \tag{5.6.6}$$

where $\lambda_0(\nu;U) = -\infty$ and $\lambda_{n+1}(\nu;U) = \infty$. Thus k characteristic fields are incoming to $\mathscr{D}$ and $n-k$ characteristic fields are outgoing from $\mathscr{D}$, through $\partial\mathscr{D}$.

We assume, further, that, for any $U \in \mathscr{M}$, the rank of $B\Lambda(\nu;U)$ is k and

$$\mathbb{E}^k(\nu;U) \oplus \ker[B\Lambda(\nu;U)] = \mathbb{R}^n, \tag{5.6.7}$$

where $\mathbb{E}^k(\nu;U)$ denotes the subspace of $\mathbb{R}^n$ spanned by $R_1(\nu;U), \cdots, R_k(\nu;U)$. To motivate this condition, we linearize the system (5.6.2) and the boundary condition (5.6.4) about any constant state $U \in \mathscr{M}$:

$$\partial_t V(x,t) + \sum_{\alpha=1}^{m} \mathrm{D}G_\alpha(U)\partial_\alpha V(x,t) = 0, \qquad x \in \mathscr{D},\ t \in (0,T), \tag{5.6.8}$$

$$B\Lambda(\nu;U)V(x,t) = 0, \qquad x \in \partial\mathscr{D},\ t \in (0,T). \tag{5.6.9}$$

Thus, roughly speaking, the role of (5.6.7) is to ensure that the trace of V on $\partial\mathscr{D}$ is determined by combining the boundary conditions with the information carried to the boundary by the $n-k$ outgoing characteristic fields.

The final assumption on the boundary conditions is the *uniform Kreiss-Lopatinski condition*, which is formulated as follows. For each state $U \in \mathscr{M}$, vector $\xi \in S^{m-1}$

tangent to the boundary, i.e., $\xi \cdot \nu = 0$, and complex number z with $\mathrm{Re} z > 0$, we define the matrix

$$M(z,\xi;U) = \Lambda(\nu;U)^{-1}[zI + i\Lambda(\xi;U)]. \tag{5.6.10}$$

We denote by $\mathbb{E}(z,\xi;U)$ the subspace of $\mathbb{R}^n$ spanned by the eigenvectors associated with the eigenvalues of $M(z,\xi;U)$ with negative real part and require that

$$|W| \le c|B\Lambda(\nu;U)W|, \qquad \text{for all } W \in \mathbb{E}(z,\xi;U), \tag{5.6.11}$$

where c is a positive constant, independent of U,ξ and z. To interpret this assumption, notice that the linear system (5.6.8) admits solutions of the form

$$V(x,t) = \exp(i\xi \cdot x + zt)W(\nu \cdot x), \tag{5.6.12}$$

where the function $W(\tau)$ satisfies the ordinary differential equation

$$\dot{W} + M(z,\xi;U)W = 0. \tag{5.6.13}$$

The role of (5.6.11) is to rule out solutions (5.6.12) that satisfy the boundary condition (5.6.9) and exhibit "tame" growth in the spatial directions but grow exponentially with time.

Finally, we turn to the initial condition (5.6.3). For $j = 0,1,\cdots$, we let H^j denote the Sobolev space $W^{j,2}(\mathscr{D};\mathbb{R}^n)$, and assume $U_0 \in H^{\ell+1}$, for $\ell > \frac{m}{2}$. One may then calculate formally, from (5.6.2), the initial values $U_1(x),\cdots,U_\ell(x)$ of the time derivatives $\partial_t U(x,0),\cdots,\partial_t^\ell U(x,0)$ of solutions. Thus

$$U_1 = -\sum_{\alpha=1}^m \mathrm{D}G_\alpha(U_0)\partial_\alpha U_0, \tag{5.6.14}$$

$$U_2 = -\sum_{\alpha=1}^m \mathrm{D}G_\alpha(U_0)\partial_\alpha U_1 - \sum_{\alpha=1}^m \mathrm{D}^2 G_\alpha(U_0)[U_1,\partial_\alpha U_0], \tag{5.6.15}$$

and so on. Moreover, $U_j \in H^{\ell+1-j}$, $j = 0,\cdots,\ell$. In particular, the trace of U_j on the hyperplane $\partial\mathscr{D}$ is well-defined, for $j = 0,\cdots,\ell$. We then require that the initial data be compatible with the boundary condition, in the sense

$$B\partial_t^j G(U(x,t))\nu = 0, \qquad t = 0,\ x \in \partial\mathscr{D}, \quad j = 0,\cdots,\ell, \tag{5.6.16}$$

namely,

$$BG(U_0(x))\nu = 0, \qquad x \in \partial\mathscr{D}, \tag{5.6.17}$$

$$B\Lambda(\nu;U_0(x))U_1(x) = 0, \qquad x \in \partial\mathscr{D}, \tag{5.6.18}$$

and so on.

We have now laid the preparation for stating the existence theorem:

5.6.1 Theorem. *Under the above assumptions on the system, the boundary conditions and the initial data, there exists a unique classical solution* $U \in C^1(\overline{\mathscr{D}} \times [0, T_\infty))$ *of the initial-boundary value problem* (5.6.2), (5.6.3), (5.6.4), *for some* $0 < T_\infty \leq \infty$. *Furthermore,*

$$U(\cdot,t) \in \bigcap_{j=0}^{\ell+1} C^j([0,T_\infty); H^{\ell+1-j}). \tag{5.6.19}$$

The interval $[0, T_\infty)$ *is maximal in that if* $T_\infty < \infty$ *then*

$$\limsup_{t \uparrow T_\infty} \|\nabla U(\cdot,t)\|_{L^\infty} = \infty \tag{5.6.20}$$

and/or the range of $U(\cdot,t)$ *escapes from every compact subset of* $\mathscr{O}$, *as* $t \uparrow T_\infty$.

The (lengthy and technical) proof proceeds from linear systems with constant coefficients to linear systems with variable coefficients, and then passes to quasilinear systems via linearization (5.6.8), (5.6.9) and a fixed point argument, for the map $U \mapsto V$.

It should be noted that the assumptions in the above theorem are too restrictive for dealing with many natural initial-boundary value problems arising in continuum physics. In the Euler equations, for isentropic or nonisentropic gas flow, the assumption that the characteristic speeds have constant multiplicity is indeed valid (see Sections 3.3.5 and 3.3.6); but the assumption that the boundary is noncharacteristic is often violated, for instance in the case of no-penetration (or slip) boundary conditions $v \cdot \nu = 0$. In the equations of isentropic or nonisentropic elastodynamics, the condition that the characteristic speeds have constant multiplicity is often violated, for example in the vicinity of the natural state of an isotropic elastic solid where the multiplicity of the characteristic speed associated with shear waves undergoes a transition. Moreover, the boundary is always characteristic, as the system possesses zero characteristic speeds. Beyond that, one needs to consider more general domains $\mathscr{D}$ and homogeneous or inhomogeneous boundary conditions on $\partial\mathscr{D}$ of more general form than (5.6.4). These issues are addressed by more sophisticated versions of Theorem 5.6.1. References are cited in Section 5.7.

5.7 Notes

A comprehensive treatment of classical solutions to the initial and initial-boundary value problem for hyperbolic systems of conservation laws is found in the recent monograph by Benzoni-Gavage and Serre [2].

Local existence of classical solutions to the Cauchy problem for symmetrizable systems of conservation laws has been established by a variety of methods, ultimately relying on the hierarchy of "energy" estimates derived by differentiating the system with respect to the spatial variables.

The earliest, and still most popular, approach, expounded in Benzoni-Gavage and Serre [2], constructs solutions to (5.1.1) by an iteration process on the linearized systems (5.1.7). It was originated by Schauder [1], in the context of the quasilinear second-order wave equation, and has attained its present general form through the contributions of several authors, in particular Friedrichs [2], Gårding [1] and Majda [3]. Godunov [3], Makino, Ukai and Kawashima [1], Chemin [1], Lax [1], M.E. Taylor [1,2] and Métivier [1] have used symmetrizers other than the Hessian of a convex entropy, or symbolic symmetrizers.

An alternative way of establishing Theorem 5.1.1, by Kato [1], is based on the theory of abstract evolution equations. The method of vanishing viscosity was adopted here because it also applies to the cases where the entropy is convex only in the direction of the involution cone (Theorem 5.4.4) or it is merely polyconvex (Theorem 5.5.3).

The discussion of the effects of damping, culminating in Theorem 5.2.1, has been adapted from Hanouzet and Natalini [1] and Yong [6]. See also Yang, Zhu and Zhao [3], and Bianchini, Hanouzet and Natalini [1]. For an application to the system of gas dynamics with damping induced by energy radiation, see Rohde and Yong [1,2]. For the effect of damping on the long time behavior of solutions, see Ruggeri and Serre [1].

The setting of the general relaxation framework, in Section 5.2, has been taken from Chen, Levermore and Liu [1]. The connection between relaxation and diffusion was first recognized in the kinetic theory of gases, where it is effected by means of the Chapman-Enskog expansion (e.g. Cercignani [1]). Chapman-Enskog type expansions have also been employed in order to relate classes of hyperbolic balance laws (5.2.1) with parabolic systems of the form (4.6.1); see Kawashima and Yong [1,2].

There is voluminous literature on various aspects of relaxation theory. Surveys and extensive bibliography are found in Natalini [3] and Yong [4]. Relevant references include Tai-Ping Liu [21], Nishibata and Yu [1], Wang and Xin [1], Donatelli and Marcati [1], Hsiao and Pan [1], Shen, Tveito and Winther [1], Yong [2,3,5], Yang and Zhu [1], Yang, Zhu and Zhao [3], Liu and Yong [1], Natalini and Terracina [1], Xin and Xu [1], DiFrancesco and Lattanzio [1], Fan and Härterich [1], Fan and Luo [1], Bedjaoui, Klingenberg and LeFloch [1], Berthelin and Bouchut [1], Junca and Rascle [1], Tadmor and Tang [2], and Lattanzio and Tzavaras [1]. In particular, the system (5.2.18) with $p(u) = a^2u$, proposed by Jin and Xin [1], has served widely as a vehicle for understanding and explaining the features of relaxation. We will visit the theory of this system in Section 16.5, and the reader may find the relevant references in Section 16.9. Baudin, Coquel and Tran [1] propose a variant of the above relaxation scheme, which bears a curious relationship to the one-dimensional Born-Infeld system; see Serre [11]. We will also come across relaxation in Section 6.6, with references in Section 6.11.

The intimate relation between relaxation and diffusion also manifests itself in the large time behavior of solutions to hyperbolic systems with "frictional" damping and in particular in the simple system governing the isentropic flow of a gas through a porous medium; see Hsiao and Liu [1], Tai-Ping Liu [25], Serre and Xiao [1], Hsiao and Luo [1], Luo and Yang [1], Nishihara and Yang [1], Hsiao and Pan [2,3], Hsiao, Li and Pan [1], Hsiao and Li [1,2], Nishihara, Wang and Yang [1,2,3], Marcati and Mei [1], He and Li [1], Liu and Natalini [1], Marcati and Pan [1], Marcati and Nishihara [1], Pan [1,2], Li and Saxton [1], Huang and Pan [1,2], Lattanzio and Rubino [1], Huang, Marcati and Pan [1], and Dafermos and Pan [1].

Out of a huge literature on nonhyperbolic systems that nevertheless exhibit behavior similar to that of hyperbolic systems with damping, here is a small representative sample: For the Euler-Poisson system, see Poupaud, Rascle and Vila [1], Dehua Wang [1,3], Wang and Chen [1], Guo [1], Engelberg, Liu and Tadmor [1], Li, Markowich and Mei [1], Feldman, Ha and Slemrod [1], Jang [1], Chae and Tadmor [1], and Tadmor and Wei [1]. For the semiconductor equations, see the monograph by Markowich, Ringhofer and Schmeiser [1], which contains a comprehensive list of references; also Guo and Strauss [1]. The monographs by Lions [2] and Feireisl [1] treat the system of equations for compressible viscoelastic fluids, in several space dimensions, and provide an exhaustive bibliography. Of course, the literature on the incompressible case, which includes the classical Navier-Stokes equations, is vast. The system of magnetohydrodynamics for viscous fluids is discussed in Chen and Wang [4,5], and Dehua Wang [4]. For the equations of radiation magnetohydrodynamics, see Rohde and Yong [2]. For the system of one-dimensional thermoviscoelasticity, see Dafermos and Hsiao [2], and Dafermos [12]. For the equations of one-dimensional thermoelasticity, see Slemrod [1], Dafermos and Hsiao [3], and the detailed survey article by Racke [1]. Finally, for the equations of one-dimensional viscoelasticity, with viscosity induced by fading memory dependence, see MacCamy [1], Dafermos and Nohel [1], Dafermos [15], and the monograph by Renardy, Hrusa and Nohel [1].

The effect of dispersion in delaying, or even preventing outright, the breakdown of classical solutions for systems satisfying the null condition is discussed in Christodoulou [1], Klainerman [1], Klainerman and Sideris [1], Sideris [2,3,4], Agemi [1], and Chae and Huh [1]. See also the monograph by Ta-tsien Li [1].

The proof of Theorem 5.3.1 combines ideas of DiPerna [7] and Dafermos [9,10]. This approach even applies in certain cases where the solution $\bar{U}$ is weak; see Gui-Qiang Chen [7], Chen and Frid [7], Gui-Qiang Chen and Yachun Li [1,2], and Gui-Qiang Chen and Jun Chen [1]. For an application in establishing the stability of rotating self-gravitating fluid masses, see Luo and Smoller [1,2]. A connection between relative entropy and relaxation is established by Tzavaras [7].

Hyperbolic systems of conservation laws with involutions were discussed by Boillat [4] and Dafermos [14]. In particular, Boillat [4] considers sufficient conditions that are more general than (5.4.2) and presents examples arising in general relativity. The analysis in Section 5.4 is intimately related to the theory of compensated compactness, as formalized by Murat and Tartar; see Tartar [1,2]. In that connection, "involution cone" corresponds to "characteristic cone." Theorem 5.4.4 originally appeared in the first edition of this book; however, typical examples, such as the system (3.3.19) of balance laws of (isentropic) elastodynamics, had been studied earlier, for example by Hughes, Kato and Marsden [1] and by Dafermos and Hrusa [1]. Theorem 5.4.5 is taken from Dafermos [24].

The notion of quasiconvexity, introduced by Definition 5.4.6, is a generalization of quasiconvexity in the sense of Morrey [1], due to Dacorogna [1]. For a detailed study and proof of Theorem 5.4.7, see Müller and Fonseca [1].

Section 5.5 follows Dafermos [27], which improves upon the treatment of this topic in earlier editions of the book. The concept (though not the name) of a contingent entropy is due to Serre [22]. The notion of polyconvexity in elastostatics was introduced by Ball [1], as a condition rendering the internal energy function weakly lower semicontinuous. It is shown there that polyconvexity implies quasiconvexity and, in turn, quasiconvexity implies rank-one convexity of the strain energy function. The question of whether, conversely, rank-one convexity generally implies quasiconvexity was settled, in the negative, by Šverak [1].

It is from P.G. LeFloch that the author originally heard the idea of extending the system of conservation laws in elastodynamics by appending conservation laws for the invariants of the stretch tensor. Explicit extensions were first published by Qin [1] and by Demoulini, Stuart and Tzavaras [2]. See also Lattanzio and Tzavaras [1]. Brenier [2] presents two distinct extensions of the equations of electrodynamics, for the Born-Infeld constitutive relations, including the one recorded here, and discusses its asymptotics in various regimes. This investigation continues in Brenier [4] and Brenier and Yong [1]. See also Neves and Serre [1]. Serre [22] devised the proper extension in electrodynamics, under general constitutive relations, by exploiting the contingent entropy-entropy flux pair (3.3.76).

A thorough discussion of initial-boundary value problems, including the details on the material sketched in Section 5.6, is found in Benzoni-Gavage and Serre [2]. See also the survey article by Higdon [1]. For perspectives on stability issues see Benzoni-Gavage, Rousset, Serre and Zumbrun [1]. The vanishing viscosity approach and the related questions on the nature and stability of resulting boundary layers have been actively investigated in recent years; see Kreiss [1], Benabdallah and Serre [1], Gisclon and Serre [1], Gisclon [1], Grenier and Guès [1], Kreiss and Kreiss [1], Xin [6], Serre and Zumbrun [1], Serre [14, 17, 24], Joseph and LeFloch [1,2,3], Rousset [1,2,3], Métivier and Zumbrun [1,2], and Guès, Métivier, Williams and Zumbrun [5,6].

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