

Chapter 2

The Probability Space of Brownian Motion

2.1 Introduction

According to Einstein's description, the Brownian motion can be defined by the following two properties: first, it has continuous trajectories (sample paths) and second, the increments of the paths in disjoint time intervals are independent zero mean Gaussian random variables with variance proportional to the duration of the time interval (it is assumed, for definiteness, that the possible trajectories of a Brownian particle start at the origin). These properties have far-reaching implications about the analytic properties of the Brownian trajectories. It can be shown, for example (see Theorem 2.4.1), that these trajectories are not differentiable at any point with probability 1 [198]. That is, the velocity process of the Brownian motion cannot be defined as a real-valued function, although it can be defined as a distribution (generalized function) [152]. Langevin's construction does not resolve this difficulty, because it gives rise to a velocity process that is not differentiable so that the acceleration process, $\Xi(t)$ in eq. (1.24), cannot be defined.

One might guess that in order to overcome this difficulty in Langevin's equation all differential equations could be converted into integral equations so that the equations contain only well defined velocities. This approach, however, fails even in the simplest differential equations that contain the process $\Xi(t)$ (which in one dimension is denoted $\Xi(t)$). For example, if we assume that $\Delta w(t) \equiv \int_t^{t+\Delta t} \Xi(s) ds \sim \mathcal{N}(0, \Delta t)$ and construct the solution of the initial value problem

$$\dot{x} = x\Xi(t), \quad x(0) = x_0 > 0 \quad (2.1)$$

by the Euler method

$$x_{\Delta t}(t + \Delta t) - x_{\Delta t}(t) = x_{\Delta t}(t)\Delta w(t), \quad x_{\Delta t}(0) = x_0 > 0, \quad (2.2)$$

the limit $x(t) = \lim_{\Delta t \rightarrow 0} x_{\Delta t}(t)$ is not the function

$$x(t) = x_0 \exp \left\{ \int_0^t \Xi(s) ds \right\}.$$

It is shown below that the solution is

$$x(t) = x_0 \exp \left\{ \int_0^t \Xi(s) ds - \frac{1}{2}t \right\}.$$

It is evident from this example that differential equations that involve the Brownian motion do not obey the rules of the differential and integral calculus.

A similar phenomenon manifests itself in other numerical schemes. Consider, for example, three different numerical schemes for integrating eq. (2.1) (or rather (2.2)), an explicit Euler, semi implicit, and implicit schemes. More specifically, consider the one-dimensional version of eq. (2.2) with $\Delta w(t) \sim \mathcal{N}(0, \Delta t)$. Discretizing time by setting $t_j = j\Delta t$ for $j = 0, 1, 2, \dots$, the random increments $\Delta w(t_j) = w(t_{j+1}) - w(t_j)$ are simulated by $\Delta w(t_j) = n_j \sqrt{\Delta t}$, where $n_j \sim \mathcal{N}(0, 1)$ are independent (zero mean standard Gaussian random numbers taken from the random number generator). The explicit Euler scheme (2.2) is written as $x_{ex}(t_{j+1}) = x_{ex}(t_j) + x_{ex}(t_j) n_j \sqrt{\Delta t}$, with $x_{ex}(0) = x_0 > 0$, the semi implicit scheme is $x_{si}(t_{j+1}) = x_{si}(t_j) + \frac{1}{2}[x_{si}(t_j) + x_{si}(t_{j+1})] n_j \sqrt{\Delta t}$, with $x_{si}(0) = x_0 > 0$, and the implicit scheme is $x_{im}(t_{j+1}) = x_{im}(t_j) + x_{im}(t_{j+1}) n_j \sqrt{\Delta t}$, with $x_{im}(0) = x_0 > 0$. In the limit $\Delta t \rightarrow 0$, $t_j \rightarrow t$ the numerical solutions converge (in probability) to the three different limits

$$\lim_{\Delta t \rightarrow 0, t_j \rightarrow t} x_{ex}(t_j) = x_0 \exp \left\{ \int_0^t \Xi(s) ds - \frac{1}{2}t \right\}$$

$$\lim_{\Delta t \rightarrow 0, t_j \rightarrow t} x_{si}(t_j) = x_0 \exp \left\{ \int_0^t \Xi(s) ds \right\}$$

$$\lim_{\Delta t \rightarrow 0, t_j \rightarrow t} x_{im}(t_j) = x_0 \exp \left\{ \int_0^t \Xi(s) ds + \frac{1}{2}t \right\}.$$

These examples indicate that naïve applications of elementary analysis and probability theory to the simulation of Brownian motion may lead to conflicting results. The study of the trajectories of the Brownian motion requires a minimal degree of mathematical rigor in the definitions and constructions of the probability space and the probability measure for the Brownian trajectories in order to gain some insight

into stochastic dynamics. Thus Section 2.2 contains a smattering of basic measure theory that is necessary for the required mathematical insight.

In this chapter the mathematical Brownian motion is defined axiomatically by the properties of the physical Brownian motion as described in Chapter 1. Two constructions of the mathematical Brownian motion are presented, the Paley–Wiener Fourier series expansion and Lévy’s method of refinements of piecewise linear approximations [150]. Some analytical properties of the Brownian trajectories are derived from the definition.

2.2 The space of Brownian trajectories

A continuous-time *random process* (or *stochastic process*) $x(t, \omega) : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ is a function of two variables, a real variable t , usually interpreted as time, and ω in a *probability space* (or *sample space*) Ω , in which events are defined. More generally, the random process $x(t, \omega)$ can take values in a set X , called *the state space*, such as the real line $\mathbb{R}$, or the Euclidean space $\mathbb{R}^d$, or any other set. When t is interpreted as time, we write $x(t, \omega) : \mathbb{R}_+ \times \Omega \rightarrow X$. For each $\omega \in \Omega$ the stochastic process is a function of t , called a *trajectory*.

We assume henceforth that the state space of a stochastic process $x(t, \omega)$ is $X = \mathbb{R}^d$ and its trajectories are continuous functions; that is, for fixed $\omega \in \Omega$ the trajectories are continuous curves in $\mathbb{R}^d$. To assign probability to events connected to trajectories, it is necessary to describe the probability space Ω . We begin with the description of the probability space and the Einstein-Langevin requirement that the trajectories of the Brownian motion be continuous. Thus, we define events in a probability space for the Brownian motion in terms of continuous functions of time. We identify all possible paths of the Brownian motion as all continuous functions. Each continuous function is an elementary event in this space. Physically, this event can represent the path of a microscopic particle in solution. The path, and thus the event in the probability space, is the outcome of the experiment of continuous recording of the path of a particle diffusing according to the Einstein-Langevin description, namely, without jumps. If jumps were found experimentally, a different theory might be needed, depending on the properties of the paths, for example, as is the case for the paths of the Poisson jump process [199, p. 290], [116, p. 22, Example 2]. In many cases, we consider sets of elementary events, which are often called “events”, for short.

We hardly ever consider elementary events, because their probability is zero. This, for example is the case of turning a roulette wheel with a needle pointing to a single point. The outcome of each spin is a single point or number (an elementary event). Each point must have probability zero, because in an honest roulette wheel all points are equally likely and there are an infinite number of them on the wheel. Of course, for every spin there is an outcome, so that events of probability zero do occur. The roulette wheel is partitioned into a finite number of intervals of finite lengths, each containing an infinite number of points (elementary events), because we want a finite nonzero estimate of the probability. Every time the roulette wheel

is spun the needle comes to rest in only one interval, which is the outcome of the game or experiment. This outcome is called “an event”, which is a composite event consisting of uncountably many elementary events, whose individual probabilities are zero; however, the probability of the outcome, the composite event, is a finite nonzero number.

In the same vein, a Brownian elementary event will have to be assigned probability zero. A typical Brownian event that corresponds to an experiment consists of (uncountably many) elementary events. It may be, for example, the set of all continuous functions that satisfy some given criteria. Thus, in an experiment one might record the ensemble of all Brownian paths that are found in a given region (under a microscope) at a given time. We formally define Brownian elementary events and events as follows. Denote by $\mathbb{R}$ and $\mathbb{R}_+$ the real numbers and the nonnegative real numbers, respectively, then

Definition 2.2.1 (The space of elementary events). *The space of elementary events for the Brownian motion is the set of all continuous real functions,*

$$\Omega = \{\omega(t) : \mathbb{R}_+ \mapsto \mathbb{R}\}.$$

Thus each continuous function is an elementary event. To define Brownian events that are more complicated than elementary events; that is, events that consist of uncountably many Brownian trajectories (each of which is an elementary event), we define first events called “cylinders”.

Definition 2.2.2 (Cylinder sets). *A cylinder set of Brownian trajectories is defined by times $0 \leq t_1 < t_2 < \dots < t_n$ and real intervals $I_k = (a_k, b_k)$, ($k = 1, 2, \dots, n$) as*

$$C(t_1, \dots, t_n; I_1, \dots, I_n) = \{\omega(t) \in \Omega \mid \omega(t_k) \in I_k, \text{ for all } 1 \leq k \leq n\}. \quad (2.3)$$

Obviously, for any $0 \leq t_1 < t$ and any interval I_1 ,

$$C(t; \mathbb{R}) = \Omega, \quad C(t_1, t; I_1, \mathbb{R}) = C(t_1; I_1). \quad (2.4)$$

Thus, for the the cylinder $C(t_1, t_2, \dots, t_n; I_1, I_2, \dots, I_n)$ not to contain a trajectory $\omega(t)$ it suffices that for at least one of the times t_k the value of $\omega(t_k)$ is not in the interval I_k , for example, the dotted trajectory in Figure 2.1 belongs to the cylinder $C(126, [-0.1, 0.5])$, but neither to $C(132, [-0.4, 0.1])$ nor to $C(136, [-0.2, -0.10])$. Thus it does not belong to the cylinder $C(126, 132, [-0.1, 0.5], [-0.4, 0.1])$, which is their intersection.

For each real x , we set $I^x = (-\infty, x]$. Then the cylinder $C(t; I^x)$ is the set of all continuous functions $\omega(\cdot)$ such that $\omega(t) \leq x$. It is the set of all Brownian trajectories that would be observed at time t to be below the level x . It is important to note that $C(t; I^x)$ consists of entire trajectories, not merely of their segments observed below the level x at time t .

The cylinder $C(t_1, t_2; I_1, I_2)$ consists of all continuous functions $\omega(\cdot)$ such that $a_1 < \omega(t_1) < b_1$ and $a_2 < \omega(t_2) < b_2$. That is, $C(t_1, t_2; I_1, I_2)$ consists of all

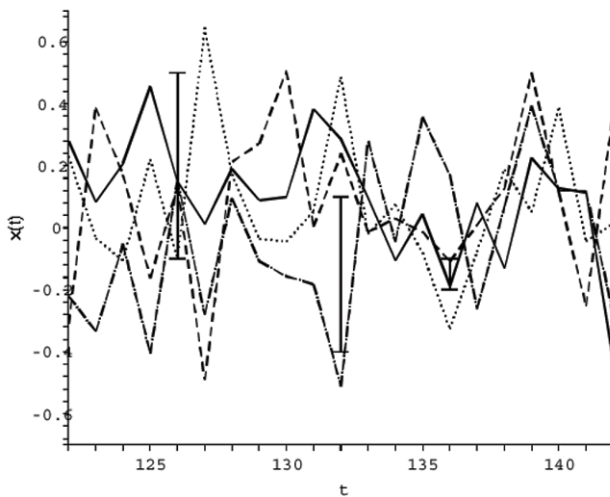


Figure 2.1. Four Brownian trajectories sampled at discrete times. Three cylinders are marked by vertical bars. The trajectories were sampled according to the scheme (2.21).

Brownian paths that are observed at time t_1 to be between the levels a_1 and b_1 and at time t_2 to be between the levels a_2 and b_2 (see Figure 2.1).

Definition 2.2.3 (Brownian events). Brownian events are all sets of Brownian trajectories that can be obtained from cylinders by the operations of countable unions, intersections, and the operation of complement.

These sets form the space of *Brownian events*, denoted $\mathcal{F}$. The space $\mathcal{F}$ is characterized by the property that if A_i are subsets of $\mathcal{F}$ ($i = 1, 2, \dots$); that is, if A_i are Brownian events, then their (countable) union, $\bigcup_{i=1}^{\infty} A_i$, is also an event and so are the complements $A_i^c = \Omega - A_i$. This space of Brownian events is an example of a σ -algebra.

Definition 2.2.4 (σ -algebra). A σ -algebra in Ω is a nonempty collection $\mathcal{F}$ of subsets of Ω such that

1. $\Omega \in \mathcal{F}$.
2. If $A \in \mathcal{F}$ then $A^c = \Omega - A \in \mathcal{F}$.
3. If $A_i \in \mathcal{F}$, ($i = 1, 2, \dots$), then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

In this notation the designation σ stands for the word “countable”. If only a finite number of unions, intersections, and complement of events are considered, we refer to the resulting set of events as an “algebra”. The elements (events) of $\mathcal{F}$ are

called *measurable sets*. Examples exist of nonmeasurable sets [183], [58]. Sigma-algebras are introduced so we can easily keep track of past, present, and future in our study of Brownian events. Causality must be included in our description of the evolution of Brownian trajectories; that is, events occurring after time t do not affect events up to time t . There are noncausal problems in probability theory, such as the problem of estimating a given random signal in a given time interval, given its noisy measurements in the past and in the future (e.g., the entire signal is recorded on a CD). This is the smoothing problem (see [215] for some discussion).

The pair $(\Omega, \mathcal{F})$ is called *probability space*. The probability measure is defined on events. In the process of constructing a probability model of the Brownian motion (or any other process) both the space of elementary events and the relevant σ -algebra of events have to be specified. There is more than one way to specify structures of events (algebras or σ -algebras) in the same space of elementary events and different pairs of spaces and σ -algebras of events are considered different probability spaces, as described below. For example, if the roulette wheel is partitioned into arcs in two different ways so that one partition cannot be obtained from the other by the operations of union, intersection, and complement, then the different partitions form different algebras in the space of elementary events for the experiment of rotating the roulette wheel. Thus, partitioning the wheel into two equal arcs or into three equal arcs results in two different algebras of events. In general, each σ -algebra in Ω specifies a different way of selecting the elementary events to form composite events that correspond to different ways of handling the same raw experimental data, when the data are the elementary events.

Definition 2.2.5 (Brownian filtration). *The σ -algebra $\mathcal{F}_t$ is defined by cylinder sets confined to times $0 \leq t_i < t$, for some fixed t . Obviously, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ if $0 \leq s < t < \infty$. The family of σ -algebras $\mathcal{F}_t$ for $t \geq 0$ is called the Brownian filtration and is said to be generated by the Brownian events up to time t .*

Note that the elementary events of the Brownian filtration $\mathcal{F}_t$ are continuous functions in the entire time range, not just the initial segments in the time interval $[0, t]$. However, only the initial segments of the Brownian paths in $\mathcal{F}_t$ that occur by time t are observed and so can be used to define the filtration. The pairs $(\Omega, \mathcal{F}_t)$ are different probability spaces for different values of t .

Up to now, we have considered only elementary events and sets of elementary events that were referred to as “events”. The events we have defined mathematically have to be assigned probabilities, to represent some measure of our uncertainty about the outcome of a given experiment [53]. The assigned probabilities form a mathematical model for the statistical processing of collected data. This was the case for the recordings of paths of Brownian particles before mathematical models were constructed by Einstein, Smoluchowski, and Langevin.

Definition 2.2.6 (Random variables in $(\Omega, \mathcal{F})$). *A random variable $X(\omega)$ in $(\Omega, \mathcal{F})$ is a real function $X(\cdot) : \Omega \rightarrow \mathbb{R}$ such that $\{\omega \in \Omega \mid X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$.*

That is, $\{\omega \in \Omega \mid X(\omega) \leq x\}$ is a Brownian event that can be expressed by countable operations of union, intersection, and complement of cylinders. In mathematical terminology a random variable in Ω is a real $\mathcal{F}$ -measurable function.

Example 2.1 (Random functions). For each $t \geq 0$ consider the random variable $X_t(\omega) = \omega(t)$ in Ω . This random variable is the outcome of the experiment of sampling the position of a Brownian particle (trajectory) at a fixed time t . Thus $X_t(\omega)$ takes different values on different trajectories. Obviously, $\{\omega \in \Omega \mid X_t(\omega) \leq x\} = \{\omega \in \Omega \mid \omega(t) \leq x\} = C(t; I^x) \in \mathcal{F}$, so that $X_t(\omega)$ is a random variable in $(\Omega, \mathcal{F})$. $\square$

Example 2.2 (Average velocity). Although, as mentioned in Section 2.1, the derivative of the Brownian path does not exist as a real-valued function, the *average velocity process* of a Brownian trajectory ω in the time interval $[t, t + \Delta t]$ can be defined as $\bar{V}_t(\omega) = [\omega(t + \Delta t) - \omega(t)]/\Delta t$. The time averaging here is not expectation, because it is defined separately on each trajectory, therefore $\bar{V}_t(\omega)$ is a random variable, which takes different values on different trajectories. To see that $\bar{V}_t(\omega)$ is a random variable in $(\Omega, \mathcal{F})$, we have to show that for every real number v the event $\{\omega \in \Omega \mid \bar{V}_t(\omega) \leq v\}$ can be expressed by countable operations of union, intersection, and complement of cylinders. To do so, we assume that $\Delta t > 0$ and write

$$\{\omega \in \Omega \mid \bar{V}_t(\omega) \leq v\} = \{\omega \in \Omega \mid \omega(t + \Delta t) - \omega(t) \leq v\Delta t\} \equiv A.$$

We denote the set of rational numbers by $\mathbb{Q}$ and the set of positive rational numbers by $\mathbb{Q}^+$ and define in $\mathcal{F}$ the set of paths

$$B \equiv \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{y \in \mathbb{Q}} C(t, t + \Delta t; [y - \varepsilon, y + \varepsilon], (-\infty, v\Delta t + y + \varepsilon]).$$

The set B is simply the set of paths in $\mathcal{F}$ such that for every rational $\varepsilon > 0$, there exists a rational y such that $|\omega(t) - y| \leq \varepsilon$ and $\omega(t + \Delta t) \leq y + v\Delta t + \varepsilon$. Showing that $A = B$ proves that $\bar{V}_t(\omega)$ is a random variable in Ω (i.e., $\bar{V}_t(\omega)$ is $\mathcal{F}$ -measurable).

To show that $A = B$, we show that $A \subset B$ and $B \subset A$. If $\omega \in A$ then

$$\omega(t + \Delta t) - \omega(t) \leq v\Delta t \tag{2.5}$$

and as is well-known from the differential calculus, for every number $\omega(t)$ and every $\varepsilon \in \mathbb{Q}^+$ there exists $y \in \mathbb{Q}$ such that

$$y - \varepsilon \leq \omega(t) \leq y + \varepsilon. \tag{2.6}$$

It follows from eqs. (2.5) and (2.6) that both inequalities

$$y - \varepsilon \leq \omega(t) \leq y + \varepsilon, \quad \omega(t + \Delta t) \leq v\Delta t + y + \varepsilon \tag{2.7}$$

hold. They mean that for every $\varepsilon \in \mathbb{Q}^+$ there exists $y \in \mathbb{Q}$ such that

$$\omega \in C(t, t + \Delta t; [y - \varepsilon, y + \varepsilon], (-\infty, v\Delta t + y + \varepsilon]).$$

This in turn means that $\omega \in B$. Hence $A \subset B$.

Conversely, if $\omega \in B$, then for every $\varepsilon \in \mathbb{Q}^+$ there exists $y \in \mathbb{Q}$ such that $\omega \in C(t, t + \Delta t; [y - \varepsilon, y + \varepsilon], (-\infty, v\Delta t + y + \varepsilon])$, which implies that the inequalities (2.7) hold and consequently $\omega(t + \Delta t) - \omega(t) \leq v\Delta t + 2\varepsilon$ for every $\varepsilon \in \mathbb{Q}^+$. It follows that $\omega(t + \Delta t) - \omega(t) \leq v\Delta t$, so that $\omega \in A$, which implies that $B \subset A$, as claimed above. $\square$

Example 2.3 (Integrals of random functions). A similar argument can be used to show, for example, that $X(\omega) = \int_0^T \omega(t) dt$ is a random variable in Ω , measurable with respect to $\mathcal{F}_T$. $\square$

Definition 2.2.7 (Markov times). A nonnegative random variable $\tau(\omega)$, defined on Ω , is called a stopping time or a Markov time relative to the filtration $\mathcal{F}_t$ for $t \geq 0$ if

$$\{\omega \in \Omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0.$$

Example 2.4 (First passage times). The first passage time (FPT) of a Brownian trajectory through a given point is a random variable in Ω and a stopping time. Indeed, assume that $\omega(0) < y$ and set $\tau_y(\omega) = \inf\{t \geq 0 \mid \omega(t) > y\}$; that is, $\tau_y(\omega)$ is the first passage time of $\omega(t)$ through the value y . To show that τ_y is $\mathcal{F}_t$ -measurable for every $t > 0$, we proceed in an analogous manner to that above. We denote by $\mathbb{Q}_t$ the set of all positive rational numbers that do not exceed t . Obviously, $\mathbb{Q}_t$ is a countable set. The event $\{\omega \in \Omega \mid \tau_y(\omega) \leq t\}$ consists of all Brownian trajectories ω that go above the level y at some time prior to t . Thus, due to the continuity of the Brownian paths,

$$\{\omega \in \Omega \mid \tau_y(\omega) \leq t\} = \bigcup_{r \in \mathbb{Q}_t} \{\omega \in \Omega \mid \omega(r) \geq y\},$$

which is a countable union of the cylinders $C(r, [y, \infty))$ for $r \leq t$ and is thus in $\mathcal{F}_t$. $\square$

Example 2.5 (Last passage time). On the other hand, the last passage time (LPT) of $x(t, \omega)$ to a given point y before time T , denoted $LPT(y, T, \omega)$, is not a stopping time, because at time t it is not a Brownian event that depends on the Brownian trajectories up to time t . Rather, it depends on events after time t , because the last passage may occur after that; that is,

$$\{\omega \in \Omega \mid LPT(y, T, \omega) \leq t\} \notin \mathcal{F}_t \text{ for all } 0 \leq t < T.$$

Although $LPT(y, T, \omega)$ is a random variable in $(\Omega, \mathcal{F})$, it is not a random variable in $(\Omega, \mathcal{F}_t)$ for $t < T$. Last passage times occur in practical problems. For example, if a Brownian particle is trapped in a finite potential well and escapes at a random time, the time between the last visit to the bottom of the well and the first passage time through the top of the well is a LPT. $\square$

Example 2.6 (Indicators). For any set $A \in \Omega$ the indicator function of A is defined by

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

For all $A \in \mathcal{F}$ the function $\mathbf{1}_A(\omega)$ is a random variable in $(\Omega, \mathcal{F})$. Indeed, if $x < 1$, then $\{\omega \in \Omega \mid \mathbf{1}_A(\omega) \leq x\} = \Omega - A = A^c$ and if $x \geq 1$, then $\{\omega \in \Omega \mid \mathbf{1}_A(\omega) \leq x\} = \Omega$ so that in either case the set $\{\omega \in \Omega \mid \mathbf{1}_A(\omega) \leq x\}$ is in $\mathcal{F}$. Thus, if A is not a measurable set, its indicator function $\mathbf{1}_A(\omega)$ is a nonmeasurable function so that $\mathbf{1}_A(\omega)$ is not a random variable. $\square$

Exercise 2.1 (Positive random variables). For a random variable $X(\omega)$ define the functions $X^+(\omega) = \max\{X(\omega), 0\}$ and $X^-(\omega) = \min\{X(\omega), 0\}$. Show that $X^+(\omega)$ and $X^-(\omega)$ are random variables (i.e., they are measurable functions). $\square$

Measurements recorded sequentially in time are often represented graphically as points in the d -dimensional real Euclidean space $\mathbb{R}^d$ ($d = 1, 2, \dots$). When the points are sampled from a curve in $\mathbb{R}^d$, they form a path. For example, recordings of trajectories of Brownian particles in $\mathbb{R}^3$ reveal that they have continuous paths, however, repeated recordings yield different paths that look completely erratic and random. When tracking charged Brownian particles (e.g., ions in solution) in the presence of an external electrostatic field, the paths remain continuous, erratic, and random; however, they tend to look different from those of uncharged Brownian particles.

Definition 2.2.8 (Stochastic processes in $(\Omega, \mathcal{F})$). A function $x(t, \omega) : \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}$ is called a stochastic process in $(\Omega, \mathcal{F})$ with continuous trajectories if

- (i) $x(t, \omega)$ is a continuous function of t for every $\omega \in \Omega$,
- (ii) for every fixed $t \geq 0$ the function $x(t, \omega) : \Omega \mapsto \mathbb{R}$ is a random variable in Ω .

The variable ω in the notation for a stochastic process $x(t, \omega)$ denotes the dependence of the value the process takes at any given time t on the elementary event ω ; that is, on the particular realization of the Brownian path ω . Point (ii) of the definition means that the sets $\{\omega \in \Omega \mid x(t, \omega) \leq y\}$ are Brownian events for each $t \geq 0$ and $y \in \mathbb{R}$; that is, they belong to $\mathcal{F}$. When they do, we say that the process $x(t, \omega)$ is *measurable* with respect to $\mathcal{F}$ or simply $\mathcal{F}$ -measurable.

Definition 2.2.9 (Adapted processes). The process $x(t, \omega)$ is said to be adapted to the Brownian filtration $\mathcal{F}_t$ if $\{\omega \in \Omega \mid x(t, \omega) \leq y\} \in \mathcal{F}_t$ for every $t \geq 0$ and $y \in \mathbb{R}$. In that case we also say that $x(t, \omega)$ is $\mathcal{F}_t$ -measurable.

This means that the events $\{\omega \in \Omega \mid x(t, \omega) \leq y\}$ can be expressed in terms of Brownian events up to time t . Thus an adapted process at time t does not depend on the future behavior of the Brownian trajectories from time t on: an adapted process is nonanticipatory. For example, for any deterministic integrable function $f(t)$ the process $x(t, \omega)$, whose trajectories are $x(t, \omega) = \int_0^t f(s)\omega(s) ds$, is adapted to the Brownian filtration. An adapted process, such as $x(t, \omega)$ above, can be viewed as the output of a causal filter operating on Brownian trajectories (which may represent a random signal).

Example 2.7 (First passage times). The first passage time (FPT) $\tau_y(\omega)$ of an adapted continuous process $x(t, \omega)$ to a given point y is a Markov time relative to

the Brownian filtration because it depends on the Brownian trajectories $\omega(t)$ up to time t . The relation of Markov times to Markov processes is discussed later. Thus, for $y > 0$ the random time $\tau_y(\omega) = \inf\{t \geq 0 \mid x(t, \omega) > y\}$ is the first passage time of $x(t, \omega)$ through the value y . To see that τ_y is a Markov time, we proceed in a manner similar to that of Example 2.4 above. The event $\{\omega \in \Omega \mid \tau_y(\omega) > t\}$ consists of all Brownian trajectories ω for which $x(t, \omega)$ stays below the level y for all times prior to t . Thus, due to the continuity of the paths of $x(t, \omega)$,

$$\{\omega \in \Omega \mid \tau_y(\omega) \leq t\} = \bigcup_{r \in \mathbb{Q}_t} \{\omega \in \Omega \mid x(r, \omega) \geq y\},$$

which is in $\mathcal{F}_t$ because $x(t, \omega)$ is an adapted process. $\square$

Example 2.8 (Last passage time). The last passage time of $x(t, \omega)$ to a given point y before time T , denoted $LPT(y, T, \omega)$, is not a Markov time relative to the Brownian filtration, because at time t it is not a Brownian event that depends on the Brownian trajectories up to time t . As mentioned in Example 2.5, it depends on events after time t , because the last passage may occur after that; that is, $\{\omega \in \Omega \mid LPT(y, T, \omega) \leq t\} \notin \mathcal{F}_t$ for all $0 \leq t < T$. $\square$

Example 2.9 (First exit time). The *first exit time* from an interval is a Markov time for the Brownian motion. It is defined for $a < 0 < b$ by

$$\tau_{[a,b]} = \inf \{t \geq 0 \mid w(t) < a \text{ or } w(t) > b\}.$$

$\square$

Exercise 2.2 (First exit time: continuation). Prove the claim of Example 2.9: show that $\tau_{[a,b]}$ is a Markov time for the Brownian motion. $\square$

There are two slightly different concepts of a *signed measure*, depending on whether one allows it to take infinite values (see, e.g., Wikipedia).

Definition 2.2.10 (Signed measure). A signed real-valued measure in the space $(\Omega, \mathcal{F})$ is a function

$$\mu : \mathcal{F} \rightarrow \mathbb{R} \tag{2.9}$$

such that for any sequence of disjoint set $\{A_n\}_{n=1}^\infty$ in $\mathcal{F}$,

$$\mu \left\{ \bigcup_{n=1}^\infty A_n \right\} = \sum_{n=1}^\infty \mu\{A_n\}.$$

Definition 2.2.11 (Real measure). If $\mu(A) \geq 0$ for all $A \in \mathcal{F}$, we say that it is a measure. The triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space.

Definition 2.2.12 (Null sets). A measurable set A such that $\mu(A) = 0$ is a μ -null set. Any property that holds for all $\omega \in \Omega$, except on a μ -null set is said to hold μ -almost everywhere (μ -a.e.) or for μ -almost all $\omega \in \Omega$.

We assume that all subsets of null sets are measurable (in measure theory this means that the measure space is *complete*). A measure is a monotone set function in the sense that if A and B are measurable sets such that $A \subset B$, then $\mu(A) \leq \mu(B)$. If $\mu(\Omega) < \infty$, we say that $\mu(A)$ is a *finite measure*. If Ω is a countable union of sets of finite measure, we say that Ω is a σ -*finite measure*.

Definition 2.2.13 (Integration with respect to a measure). A measure $\mu(A)$ defines an integral of a nonnegative measurable function $f(\omega)$ by

$$\int_{\Omega} f(\omega) d\mu(\omega) = \lim_{h \rightarrow 0} \lim_{N \rightarrow \infty} \sum_{n=0}^N nh\mu\{\omega : nh \leq f(\omega) \leq (n+1)h\}, \quad (2.10)$$

whenever the limit exists. In this case, we say that $f(\omega)$ is an integrable function.

For every measurable function $f(\omega)$ the nonnegative functions $f^+(\omega)$ and $f^-(\omega)$ are nonnegative measurable functions and $f(\omega) = f^+(\omega) - f^-(\omega)$. We say that $f(\omega)$ is an integrable function if both $f^+(\omega)$ and $f^-(\omega)$ are integrable and

$$\int_{\Omega} f(\omega) d\mu(\omega) = \int_{\Omega} f^+(\omega) d\mu(\omega) - \int_{\Omega} f^-(\omega) d\mu(\omega).$$

The function $f(\omega)$ is integrable if and only if $|f(\omega)|$ is integrable, because $|f(\omega)| = f^+(\omega) + f^-(\omega)$. For any set $A \in \mathcal{F}$ the indicator function $\mathbf{1}_A(\omega)$ (see Example 2.4) is integrable and $\int \mathbf{1}_A(\omega) d\mu(\omega) = \mu(A)$. We define an integral over a measurable set A by $\int_A f(\omega) d\mu(\omega) = \int_{\Omega} \mathbf{1}_A(\omega) f(\omega) d\mu(\omega)$. If $\int_A f(\omega) d\mu(\omega)$ exists, we say that $f(\omega)$ is *integrable in A* . In that case $f(\omega)$ is integrable in every measurable subset of A . If $\mu(A) = 0$ then $\int_A f(\omega) d\mu(\omega) = 0$. If $f(\omega)$ is an integrable function with respect to a measure $\mu(A)$, then the integral

$$\nu(A) = \int_A f(\omega) d\mu(\omega) \quad (2.11)$$

defines $\nu(A)$ as a signed measure in $\mathcal{F}$. Obviously, if $\mu(A) = 0$ then $\nu(A) = 0$.

Definition 2.2.14 (Differentiation of measures). If the measures ν and μ satisfy eq. (2.11), the function $f(\omega)$ is the Radon–Nikodym derivative of the measure ν with respect to the measure μ at the point ω and is denoted

$$f(\omega) = \frac{d\nu(\omega)}{d\mu(\omega)}. \quad (2.12)$$

Definition 2.2.15 (Absolute continuity). A signed measure ν is absolutely continuous with respect to the measure μ if μ -null sets are ν -null sets.

Thus the measure ν in (2.11) is absolutely continuous with respect to μ . If two measures are absolutely continuous with respect to each other, they are said to be *equivalent*.

Theorem 2.2.1 (Radon–Nikodym [58]). *If ν is a finite signed measure, absolutely continuous with respect to a σ -finite measure μ , then there exists a μ integrable function $f(\omega)$, uniquely defined up to μ -null sets, such that (2.11) holds for all $A \in \mathcal{F}$. For a constant c the inequality $\nu(A) \geq c\mu(A)$ for all $A \in \mathcal{F}$ implies $f(\omega) \geq c$ for μ -almost all $\omega \in \Omega$.*

The function $f(\omega)$ in the theorem is called the Radon–Nikodym derivative and is denoted as in (2.12).

Definition 2.2.16 (Probability measure). *A positive measure $\Pr$ such that $\Pr\{\Omega\} = 1$ is called a probability measure and the probability of an event $A \in \mathcal{F}$ is denoted $\Pr\{A\}$.*

Thus the probability of an event (a measurable set) is a number between 0 and 1. An event whose probability is 1 is called a *sure event*. The event Ω is a sure event. There are many ways for assigning probabilities to events, depending on the degree of uncertainty we have about a given event; different persons may assign different probabilities to the same events. We may think of the probability of an event as a measure of our uncertainty about it [53]. Recall that a measurable function on a probability space $(\Omega, \mathcal{F}, \Pr)$ is called a random variable, denoted $X(\omega)$.

Definition 2.2.17 (Expectations). *Integrals of random variables with respect to the probability measure $\Pr\{A\}$ are called expectations and are denoted*

$$\mathbb{E}X(\omega) = \int_{\Omega} X(\omega) d\Pr\{\omega\}. \quad (2.13)$$

For any set A in $\mathcal{F}$, we define

$$\begin{aligned} \mathbb{E}\{X, A\} &= \int_A X(\omega) d\Pr(\omega) \\ &= \int_{-\infty}^{\infty} x \Pr\{\omega \in A : x \leq X(\omega) \leq x + dx\}. \end{aligned} \quad (2.14)$$

Applications of integration with respect to $\Pr$ are given in the next section.

Definition 2.2.18 (PDF and pdf). *For an integrable random variable $X(\omega)$ the function*

$$F_X(x) = \Pr\{\omega : X(\omega) \leq x\}$$

is called the probability distribution function (PDF) of $X(\omega)$. The function (or generalized function [152])

$$f_X(x) = \frac{d}{dx} F_X(x)$$

is called the probability density function (pdf) of $X(\omega)$. The expectation $\mathbb{E}X(\omega)$ can be written as

$$\mathbb{E}X(\omega) = \int x dF_X(x) = \int x f_X(x) dx. \quad (2.15)$$

Note that the PDF $F_X(x)$ need not be differentiable, so the pdf $f_X(x)$ need not be a function, but rather a generalized function (a distribution).

Exercise 2.3 (Coin tossing). Construct a probability space on $\mathbb{R}$, a random variable, PDF, and pdf for the experiment of tossing a fair coin. $\square$

Definition 2.2.19 (Conditional expectations). For a sub σ -field $\mathcal{F}_1$ of $\mathcal{F}$ and a random variable $X(\omega)$ with finite variance on $\mathcal{F}$, the conditional expectation $\mathbb{E}(X | \mathcal{F}_1)$ is a random variable measurable with respect to $\mathcal{F}_1$ such that for every random variable $Y(\omega)$ measurable with respect to $\mathcal{F}_1$, whose variance is finite

$$\mathbb{E}X(\omega)Y(\omega) = \mathbb{E}[\mathbb{E}(X | \mathcal{F}_1)Y(\omega)]. \quad (2.16)$$

If we confine the functions $Y(\omega)$ to indicators of events $A \in \mathcal{F}_1$, then (2.16) implies that $\mathbb{E}(X | \mathcal{F}_1)$ is a random variable that satisfies

$$\int_A X(\omega) d\Pr(\omega) = \int_A \mathbb{E}(X | \mathcal{F}_1) d\Pr(\omega) \quad (2.17)$$

for all $A \in \mathcal{F}_1$. To determine the existence of the conditional expectation, we denote the left-hand side of eq. (2.17) $\nu(A)$ and recall that $\nu(A)$ is a signed measure on $\mathcal{F}_1$. It follows from the Radon–Nikodym theorem that the desired function is simply

$$\mathbb{E}(X | \mathcal{F}_1) = \frac{d\nu(\omega)}{d\mu(\omega)}. \quad (2.18)$$

Definition 2.2.20 (Conditional probabilities, distributions, and densities). For any event $M \in \mathcal{F}$ the conditional probability of M , given $\mathcal{F}_1$, is defined as

$$\Pr\{M | \mathcal{F}_1\} = \mathbb{E}[\mathbf{1}_M | \mathcal{F}_1].$$

The conditional probability distribution function (CPDF) of the random variable $X(\omega)$, given $\mathcal{F}_1$, is defined as

$$F_{X|\mathcal{F}_1}(x, \omega) = \Pr\{\omega : X(\omega) \leq x | \mathcal{F}_1\}.$$

Note that for every ω the CPDF $F_{X|\mathcal{F}_1}(x, \omega)$ is a probability distribution function. Its density,

$$f_{X|\mathcal{F}_1}(x, \omega) = \frac{d}{dx} F_{X|\mathcal{F}_1}(x, \omega),$$

is also a random function.

2.2.1 The Wiener measure of Brownian trajectories

Having constructed the set $\mathcal{F}$ of events for the Brownian trajectories, we proceed to construct a probability measure of these events. The probability measure is used to construct a mathematical theory of the Brownian motion that can describe experiments.

A probability measure $\Pr$ can be defined on Ω (i.e., on the events $\mathcal{F}$ in Ω) to conform with the Einstein–Langevin description of the Brownian motion. It is enough to define the probability measure $\Pr\{\cdot\}$ on cylinder sets and then to extend it to all events in $\mathcal{F}$ by the elementary properties of a probability measure (see [106] for a more detailed exposition of this topic). The following probability measure in $\mathcal{F}$ is called *the Wiener measure* [248]. Consider the cylinder $C(t; I)$, where $t \geq 0$ and $I = (a, b)$ and set

$$\Pr\{C(t; I)\} = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{-x^2/2t} dx. \quad (2.19)$$

If $0 = t_0 < t_1 < t_2 < \dots < t_n$ and I_k ($k = 1, 2, \dots, n$) are real intervals, set

$$\begin{aligned} & \Pr\{C(t_1, t_2, \dots, t_n; I_1, I_2, \dots, I_n)\} \\ &= \int_{I_1} \int_{I_2} \dots \int_{I_n} \prod_{k=1}^n \frac{dx_k}{\sqrt{2\pi(t_k - t_{k-1})}} \exp\left\{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}\right\}, \end{aligned} \quad (2.20)$$

where $x_0 = 0$ (the extension of the Wiener measure from cylinders to $\mathcal{F}$ is described in [106, 208]). The integral (2.20) is called *Wiener's discrete path integral*. The obvious features of the Wiener measure that follow from eqs. (2.4) and (2.20) are

$$\Pr\{\Omega\} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-x^2/2t} dx = 1$$

and for $t_1 < t$,

$$\begin{aligned} & \Pr\{C(t_1, t; I_1, \mathbb{R})\} \\ &= \frac{1}{2\pi\sqrt{(t - t_1)t_1}} \int_{I_1} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x - x_1)^2}{2(t - t_1)}\right\} \exp\left\{-\frac{x_1^2}{2t_1}\right\} dx dx_1 \\ &= \frac{1}{\sqrt{2\pi t_1}} \int_{I_1} \exp\left\{-\frac{x_1^2}{2t_1}\right\} dx_1 = \Pr\{C(t_1; I_1)\}. \end{aligned}$$

The Wiener measure (2.20) of a cylinder is the probability of sampling points of a trajectory in the cylinder by the simulation

$$x(t_k) = x(t_{k-1}) + \Delta w(t_k), \quad k = 1, \dots, n, \quad (2.21)$$

where t_k are ordered as above, and $\Delta w(t_k) \sim \mathcal{N}(0, t_k - t_{k-1})$ are independent normal variables. The vertices of the trajectories in Figures 2.1 and 2.2 were sampled according to (2.21) and interpolated linearly.

The axiomatic definition of the Brownian motion, consistent with the Einstein–Langevin theory is as follows

Definition 2.2.21 (The MBM). A real-valued stochastic process $w(t, \omega)$ defined on $\mathbb{R}_+ \times \Omega$ is a mathematical Brownian motion if

(1) $w(0, \omega) = 0$ w.p. 1

(2) $w(t, \omega)$ is almost surely a continuous function of t

(3) For every $t, s \geq 0$ the increment $\Delta w(s, \omega) = w(t+s, \omega) - w(t, \omega)$ is independent of $\mathcal{F}_t$, and is a zero mean Gaussian random variable with variance

$$\mathbb{E} |\Delta w(s)|^2 = s. \quad (2.22)$$

The mathematical Brownian motion, defined by axioms (1)–(3), and its velocity process are mathematical idealizations of physical processes. This idealization may lead to unexpected and counterintuitive consequences.

There are other equivalent definitions of the MBM (see Wikipedia). According to Definition 2.2.21, the cylinders (2.3) are identical to the cylinders

$$C(t_1, \dots, t_n; I_1, \dots, I_n) = \{\omega \in \Omega \mid w(t_k, \omega) \in I_k, \text{ for all } 1 \leq k \leq n\}. \quad (2.23)$$

To understand the conceptual difference between the definitions (2.3) and (2.23), we note that in (2.3) the cylinder is defined directly in terms of elementary events whereas in (2.23) the cylinder is defined in terms of a stochastic process. It is coincidental that such two different definitions produce the same cylinder. In Section 5.6 below cylinders are defined in terms of other stochastic processes, as in (2.23). It should be borne in mind, however, that the extension of the Wiener measure from cylinders to $\mathcal{F}$ is not straightforward [106].

The expectation (2.22) is meant in the sense of the definition (2.13). Properties (1)–(3) are axioms that define the Brownian motion as a mathematical entity. It has to be shown that a stochastic process satisfying these axioms actually exists. Before showing constructions of the MBM (see Section 2.3), we can derive some of its properties in a straightforward manner.

First, we note that by eq. (2.19) the PDF of the MBM is

$$\begin{aligned} F_w(x, t) &= \Pr\{\omega \in \Omega \mid w(t, \omega) \leq x\} = \Pr\{C(t, I^x)\} \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^x e^{-y^2/2t} dy \end{aligned} \quad (2.24)$$

and the pdf is

$$f_w(x, t) = \frac{\partial}{\partial x} F_w(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}. \quad (2.25)$$

It is well-known (and easily verified) that $f_w(x, t)$ is the solution of the initial value problem for the diffusion equation

$$\frac{\partial f_w(x, t)}{\partial t} = \frac{1}{2} \frac{\partial^2 f_w(x, t)}{\partial x^2}, \quad \lim_{t \downarrow 0} f_w(x, t) = \delta(x). \quad (2.26)$$

Second, we note that (1) and (2) are not contradictory, despite the fact that not all continuous functions vanish at time $t = 0$. Property (1) asserts that all trajectories of the Brownian motion that do not start at the origin are assigned probability 0. More specifically,

Theorem 2.2.2. *The Wiener measure has property (1).*

Proof. The set $\{\omega \in \Omega \mid w(0, \omega) = 0\}$ is in $\mathcal{F}$, because it can be represented as a countable intersection of cylinders. Indeed, consider two sequences t_k and ε_n that decrease to zero and define the cylinders

$$C(t_k; [-\varepsilon_n, \varepsilon_n]) = \{\omega \in \Omega \mid |w(t_k, \omega)| < \varepsilon_n\}.$$

Then

$$\{\omega \in \Omega \mid w(0, \omega) = 0\} = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} C(t_k; [-\varepsilon_n, \varepsilon_n]). \quad (2.27)$$

It follows from probability theory that

$$\begin{aligned} \Pr\{\omega \in \Omega \mid w(0, \omega) = 0\} &= \Pr\left\{\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} C(t_k; [-\varepsilon_n, \varepsilon_n])\right\} \\ &= \lim_{n \rightarrow \infty} \Pr\left\{\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} C(t_k; [-\varepsilon_n, \varepsilon_n])\right\} \\ &\geq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \Pr\left\{\bigcap_{k=m}^{\infty} C(t_k; [-\varepsilon_n, \varepsilon_n])\right\} \\ &= \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \Pr\{C(t_k; [-\varepsilon_n, \varepsilon_n])\}. \end{aligned} \quad (2.28)$$

Now, the Wiener measure (2.20) of a cylinder $C(t; [-\varepsilon, \varepsilon])$ is

$$\Pr\{C(t; [-\varepsilon, \varepsilon])\} = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx = \frac{2}{\sqrt{2\pi}} \int_0^{\varepsilon/\sqrt{t}} e^{-x^2/2} dx, \quad (2.29)$$

so that eqs.(2.28) and (2.29) give

$$\Pr\{\omega \in \Omega \mid w(0, \omega) = 0\} = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{2}{\sqrt{2\pi}} \int_0^{\varepsilon_n/\sqrt{t_k}} e^{-x^2/2} dx = 1. \quad (2.30)$$

This completes the proof of the assertion (see Figure 2.2). □

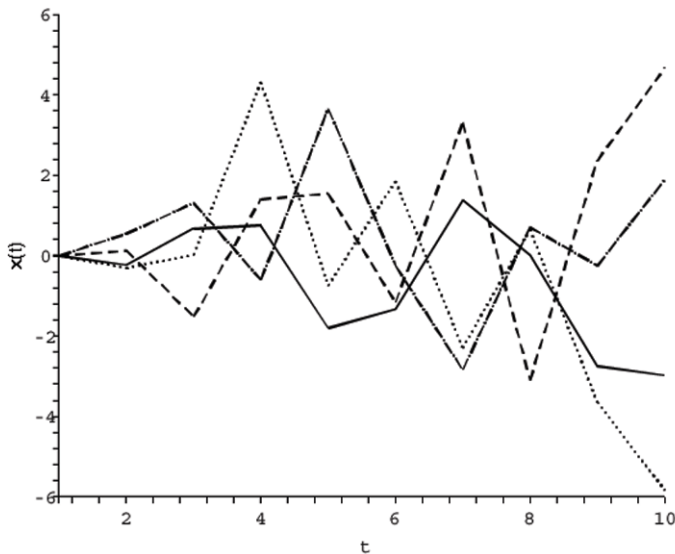


Figure 2.2. Three Brownian trajectories sampled at discrete times according to the Wiener measure $\Pr_0\{\cdot\}$ by the scheme (2.21).

The initial point

In view of the above, $x_0 = 0$ in the definition (2.20) of the Wiener measure of a cylinder means that the Brownian paths are those continuous functions that take the value 0 at time 0. That is, the Brownian paths are conditioned on starting at time $t = 0$ at the point $x_0 = w(0, \omega) = 0$. To emphasize this point, we modify the notation of the Wiener measure to $\Pr_0\{\cdot\}$. If, in eq. (2.20), this condition is replaced with $x_0 = x$, the above proof of Theorem 2.2.2 shows that $\Pr_x\{w(0, \omega) = x\} = 1$ under the modified Wiener measure, now denoted $\Pr_x\{\cdot\}$,

Thus conditioning reassigns probabilities to the Brownian paths; the set of trajectories $\{w(0, \omega) = x\}$, which was assigned the probability 0 under the measure $\Pr_0\{\cdot\}$, is now assigned the probability 1 under the measure $\Pr_x\{\cdot\}$.

Similarly, replacing the condition $t_0 = 0$ with $t_0 = s$ and setting $x_0 = x$ in eq. (2.20) shifts the Wiener measure, now denoted $\Pr_{x,s}$, so that

$$\Pr_{x,s}\{C(t; [a, b])\} = \Pr_0\{C(t - s; [a - x, b - x])\}. \quad (2.31)$$

This means that for all positive t the increment $\Delta w(s, \omega) = w(t + s, \omega) - w(t, \omega)$, as a function of s , is a MBM so that the probabilities of any Brownian event of $\Delta w(s, \omega)$ are independent of t . This property is stated in the form

(4) *The increments of the MBM are stationary.*

The above argument shows that equation (2.22) in property (3) of the MBM follows from the definition of the Wiener measure.

Theorem 2.2.3. *The Wiener measure has property (3) of Definition 2.2.21 (i.e., the increment $\Delta w(s, \omega) = w(t + s, \omega) - w(t, \omega)$ is independent of the Brownian events $\mathcal{F}_t$) and is a zero mean Gaussian variable with variance given by (2.22).*

Proof. We have to show that the joint PDF of $\Delta w(s, \omega) = w(t + s, \omega) - w(t, \omega)$ and any event $A \in \mathcal{F}_t$ is a product of the PDF of $\Delta w(s, \omega)$ and $\Pr\{A\}$. The same is true for any event $A \in \mathcal{F}_t$, because any cylinder $C \in \mathcal{F}_t$ is generated by cylinders of the form $C(s, I^x) = \{\omega \in \Omega \mid w(s, \omega) \leq x\}$ with $0 \leq s \leq t$. It suffices therefore to show the independence of the increment $\Delta w(s, \omega)$ and $w(u, \omega)$ for all $s \geq 0$ and $u \leq t$.

To show this, recall that by definition the joint PDF of $w(t_1, \omega)$, $w(t_2, \omega)$, and $w(t_3, \omega)$ for $0 < t_1 < t_2 < t_3$ is the Wiener measure of the cylinder $C = C(t_1, t_2, t_3; I^x, I^y, I^z)$. According to the definition (2.20) of the Wiener measure,

$$\begin{aligned} F_{w(t_1), w(t_2), w(t_3)}(x, y, z) &= \Pr_0(C(t_1, t_2, t_3; I^x, I^y, I^z)) \\ &= \int_{-\infty}^x d\xi \int_{-\infty}^y d\eta \int_{-\infty}^z d\zeta \frac{1}{\sqrt{(2\pi)^3 t_1 (t_2 - t_1) (t_3 - t_2)}} \\ &\quad \times \exp \left\{ -\frac{\xi^2}{2t_1} - \frac{(\eta - \xi)^2}{2(t_2 - t_1)} - \frac{(\zeta - \eta)^2}{2(t_3 - t_2)} \right\}. \end{aligned} \quad (2.32)$$

It follows that the joint pdf of $w(t_1, \omega)$, $w(t_2, \omega)$, and $w(t_3, \omega)$ is given by

$$\begin{aligned} f_{w(t_1), w(t_2), w(t_3)}(\xi, \eta, \zeta) &= \frac{\partial^3}{\partial \xi \partial \eta \partial \zeta} F_{w(t_1), w(t_2), w(t_3)}(\xi, \eta, \zeta) \\ &= \frac{1}{\sqrt{(2\pi)^3 t_1 (t_2 - t_1) (t_3 - t_2)}} \\ &\quad \times \exp \left\{ -\frac{\xi^2}{2t_1} - \frac{(\eta - \xi)^2}{2(t_2 - t_1)} - \frac{(\zeta - \eta)^2}{2(t_3 - t_2)} \right\}. \end{aligned}$$

Now, for any x and y

$$\begin{aligned} &\Pr\{\omega \in \Omega \mid w(t_3, \omega) - w(t_2, \omega) < x, w(t_1, \omega) < y\} \\ &= \int_{-\infty}^y d\xi \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{x+\eta} d\zeta f_{w(t_1), w(t_2), w(t_3)}(\xi, \eta, \zeta) \\ &= \int_{-\infty}^y d\xi \int_{\zeta-x}^{\infty} d\eta \int_{-\infty}^{\infty} d\zeta \frac{1}{\sqrt{(2\pi)^3 t_1 (t_2 - t_1) (t_3 - t_2)}} \\ &\quad \times \exp \left\{ -\frac{\xi^2}{2t_1} - \frac{(\eta - \xi)^2}{2(t_2 - t_1)} - \frac{(\zeta - \eta)^2}{2(t_3 - t_2)} \right\}. \end{aligned} \quad (2.33)$$

Substituting $\eta = \zeta - z$ and noting that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} \exp\left\{-\frac{(\zeta - z - \xi)^2}{2(t_2 - t_1)}\right\} d\zeta = 1, \quad (2.34)$$

we obtain from eq. (2.33) that

$$\begin{aligned} & \Pr\{\omega \in \Omega \mid w(t_3, \omega) - w(t_2, \omega) < x, w(t_1, \omega) < y\} \\ &= \frac{1}{\sqrt{2\pi t_1}} \int_{-\infty}^y \exp\left\{-\frac{\xi^2}{2t_1}\right\} d\xi \frac{1}{\sqrt{2\pi(t_3 - t_2)}} \int_{-\infty}^x \exp\left\{-\frac{z^2}{2(t_3 - t_2)}\right\} dz \\ &= \Pr\{\omega \in \Omega \mid w(t_1, \omega) < y\} \Pr\{\omega \in \Omega \mid w(t_3, \omega) - w(t_2, \omega) < x\}. \end{aligned} \quad (2.35)$$

Equation (2.35) means that $w(t_3, \omega) - w(t_2, \omega)$ and $w(t_1, \omega)$ are independent, as stated in property (3) of the MBM.

Next, we calculate the moments of the MBM according to the definition (2.13). Using (2.14), we find from (2.25) that

$$\begin{aligned} \mathbb{E}w(t, \omega) &= \int_{-\infty}^{\infty} x \Pr\{\omega \in \Omega \mid x \leq w(t, \omega) \leq x + dx\} \\ &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi t}} e^{-x^2/2t} dx = 0. \end{aligned}$$

Similarly, $\mathbb{E}w^2(t, \omega) = (2\pi t)^{-1/2} \int_{-\infty}^{\infty} x^2 e^{-x^2/2t} dx = t$. Now, property eq. (2.22) follows from the independence of the increments of the MBM. $\square$

We recall that the autocorrelation function of a stochastic process $x(t, \omega)$ is defined as the expectation $R_x(t, s) = \mathbb{E}x(t, \omega)x(s, \omega)$. Using the notation $t \wedge s = \min\{t, s\}$, we have the following

Theorem 2.2.4 (Property (5)). *The autocorrelation function of $w(t, \omega)$ is*

$$Ew(t, \omega)w(s, \omega) = t \wedge s. \quad (2.36)$$

Proof. Assuming that $t \geq s \geq 0$ and using property (3), we find that

$$\begin{aligned} Ew(t, \omega)w(s, \omega) &= E[w(t, \omega) - w(s, \omega)][w(s, \omega) - w(0, \omega)] + Ew(s, \omega)w(s, \omega) \\ &= s = t \wedge s. \end{aligned}$$

$\square$

2.2.2 The MBM in $\mathbb{R}^d$

If $w_1(t), w_2(t), \dots, w_d(t)$ are independent Brownian motions, the vector process

$$\mathbf{w}(t) = (w_1(t), w_2(t), \dots, w_d(t))^T$$

is defined as the *d-dimensional Brownian motion*. The probability space Ω for the *d*-dimensional Brownian motion consists of all $\mathbb{R}^d$ -valued continuous functions of t . Thus

$$\boldsymbol{\omega}(t) = (\omega_1(t), \omega_2(t), \dots, \omega_d(t))^T,$$

where $\omega_j(t) \in \Omega$. Cylinder sets are defined by the following

Definition 2.2.22 (Cylinder sets in $\mathbb{R}^d$). A cylinder set of *d*-dimensional Brownian trajectories is defined by times $0 \leq t_1 < t_2 < \dots < t_k$ and open sets $\mathbf{I}_k$, ($k = 1, 2, \dots, k$) as

$$C(t_1, \dots, t_k; \mathbf{I}_1, \dots, \mathbf{I}_k) = \{\boldsymbol{\omega} \in \Omega \mid \boldsymbol{\omega}(t_j) \in \mathbf{I}_j \text{ for } j = 1, \dots, k\}. \quad (2.37)$$

The open sets $\mathbf{I}_j$ can be, for example, open boxes or balls in $\mathbb{R}^d$. In particular, we write $\mathbf{I}_{\mathbf{x}} = \{\boldsymbol{\omega} \leq \mathbf{x}\} = \{\omega_1 \leq x_1, \dots, \omega_d \leq x_d\}$. The appropriate σ -algebra $\mathcal{F}$ and the filtration $\mathcal{F}_t$ are constructed as in Section 2.2.

Definition 2.2.23 (The Wiener measure for the *d*-dimensional MBM). The *d*-dimensional Wiener measure of a cylinder is defined as

$$\begin{aligned} & \Pr\{C(t_1, t_2, \dots, t_k; \mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_k)\} \\ &= \int_{\mathbf{I}_1} \int_{\mathbf{I}_2} \dots \int_{\mathbf{I}_k} \prod_{j=1}^k \frac{d\mathbf{x}_j}{[2\pi(t_j - t_{j-1})]^{n/2}} \exp\left\{-\frac{|\mathbf{x}_j - \mathbf{x}_{j-1}|^2}{2(t_j - t_{j-1})}\right\}. \end{aligned} \quad (2.38)$$

The PDF of the *d*-dimensional MBM is

$$\begin{aligned} F_{\mathbf{w}}(\mathbf{x}, t) &= \Pr\{\boldsymbol{\omega} \in \Omega \mid \mathbf{w}(t, \boldsymbol{\omega}) \leq \mathbf{x}\} \\ &= \frac{1}{(2\pi t)^{n/2}} \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_d} e^{-|\mathbf{y}|^2/2t} dy_1 \dots dy_d \end{aligned} \quad (2.39)$$

and the pdf is

$$f_{\mathbf{w}}(\mathbf{x}, t) = \frac{\partial^d F_{\mathbf{w}}(\mathbf{x}, t)}{\partial x_1 \partial x_2 \dots \partial x_d} = \frac{1}{(2\pi t)^{n/2}} e^{-|\mathbf{x}|^2/2t}. \quad (2.40)$$

Equations (2.26) imply that $f_{\mathbf{w}}(\mathbf{x}, t)$ satisfies the d -dimensional diffusion equation and the initial condition

$$\frac{\partial f_{\mathbf{w}}(\mathbf{x}, t)}{\partial t} = \frac{1}{2} \Delta f_{\mathbf{w}}(\mathbf{x}, t), \quad \lim_{t \downarrow 0} f_{\mathbf{w}}(\mathbf{x}, t) = \delta(\mathbf{x}). \quad (2.41)$$

It can be seen from eq. (2.38) that any rotation of the d -dimensional Brownian motion is d -dimensional Brownian motion.

Higher-dimensional stochastic processes are defined by the following

Definition 2.2.24 (Vector valued processes). A vector-valued function $\mathbf{x}(t, \omega) : \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}^d$ is called a stochastic process in $(\Omega, \mathcal{F})$ with continuous trajectories if

- (i) $\mathbf{x}(t, \omega)$ is a continuous function of t for every $\omega \in \Omega$,
- (ii) for every $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^d$ the sets $\{\omega \in \Omega \mid \mathbf{x}(t, \omega) \leq \mathbf{x}\}$ are Brownian events; that is, if $\{\omega \in \Omega \mid \mathbf{x}(t, \omega) \leq \mathbf{x}\} \in \mathcal{F}$.

Note that the dimension of the elementary events $\omega(\cdot) : \mathbb{R}_+ \mapsto \mathbb{R}^n$ and the dimension of the space in which the trajectories $\mathbf{x}(t, \omega)$ move, d , are not necessarily the same. As above, when $\{\omega \in \Omega \mid \mathbf{x}(t, \omega) \leq \mathbf{x}\} \in \mathcal{F}$, we say that the process $\mathbf{x}(t, \omega)$ is $\mathcal{F}$ -measurable. The PDF of $\mathbf{x}(t, \omega)$ is defined as

$$F_{\mathbf{x}}(\mathbf{y}, t) = \Pr\{\omega \in \Omega \mid \mathbf{x}(t, \omega) \leq \mathbf{y}\} \quad (2.42)$$

and the pdf is defined as

$$f_{\mathbf{x}}(\mathbf{y}, t) = \frac{\partial^d F_{\mathbf{x}}(\mathbf{y}, t)}{\partial y^1 \partial y^2 \dots \partial y^d}. \quad (2.43)$$

The expectation of a matrix-valued function $\mathbf{g}(\mathbf{x})$ of a vector-valued process $\mathbf{x}(t, \omega)$ is the matrix

$$\mathbb{E} \mathbf{g}(\mathbf{x}(t, \omega)) = \int \mathbf{g}(\mathbf{y}) f_{\mathbf{x}}(\mathbf{y}, t) d\mathbf{y}. \quad (2.44)$$

Definition 2.2.25 (Autocorrelation and autocovariance). The autocorrelation matrix of $\mathbf{x}(t, \omega)$ is defined as the $n \times n$ matrix

$$\mathbf{R}_{\mathbf{x}}(t, s) = \mathbb{E} \mathbf{x}(t) \mathbf{x}^T(s) \quad (2.45)$$

and the autocovariance matrix is defined as

$$\mathbf{Cov}_{\mathbf{x}}(t, s) = \mathbb{E} [\mathbf{x}(t) - \mathbb{E} \mathbf{x}(t)] [\mathbf{x}(s) - \mathbb{E} \mathbf{x}(s)]^T. \quad (2.46)$$

The autocovariance matrix of the d -dimensional Brownian motion is found from (2.36) as

$$\mathbf{Cov}_{\mathbf{w}}(t, s) = \mathbf{I}(t \wedge s), \quad (2.47)$$

where $\mathbf{I}$ is the identity matrix.

Exercise 2.4 (Transformations preserving the MBM). Show, by verifying properties (1)–(3), that the following processes are Brownian motions:

(i) $w_1(t) = w(t+s) - w(s)$

(ii) $w_2(t) = cw(t/c^2)$, where c is any positive constant

(iii) $w_3(t) = tw(1/t)$,

(iv) $w_4(t) = w(T) - w(T-t)$ for $0 \leq t \leq T$,

(v) $w_5(t) = -w(t)$. □

Exercise 2.5 (Changing scale). Give necessary and sufficient conditions on the functions $f(t)$ and $g(t)$ such that the process $w_4(t) = f(t)w(g(t))$ is MBM. □

Exercise 2.6 (The joint pdf of the increments). Define

$$\Delta \mathbf{w} = (\Delta w(t_1), \Delta w(t_2), \dots, \Delta w(t_n))^T.$$

Find the joint pdf of $\Delta \mathbf{w}$. □

Exercise 2.7 (The radial MBM). Define the *radial MBM* by $y(t) = |\mathbf{w}(t)|$, where $\mathbf{w}(t)$ is the d -dimensional MBM. Find the pdf of $y(t)$, the partial differential equation, and the initial condition it satisfies. □

2.3 Constructions of the MBM

Two mathematical problems arise with the axiomatic definition of the Brownian motion. One is the question of existence, or construction of such a process and the other is of computer simulations of Brownian trajectories with different refinements of time discretization. The first proof of existence and a mathematical construction of the Brownian motion is due to Paley and Wiener [197] and is presented in Section 2.3.1. The second construction, due to P. Lévy [150], and the method of refinement of computer simulations of the Brownian paths are presented in Section 2.3.2.

2.3.1 The Paley–Wiener construction of the Brownian motion

Assume that X_k, Y_k ($k = 0, \pm 1, \pm 2, \dots$) are zero mean and unit variance Gaussian i.i.d. random variables defined in a probability space $\tilde{\Omega}$. The probability space $\tilde{\Omega}$ can be chosen, for example, as the interval $[0, 1]$ or $\mathbb{R}$ (see [106]). The variable $Z_k = (X_k + iY_k)/\sqrt{2}$ is called a *complex Gaussian variable*. The simplest properties of Z_k are $\mathbb{E}Z_k Z_l = 0$ and $\mathbb{E}Z_k \bar{Z}_l = \delta_{kl}$. The series

$$\tilde{Z}_1(t) = tZ_0 + \sum_{n \neq 0} \frac{Z_n (e^{int} - 1)}{in} \quad (2.48)$$

converges in $L^2([0, 2\pi] \times \Omega)$, because the coefficients are $O(n^{-1})$. Each trajectory of $\tilde{Z}_1(t)$ is obtained from an infinite sequence of the numbers Z_k that are drawn independently from a standard Gaussian distribution. That is, every trajectory of $\tilde{Z}_1(t)$

is obtained from a realization of the infinite sequence of the random variables Z_k . Every continuous function has a unique Fourier series representation in $L^2[0, 2\pi]$, however not all Fourier series that converge in the $L^2[0, 2\pi]$ sense are continuous functions. Thus the space Ω of continuous functions is a subset of all the possible realizations of $\tilde{Z}_1(t)$ obtained from realizations of infinite sequences of the independent Gaussian random variables Z_k . It follows that $\tilde{Z}_1(t)$ may be a stochastic process on $L^2[0, 2\pi]$ rather than on Ω . Note that (2.48) embeds infinite-dimensional Gaussian vectors $\{Z_k\}_{k=1}^\infty$ in $L^2[0, 2\pi]$.

We set $Z(t) = \tilde{Z}_1(t)/\sqrt{2\pi}$ and note that $Z(t)$ satisfies the properties (1), (3)–(5) of the Brownian motion. Indeed, $Z(t)$ is obviously a zero mean Gaussian process with independent increments on the probability space $\tilde{\Omega}$. The autocorrelation function is calculated from

$$\mathbb{E}\tilde{Z}_1(t)\overline{\tilde{Z}_1(s)} = ts + \sum_{n \neq 0} \frac{(e^{int} - 1)(e^{-ins} - 1)}{n^2}.$$

It is well-known from Fourier series theory that

$$\sum_{n \neq 0} \frac{(e^{int} - 1)(e^{-ins} - 1)}{n^2} = \begin{cases} s(2\pi - t) & \text{for } 0 \leq s \leq t \leq 2\pi \\ t(2\pi - s) & \text{for } 0 \leq t \leq s \leq 2\pi. \end{cases} \quad (2.49)$$

Exercise 2.8 (Proof of (2.49)). Prove eq. (2.49). □

It follows that

$$\mathbb{E}\tilde{Z}_1(t)\overline{\tilde{Z}_1(s)} = \begin{cases} st + s(2\pi - t) = 2\pi s & \text{for } 0 \leq s \leq t \leq 2\pi \\ ts + t(2\pi - s) = 2\pi t & \text{for } 0 \leq t \leq s \leq 2\pi \end{cases} = 2\pi(t \wedge s).$$

Separating the complex process into its real and imaginary parts, we get two independent Brownian motions.

Now that $Z(t)$ has been shown to satisfy all the requirements of the definition of a Brownian motion, but the continuity property, it remains to show that almost all its paths are continuous. To show that the paths of $Z(t)$ are continuous, the almost sure (in $\tilde{\Omega}$) uniform convergence of the series (2.48) has to be demonstrated. Once this is done, the space of realizations of the infinite sequence Z_k can be identified with the space Ω of continuous functions through the one-to-one correspondence (2.48). Thus, we write $Z(t, \omega)$ to denote any realization of the path. For any ω denote

$$Z_{m,n}(t, \omega) = \sum_{k=m+1}^n \frac{Z_k e^{ikt}}{ik} \quad \text{for } n > m. \quad (2.50)$$

Theorem 2.3.1 (Paley–Wiener). *The sum $\sum_{n \neq 0} Z_{2^n, 2^{n+1}}(t)$ converges uniformly for $t \in \mathbb{R}$ to a Brownian motion, except possibly on a set of probability 0 in $\tilde{\Omega}$.*

Proof. According to eq. (2.50),

$$\begin{aligned} |Z_{m,n}(t, \omega)|^2 &= \sum_{k=m+1}^n \frac{|Z_k(\omega)|^2}{k^2} + 2\operatorname{Re} \left\{ \sum_{j=1}^{n-m-1} e^{ijt} \sum_{k=m+1+j}^n \frac{Z_k \bar{Z}_{k-j}}{k(k-j)} \right\} \\ &\leq \sum_{k=m+1}^n \frac{|Z_k(\omega)|^2}{k^2} + 2 \sum_{j=1}^{n-m-1} \left| \sum_{k=m+1+j}^n \frac{Z_k \bar{Z}_{k-j}}{k(k-j)} \right|. \end{aligned}$$

Setting $T_{m,n}(\omega) = \max_{0 \leq t \leq 2\pi} |Z_{m,n}(t, \omega)|$, we get

$$\mathbb{E} T_{m,n}^2 \leq \sum_{k=m+1}^n \frac{1}{k^2} + 2 \sum_{j=1}^{n-m-1} \mathbb{E} \left| \sum_{k=m+1+j}^n \frac{Z_k \bar{Z}_{k-j}}{k(k-j)} \right|.$$

Using the Cauchy-Schwarz inequality $\mathbb{E} |\Sigma| \leq \left(\mathbb{E} |\Sigma|^2 \right)^{1/2}$, we obtain the inequality

$$\begin{aligned} &\sum_{j=1}^{n-m-1} \left(\mathbb{E} \left| \sum_{k=m+1+j}^n \frac{Z_k \bar{Z}_{k-j}}{k(k-j)} \right|^2 \right)^{1/2} \\ &\leq \mathbb{E} \sum_{k=m+1+j}^n \frac{|Z_k|^2 |Z_{k-j}|^2}{k^2 (k-j)^2} + 2\operatorname{Re} \sum_{m+1+j < l < k \leq n} \mathbb{E} \frac{Z_k \bar{Z}_{k-j} Z_l \bar{Z}_{l-j}}{k(k-j)l(l-j)} \\ &= \sum_{k=m+1+j}^n \frac{1}{k^2 (k-j)^2}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E} T_{m,n}^2 &\leq \sum_{k=m+1}^n \frac{1}{k^2} + 2 \sum_{j=1}^{n-m-1} \left\{ \sum_{k=m+1+j}^n \frac{1}{k^2 (k-j)^2} \right\}^{1/2} \\ &\leq \frac{n-m}{m^2} + 2(n-m) \left(\frac{n-m}{m^4} \right)^{1/2}. \end{aligned}$$

Now, we choose $n = 2m$ and apply the Cauchy-Schwarz inequality again to get

$$\mathbb{E} \left(\max_t |Z_{m,2m}(t, \omega)| \right) \leq (\mathbb{E} T_{m,2m}^2)^{1/2} \leq \sqrt{\frac{1}{m} + \frac{2}{\sqrt{m}}} \leq 2m^{-1/4}.$$

It follows that

$$\sum_{n=1}^{\infty} \mathbb{E} \left(\max_t |Z_{2^n, 2^{n+1}}(t, \omega)| \right) \leq 2 \sum_{n=1}^{\infty} 2^{-n/4} < \infty.$$

Lebesgue's monotone convergence theorem [183] asserts that if the nonnegative functions f_n are integrable such that $\int \sum_{n=1}^m f_n(x) dx \rightarrow \text{limit as } m \rightarrow \infty$, then $\sum_{n=1}^m f_n(x) \rightarrow \text{limit a.e.}$ It follows that the sum $\sum_{n \neq 0} Z_{2^n, 2^{n+1}}(t)$ converges uniformly in t for almost all ω . $\square$

2.3.2 P. Lévy's method and refinements

P. Lévy [150] proposed a construction of the Brownian motion that is particularly useful in computer simulations of Brownian trajectories. The Brownian paths are constructed by a process that mimics the sampling of the Brownian path on different time scales, beginning with a coarse scale through consecutive refinements.

Consider a sequence of standard Gaussian i.i.d. random variables $\{Y_k\}$, for $k = 0, 1, \dots$ defined in $\tilde{\Omega}$. We denote by ω any realization of the infinite sequence $\{Y_k\}$ and construct a continuous path corresponding to this realization. We consider a sequence of binary partitions of the unit interval,

$$T_1 = \{0, 1\}, \quad T_2 = \left\{0, \frac{1}{2}, 1\right\}, \quad T_3 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\} \dots, \\ T_{n+1} = \left\{\frac{k}{2^n}, k = 0, 1, \dots, 2^n\right\}.$$

The set $T_0 = \bigcup_{n=1}^{\infty} T_n$ contains all the binary numbers in the unit interval. The binary numbers are dense in the unit interval in the sense that for every $0 \leq x \leq 1$ there is a sequence of binary numbers $x_j = k_j 2^{-n_j}$ with $0 \leq k_j \leq 2^{n_j}$ such that $x_j \rightarrow x$ as $j \rightarrow \infty$.

Define $X_1(\omega) = tY_1(\omega)$ for $0 \leq t \leq 1$. Keeping in mind that $T_2 = \{0, \frac{1}{2}, 1\}$ and $T_1 \setminus T_1 = \{\frac{1}{2}\}$, we refine by keeping the "old" points; that is, by setting $X_2(t, \omega) = X_1(t, \omega)$ for $t \in T_1$ and in the "new" point, $T_2 \setminus T_1 = \{\frac{1}{2}\}$, we set $X_2(\frac{1}{2}, \omega) = \frac{1}{2}[X_1(0, \omega) + X_1(1, \omega)] + \frac{1}{2}Y_2(\omega)$. The process $X_2(t, \omega)$ is defined in the interval by linear interpolation between the points of T_2 . We proceed by induction,

$$X_{n+1}(t, \omega) = \begin{cases} X_n(t, \omega) & \text{for } t \in T_n \text{ (old points)} \\ \frac{1}{2} \left\{ X_n\left(t + \frac{1}{2^n}, \omega\right) + X_n\left(t - \frac{1}{2^n}, \omega\right) \right\} + \frac{1}{2^{(n+1)/2}} Y_k(\omega) & \text{for } t \in T_{n+1} \setminus T_n, k = 2^{n-1} + \frac{1}{2}(2^n t - 1) \text{ (new points)} \\ \text{connect linearly between consecutive points} \end{cases}$$

(see Figure 2.3). A Brownian trajectory sampled at 1024 points is shown in Figure 2.4.

Thus $X_{n+1}(t)$ is a refinement of $X_n(t)$. Old points stay put! So far, for every realization ω , we constructed an infinite sequence of continuous functions. We show below that for almost all (in the sense of $\tilde{\Omega}$) realizations ω the sequence $X_n(t)$ converges uniformly to a continuous function, thus establishing a correspondence

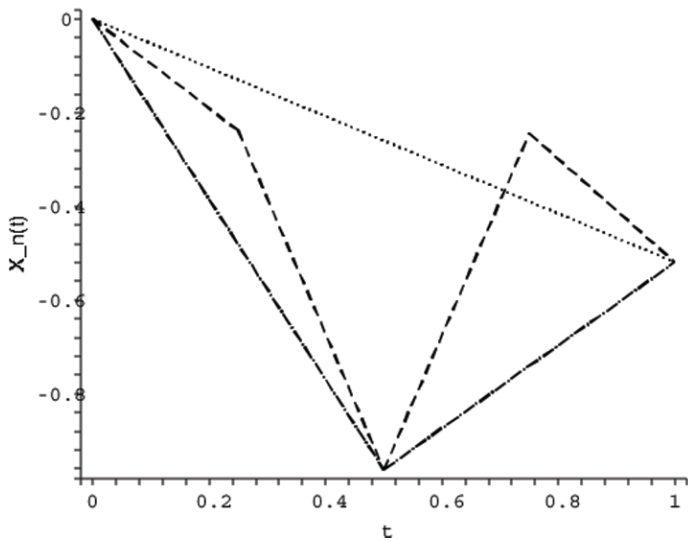


Figure 2.3. The graphs of $X_1(t)$ (dots), its first refinement $X_2(t)$ (dot-dash), and second refinement $X_3(t)$ (dash).

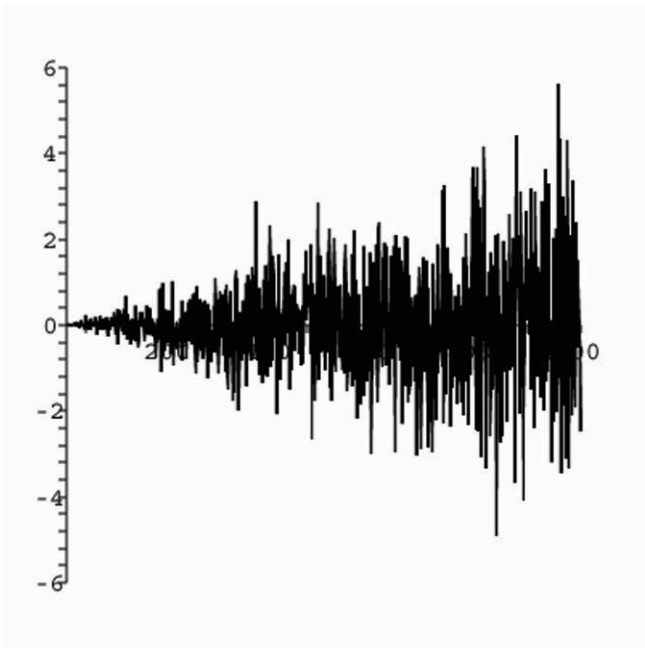


Figure 2.4. A Brownian trajectory sampled at 1024 points.

between ω and a continuous function. Obviously, the correspondence can be reversed in this construction.

Exercise 2.9 (MBM at binary points). Show that at binary points, $t_{k,n} = k2^{-n}$, $0 \leq k \leq 2^n$, the process $X_n(t, \omega)$ has the properties of the Brownian motion $w(t)$. $\square$

Exercise 2.10 (Refinements). If a Brownian trajectory is sampled at points $0 = t_0 < t_1 < \dots < t_n = T$ according to the scheme (2.21) or otherwise, how should the sampling be refined by introducing an additional sampling point $\tilde{t}_i$ such that $t_i < \tilde{t}_i < t_{i+1}$? $\square$

Exercise 2.11 (L^2 convergence*). Show that $X_n(t, \omega) \xrightarrow{L^2} X(t, \omega)$, where $X(t, \omega)$ has continuous paths [101]. $\square$

Exercise 2.12 (A_n i.o.). Consider an infinite sequence of events (sets) $\{A_n\}$. The set A_n i.o. (infinitely often) is the set of all elementary events (points) that occur in infinitely many of the events A_n (that belong to infinitely many sets A_n). Thus $x \in A_n$ i.o. if and only if there is an infinite sequence of indices n_k , ($k = 1, 2, \dots$), such that $x \in A_{n_k}$ for all $k = 1, 2, \dots$. Show that A_n i.o. $= \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$. $\square$

We need the following

Lemma 2.3.1 (Borel–Cantelli). If $\sum_{n=1}^{\infty} \Pr\{A_n\} < \infty$, then $\Pr\{A_n \text{ i.o.}\} = 0$.

Proof. From Exercise 2.12 it follows that $A_n \text{ i.o.} \subset \bigcup_{n=m}^{\infty} A_n$ for all $m > 0$. The convergence of the series $\sum_{n=1}^{\infty} \Pr\{A_n\}$ implies the inequalities $\Pr\{A_n \text{ i.o.}\} \leq \Pr\{\bigcup_{n=m}^{\infty} A_n\} \leq \sum_{n=m}^{\infty} \Pr\{A_n\} \rightarrow 0$ as $m \rightarrow \infty$. $\square$

The lemma means that if the sum of the probabilities converges, then, with probability one, only a finite number of the events occur.

Theorem 2.3.2 (P. Lévy). $X_n(t, \omega) \rightarrow X(t, \omega)$ almost surely in $\tilde{\Omega}$, where $X(t, \omega)$ is continuous for $0 \leq t \leq 1$.

Proof. Set $Z_n = X_{n+1} - X_n$; then $Z_n(t) = 0$ for $t \in T_n$ and

$$\max_{0 \leq t \leq 1} |Z_n(t, \omega)| = \max_{2^{n-1} \leq k \leq 2^n} 2^{-n+1} |Y_k|.$$

It follows that for every $\lambda_n > 0$,

$$\begin{aligned} P_n &= \Pr \left\{ \max_{0 \leq t \leq 1} |Z_n(t, \omega)| > \lambda_n \right\} \leq 2^{n-1} \Pr \left\{ |Y_k| \geq 2^{\frac{n+1}{2}} \lambda_n \right\} \\ &\leq 2^{n-1} \frac{2}{\sqrt{2\pi}} \int_{2^{(n+1)/2} \lambda_n}^{\infty} e^{-x^2/2} dx \leq \frac{2^{n-1}}{\sqrt{2\pi}} \frac{1}{\lambda_n} \exp \left\{ -\frac{1}{2} \left(2^{(n+1)/2} \lambda_n \right)^2 \right\} \end{aligned}$$

because (show!) $(2\pi)^{-1/2} \int_a^\infty e^{-x^2/2} dx < (2\pi)^{-1/2} a^{-1} e^{-a^2/2}$. Choosing $\lambda_n = 2^{-(n+1)/2} \sqrt{2cn \log 2}$, ($c > 1$), we get

$$\begin{aligned} \sum_n P_n &< \frac{1}{\sqrt{2\pi}} \sum_n \frac{2^{(n-1)/2} 2^{(n+1)/2}}{\sqrt{2cn \log 2}} \exp \left\{ - \left(2^{(n+1)/2} \frac{\sqrt{2cn \log 2}}{2^{(n+1)/2}} \right)^2 \right\} \\ &= \sum_n \frac{2^n 2^{-(cn+1/2)}}{2\sqrt{\pi cn \log 2}} = C_1 \sum_n 2^{(1-c)n-1/2} < \infty, \end{aligned}$$

where C_1 is a constant. It follows from the Borel–Cantelli lemma that with probability one, only a finite number of the events $\{\max_{0 \leq t \leq 1} |Z_n(t, \omega)| > \lambda_n\}$ occur. Thus, for n sufficiently large,

$$\max_{0 \leq t \leq 1} |Z_n(t, \omega)| \leq \lambda_n 2^{-(n+1)/2} \sqrt{2cn \log 2} \quad \text{w.p. 1.}$$

It follows from the Weierstrass M-test that for almost all trajectories ω the series $\sum_{n=1}^\infty Z_n(t, \omega)$ converges uniformly for all $t \in [0, 2\pi]$. Cauchy's theorem asserts that the sum of a uniformly convergent series of continuous functions is continuous. The trajectories of the process $X(t, \omega) = \sum_{n=1}^\infty Z_n(t, \omega)$ are continuous with probability one, because, with probability one, each function $Z_n(t, \omega)$ is continuous. $\square$

Exercise 2.13 (Lévy's construction gives a MBM). Show that if $X_1(t)$ and $X_2(t)$ are independent Brownian motions on the interval $[0, 1]$, then the process

$$X(t) = \begin{cases} X_1(t) & \text{for } 0 \leq t \leq 1 \\ X_1(1) + tX_2\left(\frac{1}{t}\right) - X_2(1) & \text{for } t > 1 \end{cases}$$

is a Brownian motion on $\mathbb{R}_+$. $\square$

2.4 Analytical and statistical properties of Brownian paths

The Wiener measure assigns probability 0 to several important classes of Brownian paths. These classes include all differentiable paths, all paths that satisfy the Lipschitz condition at any point, all continuous paths with bounded variation on any interval, and so on. The Brownian paths have many interesting properties, whose investigation exceeds the scope of this book. For a more detailed description of the Brownian paths see, for example, [106], [101], [208]. Here we list only a few of the most prominent features of the Brownian paths. As shown below, although continuous, the Brownian paths are nondifferentiable at any point with probability 1. This means that the Wiener measure assigns probability 0 to all differentiable paths. This fact implies that the white noise process $\dot{w}(t)$ does not exist, so that strictly speaking, none of the calculations carried out under the assumption that $\dot{w}(t)$ exists are

valid. This means that the velocity process of the MBM (white noise) should be interpreted as the overdamped limit [214] of the Brownian velocity process described in Chapter 1. In 1933 Paley, Wiener, and Zygmund [198] proved the following

Theorem 2.4.1 (Paley, Wiener, Zygmund). *The Brownian paths are nondifferentiable at any point with probability 1.*

Proof. (Dvoretzky, Erdős, Kakutani [60]) We construct a set B of trajectories that contains all the trajectories such that the derivative $\dot{w}(t, \omega)$ exists for some $t \in [0, 1]$ and show that $\Pr\{B\} = 0$. The set B is constructed in the form $B = \bigcup_{l=1}^{\infty} \bigcup_{m=1}^{\infty} B_{l,m}$, such that $\Pr\{B_{l,m}\} = 0$ for all l, m . Consider a trajectory ω such that $\dot{w}(t, \omega)$ exists for some $t \in [0, 1]$. This means that $\lim_{s \rightarrow t} [w(t, \omega) - w(s, \omega)]/(t - s) = \dot{w}(t, \omega) \neq \pm\infty$, or, equivalently, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|[w(t, \omega) - w(s, \omega)]/(t - s) - \dot{w}(t, \omega)| < \varepsilon$ if $|t - s| < \delta$. It follows that $|[w(t, \omega) - w(s, \omega)]/(t - s)| \leq |\dot{w}(t, \omega)| + \varepsilon < l$, for some $l \geq 1$. Define $i = [nt] + 1$, then

$$\left| t - \frac{i}{n} \right| < \frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.51)$$

Now choose $s_{i,j} = (i + j)/n$ for $j = 0, 1, 2, 3$, then

$$|w(s_{i,0}, \omega) - w(s_{i,1}, \omega)| \quad (2.52)$$

$$\leq |w(s_{i,0}, \omega) - w(t, \omega)| + |w(t, \omega) - w(s_{i,1}, \omega)| < \frac{3l}{n}.$$

Similarly, $|w(s_{i,j}, \omega) - w(s_{i,j+1}, \omega)| \leq 7l/n$ for $j = 1, 2$. Thus, if $\dot{w}(t, \omega)$ exists for some ω and t , then there exists $l \in \{1, 2, \dots, n\}$ such that for all sufficiently large n there exists an i such that inequalities (2.51) and (2.52) hold for $j = 0, 1, 2$. It follows that the set of ω such that the derivative exists at some point, $\{\omega \in \Omega \mid \dot{w}(t, \omega) \text{ exists for some } t \in [0, 1]\}$, is contained in the set

$$\left\{ \omega \in \Omega \mid \bigcup_{l=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \bigcup_{i=1}^n \bigcap_{j=0}^2 \left\{ |w(s_{i,j}, \omega) - w(s_{i,j+1}, \omega)| \leq \frac{7l}{n} \right\} \right\}.$$

We denote $B_{l,m} = \bigcap_{n=m}^{\infty} \bigcup_{i=1}^n \bigcap_{j=0}^2 \{ |w(s_{i,j}, \omega) - w(s_{i,j+1}, \omega)| \leq 7l/n \}$ and show $\Pr\{B_{l,m}\} = 0$. Indeed, the increments $w(s_{i,j}) - w(s_{i,j+1})$ are independent random zero mean Gaussian variables with variance $1/n$. It follows that the probability that the three events $\{ |w(s_{i,j}, \omega) - w(s_{i,j+1}, \omega)| \leq 7l/n \}$ for $j = 0, 1, 2$ occur simultaneously is the product of the probabilities of each event occurring separately; that is,

$$\Pr \left\{ \bigcap_{j=0}^2 \left\{ |w(s_{i,j}, \omega) - w(s_{i,j+1}, \omega)| \leq \frac{7l}{n} \right\} \right\} = \left(\sqrt{\frac{n}{2\pi}} \int_{-7l/n}^{7l/n} e^{-nx^2/2} dx \right)^3.$$

The probability of a union of n such events does not exceed the sum of the probabilities; that is,

$$\begin{aligned} & \Pr \left\{ \bigcup_{i=1}^n \bigcap_{j=0}^2 \left\{ |w(s_{i,j}, \omega) - w(s_{i,j+1}, \omega)| \leq \frac{7l}{n} \right\} \right\} \\ & \leq n \left(\sqrt{\frac{n}{2\pi}} \int_{-7l/n}^{7l/n} e^{-nx^2/2} dx \right)^3. \end{aligned}$$

The probability of the intersection $\bigcap_{n \geq m}$ does not exceed the probability of any of the intersected sets, thus, changing the variable of integration to $y = \sqrt{n}x$, we obtain that for all $n \geq m$,

$$\Pr \{B_{l,m}\} \leq \frac{8n}{\sqrt{8\pi^3}} \left(\int_0^{7l/\sqrt{n}} e^{-y^2/2} dy \right)^3 \leq \frac{8n}{\sqrt{8\pi^3}} \left(\frac{7l}{\sqrt{n}} \right)^3 = \text{const.} n^{-1/2} \rightarrow 0$$

as $n \rightarrow \infty$. That is, $\Pr \{B_{l,m}\} = 0$. $\square$

The proofs of the following theorems are given, for example, in [106], [101], [208].

Theorem 2.4.2 (The Khinchine–Lévy law of the iterated logarithm).

$$\limsup_{t \rightarrow \infty} \frac{w(t)}{\sqrt{2t \log \log t}} = 1, \quad \liminf_{t \rightarrow \infty} \frac{w(t)}{\sqrt{2t \log \log t}} = -1. \quad (2.53)$$

Theorem 2.4.3 (Modulus of continuity).

$$\limsup_{h \rightarrow 0} \frac{w(t+h) - w(t)}{\sqrt{2|h| \log \log |h^{-1}|}} = 1, \quad \liminf_{h \rightarrow 0} \frac{w(t+h) - w(t)}{\sqrt{2|h| \log \log |h^{-1}|}} = -1. \quad (2.54)$$

In particular

$$\limsup_{h \rightarrow 0} \frac{|\Delta w|}{|h|^\alpha} = \begin{cases} \infty & \text{if } \alpha \geq \frac{1}{2} \\ 0 & \text{if } \alpha < \frac{1}{2}. \end{cases} \quad (2.55)$$

Theorem 2.4.4 (The level-crossing property). *For any level a the times t such that $w(t) = a$ form a perfect set (i.e., every point of this set is a limit of points in this set).*

Thus, when a Brownian path reaches a given level at time t it recrosses it infinitely many times in every interval $[t, t + \Delta t]$.

Exercise 2.14 (Properties of Brownian trajectories). Compare plots of simulated trajectories $w(t)$ with $\sqrt{2t \log \log t}$. $\square$

2.4.1 The Markov property of the MBM

Definition 2.4.1 (Markov process). A stochastic process $\zeta(t)$ on $[0, T]$ is called a Markov process if for any sequences $0 \leq t_0 < \dots < t_n \leq T$ and $x_0, x_1, \dots, x_n$, its transition probability distribution function has the property

$$\begin{aligned} & \Pr\{\zeta(t_n) < x_n \mid \zeta(t_{n-1}) < x_{n-1}, \zeta(t_{n-2}) < x_{n-2}, \dots, \zeta(t_0) < x_0\} \\ &= \Pr\{\zeta(t_n) < x_n \mid \zeta(t_{n-1}) < x_{n-1}\}. \end{aligned} \quad (2.56)$$

The transition probability density function, defined by

$$\begin{aligned} & p(x_n, t_n \mid x_{n-1}, t_{n-1}, \dots, x_1, t_1) \\ &= \frac{\partial}{\partial x_n} \Pr\{\zeta(t_n) < x_n \mid \zeta(t_{n-1}) = x_{n-1}, \zeta(t_{n-2}) = x_{n-2}, \dots, \zeta(t_0) = x_0\}, \end{aligned}$$

then satisfies

$$p(x_n, t_n \mid x_{n-1}, t_{n-1}, \dots, x_1, t_1) = p(x_n, t_n \mid x_{n-1}, t_{n-1}). \quad (2.57)$$

The Markov property eq. (2.56) means that the process “forgets” the past in the sense that if the process is observed at times $t_0, t_1, \dots, t_{n-1}$ such that $0 \leq t_0 < \dots < t_{n-1} \leq T$, its “future” evolution (at times $t > t_{n-1}$) depends only on the “latest” observation (at time t_{n-1}).

For any three times $t > \tau > s$ and any points x, y, z , we can write the identities

$$\begin{aligned} p(y, t, z, \tau \mid x, s) &= p(y, t \mid z, \tau, x, s) p(z, \tau \mid x, s) \\ &= p(y, t \mid z, \tau) p(z, \tau \mid x, s), \end{aligned} \quad (2.58)$$

the last equation being a consequence of the Markov property. Now, using the identities (2.58) and writing $p(y, t \mid x, s)$ as a marginal density of $p(y, t, z, \tau \mid x, s)$, we obtain

$$\begin{aligned} p(y, t \mid x, s) &= \int p(y, t, z, \tau \mid x, s) dz = \int p(y, t \mid z, \tau, x, s) p(z, \tau \mid x, s) dz \\ &= \int p(y, t \mid z, \tau) p(z, \tau \mid x, s) dz. \end{aligned} \quad (2.59)$$

Equation (2.59) is called the *Chapman–Kolmogorov equation* (CKE). More general properties of Markov processes are described in Chapter 9.

Theorem 2.4.5. *The MBM is a Markov process.*

Proof. To determine the Markov property of the Brownian motion, consider any sequences $0 = t_0 < t_1 < \dots < t_n$ and $x_0 = 0, x_1, \dots, x_n$. The joint pdf of the vector

$$\mathbf{w} = (w(t_1), w(t_2), \dots, w(t_n))^T \quad (2.60)$$

is given by (see eq. (2.20))

$$\begin{aligned} p(x_1, t_1; x_2, t_2; \dots; x_n, t_n) &= \Pr\{w(t_1) = x_1, w(t_2) = x_2, \dots, w(t_n) = x_n\} \\ &= \prod_{k=1}^n \left[\{2\pi(t_k - t_{k-1})\}^{-1/2} \exp\left\{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}\right\} \right], \end{aligned} \quad (2.61)$$

so that for $0 = t_0 < t_1 < \dots < t_n < t = t_{n+1}$ and $0 = x_0, x_1, \dots, x_n, x = x_{n+1}$,

$$\begin{aligned} &\Pr\{w(t) = x \mid w(t_n) = x_n, \dots, w(t_1) = x_1\} \\ &= \frac{\Pr\{w(t_{n+1}) = x_{n+1}, w(t_n) = x_n, \dots, w(t_1) = x_1\}}{\Pr\{w(t_n) = x_n, \dots, w(t_1) = x_1\}} \\ &= \frac{\prod_{k=1}^{n+1} \left[\{2\pi(t_k - t_{k-1})\}^{-1/2} \exp\left\{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}\right\} \right]}{\prod_{k=1}^n \left[\{2\pi(t_k - t_{k-1})\}^{-1/2} \exp\left\{-\frac{(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}\right\} \right]} \\ &= \frac{1}{\sqrt{2\pi(t - t_n)}} \exp\left\{-\frac{(x_{n+1} - x_n)^2}{2(t_{n+1} - t_n)}\right\} = \Pr\{w(t) = x \mid w(t_n) = x_n\}; \end{aligned}$$

that is, the Brownian motion is a Markov process. $\square$

It follows that it suffices to know the two-point transition pdf of the Brownian motion, $p(y, t \mid x, s) = \Pr\{w(t) = y \mid w(s) = x\}$ for $t > s$, to calculate the joint and conditional probability densities of the vector (2.60); that is, $p(x_1, t_1; x_2, t_2; \dots; x_n, t_n) = \prod_{k=1}^n p(x_k, t_k \mid x_{k-1}, t_{k-1})$.

Theorem 2.4.6 (The strong Markov property of the MBM). *If τ is a Markov time for the Brownian motion, then the process $\tilde{w}(t) = w(t + \tau) - w(\tau)$ is a Brownian motion.*

Exercise 2.15. (Strong Markov property of MBM)

(i) Verify the Chapman–Kolmogorov equation for the MBM.

(ii) Prove Theorem 2.4.6 [106], [101]. $\square$

Exercise 2.16 (The velocity process). Consider the velocity process $y(t)$ in Definition 1.4.1, $y(t) = w(t) - \int_0^t e^{-(t-s)} w(s) ds$, and define the displacement process $x(t) = \int_0^t y(s) ds$.

(i) Prove that $y(t)$ is a Markov process.

(ii) Prove that $x(t)$ is not a Markov process.

(iii) Prove that the two-dimensional process $z(t) = (x(t), y(t))$ is Markovian. $\square$

2.4.2 Reflecting and absorbing walls

The variants of the Brownian motion defined below appear in many applications of diffusion theory. The calculation of the transition probability density functions of

the variants can be done directly from the definition (see, e.g., [199]) and also by solving boundary value problems for partial differential equations that the transition probability density functions satisfy. The partial differential equations and the boundary conditions that these transition probability density functions satisfy are derived in Section 3.4. With these equations the calculations become straightforward. In this chapter the calculations based on the definition of the variants of the Brownian motion are presented as exercises.

Definition 2.4.2 (Reflected MBM). *The process $w^+(t) = |w(t)|$ is called the reflected Brownian motion.*

The reflected Brownian motion is obtained from the Brownian motion by observing its trajectories in the negative x -axis in a mirror placed at the origin. The reflected Brownian motion is used to describe the motion of freely diffusing particles in the presence of an impermeable wall.

Exercise 2.17. (The transition pdf of the reflected MBM).

- (i) Find the transition probability density function of the reflected Brownian motion.
- (ii) Prove that the transition pdf of the reflected Brownian motion, $f_{|w|}(x, t)$, satisfies the diffusion equation and the initial condition (2.26) on the positive ray, and the boundary condition $f'_{|w|}(0, t) = 0$ for $t > 0$.
- (iii) Prove that (i) and (ii) imply that if $x_0 > 0$, then $\int_0^\infty f_{|w|}(x, t) dx = 1$ for all $t \geq 0$. □

Exercise 2.18 (The reflected MBM is Markovian). Show that the reflected MBM is a Markov process. □

Exercise 2.19 (The reflection principle). Prove the following reflection principle. Let τ_a be the first passage time to a ; then for every Brownian path $w(t, \omega_1)$, $t \geq \tau_a$, there is another path, $w(t, \omega_2)$, $t \geq \tau_a$, which is the mirror image of $w(t, \omega_1)$ about the line $L_a : w = a$ [199]. □

Exercise 2.20 (The joint PDF of the FPT and the maximum of the MBM motion). Setting $M(t) = \max_{0 \leq s \leq t} w(s)$, we find, by definition,

$$\Pr\{M(t) \leq a\} = \Pr\{\tau_a \geq t\}. \quad (2.62)$$

Thus the PDFs of $M(t)$ and τ_a are related through eq. (2.62). Prove that if $x \leq a$, then

$$\begin{aligned} \Pr\{w(t) \leq x, M(t) \geq a\} &= \frac{1}{\sqrt{2\pi t}} \int_{2a-x}^{\infty} e^{-y^2/2t} dy \\ \Pr\{w(t) \leq x, M(t) \leq a\} &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^x e^{-y^2/2t} dy - \frac{1}{\sqrt{2\pi t}} \int_{2a-x}^{\infty} e^{-y^2/2t} dy. \end{aligned}$$

□

Exercise 2.21. (The PDF of the FPT).

(i) Prove that $\Pr\{\tau_a \leq t\} = 2(2\pi t)^{-1/2} \int_a^\infty e^{-y^2/2t} dy$. Conclude that the first passage time to a given point is finite with probability 1 but its mean is infinite. This is the continuous time version of the Gambler's Ruin "Paradox": gambling with even odds against an infinitely rich adversary leads to sure ruin in a finite number of games, but on the average, the gambler can play forever (see [72, Ch. XIV.3]).

(ii) Use the above result to conclude that the one-dimensional MBM is *recurrent* in the sense that $\Pr\{w(t, \omega) = x \text{ for some } t > T\} = 1$ for every x and every T . This means that the MBM returns to every point infinitely many times for arbitrary large times.

(iii) Consider two independent Brownian motions, $w_1(t)$ and $w_2(t)$ that start at x_1 and x_2 on the positive axis and denote by τ_1 and τ_2 their first passage times to the origin, respectively. Define $\tau = \tau_1 \wedge \tau_2$, the first passage time of the first Brownian motion to reach the origin. Find the PDF and mean value of τ . $\square$

Exercise 2.22 (Absorbed MBM). If the Brownian motion is stopped at the moment it reaches a for the first time, the process $y(t) = w(t)$ for $t \leq \tau_a$ and $y(t) = a$ for $t \geq \tau_a$ is called the absorbed Brownian motion.

(i) Prove

$$\Pr\{y(t) \leq y\} = \begin{cases} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^y e^{-z^2/2t} dz - \frac{1}{\sqrt{2\pi t}} \int_{2a-y}^{\infty} e^{-z^2/2t} dz & \text{for } y < a \\ 1 & \text{for } y \geq a. \end{cases}$$

Prove that the pdf of $y(t)$, denoted $f_{y(\cdot)}(x, t)$, satisfies for $x < a$ the diffusion equation and the initial condition (2.26) and the boundary condition $f_{y(\cdot)}(x, t) = 0$ for $x \geq a$.

(ii) Assume that the trajectories of the MBM begin at time t_0 at a point $x_0 < a$. Find the pdf of the absorbed MBM for this case.

(iii) Prove that the pdf of the absorbed MBM in (ii), denoted $f_{y(\cdot)}(x, t | x_0, t_0)$, is the solution of the initial and boundary value problem for the diffusion equation

$$\begin{aligned} \frac{\partial f_{y(\cdot)}}{\partial t} &= \frac{1}{2} \frac{\partial^2 f_{y(\cdot)}}{\partial x^2} \text{ for } x < a \\ \lim_{t \downarrow t_0} f_{y(\cdot)} &= \delta(x - x_0), \quad f_{y(\cdot)} = 0 \text{ for } x \geq a, \quad t > t_0. \end{aligned} \tag{2.63}$$

(iv) Find the partial differential equation and the terminal and boundary conditions that the function $f_{y(\cdot)}(x, t | x_0, t_0)$ satisfies with respect to the initial point and time (x_0, t_0) .

(v) Verify that eq. (2.62) holds.

(vi) It is known from the theory of parabolic partial differential equations [78] that the partial differential equations and boundary conditions in (iii) and (iv) have a unique solutions. Use this information to derive and solve partial differential equations and boundary conditions for the PDF of the FPT from a point x to a point y .

(vii) Prove that $\Pr\{\tau_a > t \mid x_0, t_0\} = \int_{-\infty}^a f_{y(\cdot)}(x, t \mid x_0, t_0) dx$.

(viii) Prove in the presence of an absorbing boundary at a the population of Brownian trajectories in $(-\infty, a)$ decays in time; that is,

$$\lim_{t \rightarrow \infty} \int_{-\infty}^a f_{y(\cdot)}(x, t \mid x_0, t_0) dx = 0,$$

which is equivalent to the decay of $\Pr\{\tau_a > t \mid x_0, t_0\}$ as $t \rightarrow \infty$. Reconcile this with (i).

(ix) Prove that the function $u(t \mid x_0, t_0) = \Pr\{\tau_a \leq t \mid x_0, t_0\}$ is the solution of the terminal and boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t_0} + \frac{1}{2} \frac{\partial^2 u}{\partial x_0^2} &= -1 \quad \text{for } x_0 < a, \quad t_0 < t \\ \lim_{t_0 \uparrow t} u &= 0 \quad \text{for } x_0 < a, \quad u(t \mid a, t_0) = 1. \end{aligned}$$

(x*) Consider the d -dimensional MBM that starts at a distance r from the origin and assume that an absorbing sphere of radius a is centered at the origin. The FPT to the sphere $|\mathbf{w}| = a$ is defined by $\tau_a = \inf\{t \mid |\mathbf{w}| = a\}$. $\square$

Exercise 2.23 (Absorbed MBM in $\mathbb{R}^d$). The d -dimensional Brownian motion with absorption at the sphere is defined by

$$\mathbf{y}(t) = \begin{cases} \mathbf{w}(t) & \text{for } t \leq \tau_a \\ \mathbf{w}(\tau_a) & \text{for } t \geq \tau_a. \end{cases}$$

Denote $y(t) = |\mathbf{y}(t)|$.

(i) Formulate and prove a reflection principle for the process $y(t)$ (use reflection of the d -dimensional MBM in a sphere of radius a , as defined in eq. (2.64) below).

(ii) Formulate and solve initial and boundary value problems for the pdf and the FPT of $y(t)$, analogous to (i)–(ix) above. $\square$

Exercise 2.24 (The absorbed MBM is Markovian). Show that the absorbed MBM is a Markov process. $\square$

Exercise 2.25 (Reflecting wall). If upon hitting the line $L_a : w = a$ for the first time, the Brownian path is reflected in L_a , the resulting process is defined by

$$x(t) = \begin{cases} w(t) & \text{for } w(t) \leq a \\ 2a - w(t) & \text{for } w(t) > a. \end{cases}$$

Thus $x(t) \leq a$ for all t .

(i) Prove that for $x \leq a$,

$$\Pr\{x(t) \leq x\} = 1 - \frac{1}{\sqrt{2\pi t}} \int_x^{2a-x} e^{-y^2/2t} dy$$

(see [199]).

(ii) For the d -dimensional MBM $w(t)$, define a reflected d -dimensional MBM in a sphere of radius a centered at the origin by

$$w^*(t) = \begin{cases} w(t) & \text{for } |w(t)| \leq a \\ a^2 \frac{w(t)}{|w(t)|^2} & \text{for } |w(t)| \geq a. \end{cases} \quad (2.64)$$

Obviously, $|w^*(t)| \leq a$. Find the pdf of $w^*(t)$, the partial differential equation, the initial conditions, and boundary conditions it satisfies on the sphere. $\square$

Exercise 2.26 (The Brownian bridge). Start the Brownian motion at the point x ; that is, set $x(t) = w(t) + x$, and consider only the trajectories that pass through the point y at time t_0 . That is, condition $x(t)$ on $w(t_0) + x = y$. Thus the paths of the Brownian bridge $x(t)$ are those paths of the Brownian motion that satisfy the given condition.

(i) Show that $x(t) = w(t) - (t/t_0)[w(t_0) - y + x] + x$ for $0 \leq t \leq t_0$.

(ii) Show that $x(t)$ is a Gaussian process.

(iii) Calculate the mean and the autocorrelation function of the Brownian bridge.

(iv) Show that $x(t)$ and $x(t_0 - t)$ have the same PDF for $0 \leq t \leq t_0$.

(v) Show that $x(t)$ is a Markov process [117]. $\square$

Exercise 2.27 (Maximum of the Brownian bridge). Find the distribution of the maximum of the Brownian bridge in the interval $[0, t]$ for $0 < t < t_0$ (see also Exercise 6.11 below). $\square$

2.4.3 MBM and martingales

Definition 2.4.3 (Martingales). A martingale is a stochastic process $x(t)$ such that $\mathbb{E}|x(t)| < \infty$ for all t and for every $t_1 < t_2 < \dots < t_n < t$ and $x_1, x_2, \dots, x_n$,

$$\mathbb{E}[x(t) | x(t_1) = x_1, x(t_2) = x_2, \dots, x(t_n) = x_n] = x_n \quad (2.65)$$

(see [208], [115], [178]). In gambling theory $x(t)$ often represents the capital of a player at time t . The martingale property (2.65) means that the game is *fair*; that is, not biased.

Theorem 2.4.7. The MBM is a martingale.

Proof. Indeed,

$$\mathbb{E}|w(t)| = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} |x| e^{-x^2/2t} dx = \sqrt{\frac{2t}{\pi}} < \infty$$

and due to the Markov property of the MBM (see Section 2.4.1)

$$\begin{aligned}
 & \mathbb{E}[w(t) \mid w(t_1) = x_1, w(t_2) = x_2, \dots, w(t_n) = x_n] \\
 &= \int_{-\infty}^{\infty} xp(x, t \mid x_1, t_1, x_2, t_2, \dots, x_n, t_n) dx \\
 &= \int_{-\infty}^{\infty} xp(x, t \mid x_n, t_n) dx = \frac{1}{\sqrt{2\pi(t-t_n)}} \int_{-\infty}^{\infty} xe^{-(x-x_n)^2/2(t-t_n)} dx = x_n.
 \end{aligned}$$

□

Theorem 2.4.8. *For every α , the process $x(t) = \exp\{\alpha w(t) - \alpha^2 t/2\}$ is a martingale.*

Proof. The property $\mathbb{E}|x(t)| < \infty$ is obtained from

$$\begin{aligned}
 \mathbb{E}|x(t)| &= \mathbb{E}x(t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left\{\alpha x - \frac{\alpha^2}{2}t\right\} \exp\left\{-\frac{x^2}{2t}\right\} dx \\
 &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x - \alpha t)^2}{2t}\right\} dx = 1.
 \end{aligned} \tag{2.66}$$

To verify the martingale property (2.65), we use the identity

$$x(t) = \exp\left\{\alpha[w(t) - w(s)] - \frac{\alpha^2}{2}(t-s)\right\} x(s). \tag{2.67}$$

Setting $\tilde{w}(t-s) = w(t) - w(s)$, we rewrite (2.67) as $x(t) = \tilde{x}(t-s)x(s)$ and recall that the increment $\tilde{w}(t-s)$ is a MBM independent of $x(s)$. It follows that the process

$$\tilde{x}(t-s) = \exp\left\{\alpha\tilde{w}(t-s) - \frac{\alpha^2}{2}(t-s)\right\}$$

has the same probability law as $x(t-s)$, but is independent of $x(\tau)$ for all $\tau \leq s$. Hence, using (2.66), we obtain

$$\begin{aligned}
 & \mathbb{E}[x(t) \mid x(t_1) = x_1, x(t_2) = x_2, \dots, x(t_n) = x_n] \\
 &= \mathbb{E}[\tilde{x}(t-t_n)x(t_n) \mid x(t_1) = x_1, x(t_2) = x_2, \dots, x(t_n) = x_n] \\
 &= x_n \mathbb{E}\tilde{x}(t-t_n) = x_n.
 \end{aligned} \tag{2.68}$$

□

Definition 2.4.4 (Submartingales). *If instead of (2.65) $x(t)$ satisfies the inequality*

$$\mathbb{E}[x(t) \mid x(t_1) = x_1, x(t_2) = x_2, \dots, x(t_n) = x_n] \geq x_n \tag{2.69}$$

(in addition to $\mathbb{E}|x(t)| < \infty$), then $x(t)$ is said to be a submartingale.

For example, if $x(t)$ is a martingale, then $y(t) = |x(t)|$ is a *submartingale*, because $y(t) \geq \pm x(t)$ and

$$\begin{aligned} & \mathbb{E}[y(t) \mid y(t_1) = y_1, y(t_2) = y_2, \dots, y(t_n) = y_n] \\ & \geq \mathbb{E}[\pm x(t) \mid x(t_1) = \pm y_1, x(t_2) = \pm y_2, \dots, x(t_n) = \pm y_n] = \pm y_1 \end{aligned} \quad (2.70)$$

for all possible combinations of $+$ and $-$, which is the submartingale condition (2.69). Thus the reflected Brownian motion $|w(t)|$ is a submartingale.

Exercise 2.28 (Martingales). Show that the following are martingales.

(i) $w^2(t) - t$

(ii) $\exp\{-\alpha^2 t \cosh[\sqrt{2}\alpha w(t)]\}.$

□

Exercise 2.29 (Martingales: continued). Assume that α in Theorem 2.4.8 is a random variable in Ω such that $\alpha(\omega)$ is $\mathcal{F}_t$ -measurable for $a \leq t \leq b$ (it is independent of the Brownian increments $w(s + \Delta s, \omega) - w(s, \omega)$ for all $s \geq b$ and $\Delta s > 0$).

(i) Show that if $\alpha(\omega)$ is bounded, then $x(t, \omega)$ in Theorem 2.4.8 is a martingale in the interval $[a, b]$.

(ii) Can the boundedness condition be relaxed? How?

(iii) Consider the following generalization: Let $C_1(\omega)$ and $C_2(\omega)$ be random variables in Ω and let $0 < t_1 < T$. Assume that $C_1(\omega)$ is independent of $w(t, \omega)$ for all t and $C_2(\omega)$ is $\mathcal{F}_{t_1}$ -measurable (independent of $w(t, \omega) - w(t_1, \omega)$ for all $t > t_1$). Consider the following process in the interval $[0, T]$,

$$x(t, \omega) = \begin{cases} \exp\left\{C_1(\omega)w(t, \omega) - \frac{C_1^2(\omega)}{2}t\right\} & \text{if } 0 \leq t \leq t_1 \\ \exp\{C_1(\omega)w(t_1, \omega) + C_2(\omega)[w(t, \omega) - w(t_1, \omega)]\} \\ \quad \times \exp\left\{-\frac{1}{2}[C_1^2(\omega)t_1 + C_2^2(\omega)(t - t_1)]\right\} & \text{if } t_1 < t \leq T. \end{cases}$$

Show that if $C_1(\omega)$ and $C_2(\omega)$ are bounded, then $x(t, \omega)$ is a martingale in $[0, T]$.

(iv) Find a more general condition than boundedness that ensures the same result [115].

□

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