Lecture Notes in Physics 779

Introduction to Conformal Field Theory

With Applications to String Theory

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Chapter 2
Basics in Conformal Field Theory

The approach for studying conformal field theories is somewhat different from the usual approach for quantum field theories. Because, instead of starting with a classical action for the fields and quantising them via the canonical quantisation or the path integral method, one employs the symmetries of the theory. In the spirit of the so-called boot-strap approach, for CFTs one defines and for certain cases even solves the theory just by exploiting the consequences of the symmetries. Such a procedure is possible in two dimensions because the algebra of infinitesimal conformal transformations in this case is very special: it is infinite dimensional.

In this chapter, we will introduce the basic notions of two-dimensional conformal field theory from a rather abstract point of view. However, in Sect. 2.9, we will study in detail three simple examples important for string theory which are given by a Lagrangian action.

2.1 The Conformal Group

We start by introducing conformal transformations and determining the condition for conformal invariance. Next, we are going to consider flat space in $d \geq 3$ dimensions and identify the conformal group. Finally, we study in detail the case of Euclidean two-dimensional flat space $\mathbb{R}^{2,0}$ and determine the conformal group and the algebra of infinitesimal conformal transformations. We also comment on two-dimensional Minkowski space $\mathbb{R}^{1,1}$ in the end.

2.1.1 Conformal Invariance

Conformal Transformations

Let us consider a flat space in d dimensions and transformations thereof which locally preserve the angle between any two lines. Such transformations are illustrated in Fig. 2.1 and are called *conformal transformations*.

In more mathematical terms, a conformal transformation is defined as follows. Let us consider differentiable maps $\varphi : U \to V$, where $U \subset M$ and $V \subset M'$ are

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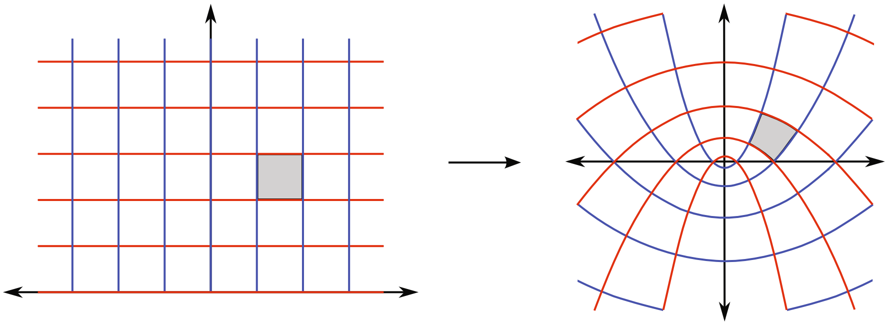


Fig. 2.1 Conformal transformation in two dimensions

open subsets. A map φ is called a conformal transformation, if the metric tensor satisfies $\varphi^* g' = \Lambda g$. Denoting $x' = \varphi(x)$ with $x \in U$, we can express this condition in the following way:

$$g'_{\rho\sigma}\left(x'\right) \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\, g_{\mu\nu}(x) \ ,$$

where the positive function $\Lambda(x)$ is called the scale factor and Einstein's sum convention is understood. However, in these lecture notes, we focus on $M' = M$ which implies $g' = g$, and we will always consider flat spaces with a constant metric of the form $\eta_{\mu\nu} = \text{diag}(-1, \ldots, +1, \ldots)$. In this case, the condition for a conformal transformation can be written as

$$\boxed{\eta_{\rho\sigma} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\, \eta_{\mu\nu}} \ . \tag{2.1}$$

Note furthermore, for flat spaces the scale factor $\Lambda(x) = 1$ corresponds to the Poincaré group consisting of translations and rotations, respectively Lorentz transformations.

Conditions for Conformal Invariance

Let us next study infinitesimal coordinate transformations which up to first order in a small parameter $\epsilon(x) \ll 1$ read

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}(\epsilon^2) \ . \tag{2.2}$$

Noting that $\epsilon_{\mu} = \eta_{\mu\nu}\epsilon^{\nu}$ as well as that $\eta_{\mu\nu}$ is constant, the left-hand side of Eq. (2.1) for such a transformation is determined to be of the following form:

$$\begin{aligned}
\eta_{\rho\sigma}\,\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} &= \eta_{\rho\sigma}\left(\delta^{\rho}_{\mu}+\frac{\partial\epsilon^{\rho}}{\partial x^{\mu}}+\mathcal{O}(\epsilon^2)\right)\left(\delta^{\sigma}_{\nu}+\frac{\partial\epsilon^{\sigma}}{\partial x^{\nu}}+\mathcal{O}(\epsilon^2)\right)\\
&= \eta_{\mu\nu}+\eta_{\mu\sigma}\,\frac{\partial\epsilon^{\sigma}}{\partial x^{\nu}}+\eta_{\rho\nu}\,\frac{\partial\epsilon^{\rho}}{\partial x^{\mu}}+\mathcal{O}(\epsilon^2)\\
&= \eta_{\mu\nu}+\left(\frac{\partial\epsilon_{\mu}}{\partial x^{\nu}}+\frac{\partial\epsilon_{\nu}}{\partial x^{\mu}}\right)+\mathcal{O}(\epsilon^2)\ .
\end{aligned}$$

The question we want to ask now is, under what conditions is the transformation (2.2) conformal, i.e. when is Eq. (2.1) satisfied? Introducing the short-hand notation $\frac{\partial}{\partial x^{\mu}}=\partial_{\mu}$, from the last formula we see that, up to first order in ϵ, we have to demand that

$$\partial_{\mu}\epsilon_{\nu}+\partial_{\nu}\epsilon_{\mu}=K(x)\,\eta_{\mu\nu}\ ,$$

where $K(x)$ is some function. This function can be determined by tracing the equation above with $\eta^{\mu\nu}$

$$\begin{aligned}
\eta^{\mu\nu}\Big(\partial_{\mu}\epsilon_{\nu}+\partial_{\nu}\epsilon_{\mu}\Big) &= K(x)\,\eta^{\mu\nu}\eta_{\mu\nu}\\
2\,\partial^{\mu}\epsilon_{\mu} &= K(x)\,d\ .
\end{aligned}$$

Using this expression and solving for $K(x)$, we find the following restriction on the transformation (2.2) to be conformal:

$$\boxed{\partial_{\mu}\epsilon_{\nu}+\partial_{\nu}\epsilon_{\mu}=\frac{2}{d}\,\big(\partial\cdot\epsilon\big)\,\eta_{\mu\nu}}\ , \tag{2.3}$$

where we employed the notation $\partial^{\mu}\epsilon_{\mu}=\partial\cdot\epsilon$. Finally, the scale factor can be read off as $\Lambda(x)=1+\frac{2}{d}(\partial\cdot\epsilon)+\mathcal{O}(\epsilon^2)$.

Some Useful Relations

Let us now derive two useful equations for later purpose. First, we modify Eq. (2.3) by taking the derivative ∂^{ν} and summing over ν. We then obtain

$$\begin{aligned}
\partial^{\nu}\Big(\partial_{\mu}\epsilon_{\nu}+\partial_{\nu}\epsilon_{\mu}\Big) &= \frac{2}{d}\,\partial^{\nu}\big(\partial\cdot\epsilon\big)\,\eta_{\mu\nu}\\
\partial_{\mu}\big(\partial\cdot\epsilon\big)+\Box\,\epsilon_{\mu} &= \frac{2}{d}\,\partial_{\mu}\big(\partial\cdot\epsilon\big)
\end{aligned}$$

with $\Box=\partial^{\mu}\partial_{\mu}$. Furthermore, we take the derivative ∂_{ν} to find

$$\partial_{\mu}\partial_{\nu}\big(\partial\cdot\epsilon\big)+\Box\,\partial_{\nu}\epsilon_{\mu}=\frac{2}{d}\,\partial_{\mu}\partial_{\nu}\big(\partial\cdot\epsilon\big)\ . \tag{2.4}$$

After interchanging $\mu \leftrightarrow \nu$, adding the resulting expression to Eq. (2.4) and using Eq. (2.3) we get

$$\begin{aligned} & 2\,\partial_\mu\partial_\nu\left(\partial\cdot\epsilon\right)+\Box\left(\frac{2}{d}\left(\partial\cdot\epsilon\right)\eta_{\mu\nu}\right)=\frac{4}{d}\,\partial_\mu\partial_\nu\big(\partial\cdot\epsilon\big)\,,\\ \Rightarrow \qquad & \Big(\eta_{\mu\nu}\,\Box+(d-2)\,\partial_\mu\partial_\nu\Big)\,(\partial\cdot\epsilon)=0\,. \end{aligned}$$

Finally, contracting this equation with $\eta^{\mu\nu}$ gives

$$\boxed{(d-1)\,\Box\,(\partial\cdot\epsilon)=0}\,. \tag{2.5}$$

The second expression we want to use later is obtained by taking derivatives ∂_ρ of Eq. (2.3) and permuting indices

$$\begin{aligned} \partial_\rho\partial_\mu\epsilon_\nu+\partial_\rho\partial_\nu\epsilon_\mu &= \frac{2}{d}\,\eta_{\mu\nu}\,\partial_\rho(\partial\cdot\epsilon)\,,\\ \partial_\nu\partial_\rho\epsilon_\mu+\partial_\mu\partial_\rho\epsilon_\nu &= \frac{2}{d}\,\eta_{\rho\mu}\,\partial_\nu(\partial\cdot\epsilon)\,,\\ \partial_\mu\partial_\nu\epsilon_\rho+\partial_\nu\partial_\mu\epsilon_\rho &= \frac{2}{d}\,\eta_{\nu\rho}\,\partial_\mu(\partial\cdot\epsilon)\,. \end{aligned}$$

Subtracting then the first line from the sum of the last two leads to

$$2\,\partial_\mu\partial_\nu\epsilon_\rho=\frac{2}{d}\,\big(-\eta_{\mu\nu}\partial_\rho+\eta_{\rho\mu}\partial_\nu+\eta_{\nu\rho}\partial_\mu\big)\big(\partial\cdot\epsilon\big)\,. \tag{2.6}$$

2.1.2 Conformal Group in d ≥ 3

After having obtained the condition for an infinitesimal transformations to be conformal, let us now determine the conformal group in the case of dimension $d \geq 3$.

Conformal Transformations and Generators

We note that Eq. (2.5) implies that $(\partial\cdot\epsilon)$ is at most linear in x^μ, i.e. $(\partial\cdot\epsilon) = A + B_\mu x^\mu$ with A and B_μ constant. Then it follows that ϵ_μ is at most quadratic in x^ν and so we can make the ansatz:

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}\,x^\nu x^\rho\,, \tag{2.7}$$

where $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho} \ll 1$ are constants and the latter is symmetric in the last two indices, i.e. $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. We now study the various terms in Eq. (2.7) separately because the constraints for conformal invariance have to be independent of the position x^μ.

- The constant term a_μ in Eq. (2.7) is not constrained by Eq. (2.3). It describes infinitesimal translations $x'^\mu = x^\mu + a^\mu$, for which the generator is the momentum operator $P_\mu = -i\partial_\mu$.
- In order to study the term of Eq. (2.7) which is linear in x, we insert (2.7) into the condition (2.3) to find

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}\,\big(\eta^{\rho\sigma} b_{\sigma\rho}\big)\eta_{\mu\nu}\ .$$

  From this expression, we see that $b_{\mu\nu}$ can be split into a symmetric and an antisymmetric part

$$b_{\mu\nu} = \alpha\,\eta_{\mu\nu} + m_{\mu\nu}\ ,$$

  where $m_{\mu\nu} = -m_{\nu\mu}$. The symmetric term $\alpha\,\eta_{\mu\nu}$ describes infinitesimal scale transformations $x'^\mu = (1+\alpha)\,x^\mu$ with generator $D = -ix^\mu\partial_\mu$. The antisymmetric part $m_{\mu\nu}$ corresponds to infinitesimal rotations $x'^\mu = (\delta^\mu_\nu + m^\mu{}_\nu)\,x^\nu$ with generator being the angular momentum operator $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$.
- The term of Eq. (2.7) at quadratic order in x can be studied by inserting Eq. (2.7) into expression (2.6). We then calculate

$$\partial\cdot\epsilon = b^\mu{}_\mu + 2\,c^\mu_{\mu\rho}\,x^\rho \qquad \Rightarrow \qquad \partial_\nu\big(\partial\cdot\epsilon\big) = 2\,c^\mu_{\mu\nu}\ ,$$

  from which we find that

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu \qquad \text{with} \qquad b_\mu = \frac{1}{d}\,c^\rho{}_{\rho\mu}\ .$$

  The resulting transformations are called *Special Conformal Transformations* (SCT) and have the following infinitesimal form:

$$x'^\mu = x^\mu + 2\,\big(x\cdot b\big)x^\mu - \big(x\cdot x\big)\,b^\mu\ . \tag{2.8}$$

  The corresponding generator is written as $K_\mu = -i\,\big(2\,x_\mu\,x^\nu\partial_\nu - \big(x\cdot x\big)\partial_\mu\big)$.

We have now identified the infinitesimal conformal transformations. However, in order to determine the conformal group, we will need the finite conformal transformations which are summarised in Table 2.1 together with the corresponding generators.

Focus on Special Conformal Transformations

For the finite Special Conformal Transformation shown in Table 2.1, one can check that expression (2.8) is its infinitesimal version by expanding the denominator for small b^μ. Furthermore, the scale factor for SCTs is computed as

$$\Lambda(x) = \Big(1 - 2\,(b\cdot x) + (b\cdot b)(x\cdot x)\Big)^2\ .$$

Table 2.1 Finite conformal transformations and corresponding generators

Transformations		Generators
translation	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_{\mu} = -i\,\partial_{\mu}$
dilation	$x'^{\mu} = \alpha\, x^{\mu}$	$D = -i\, x^{\mu}\partial_{\mu}$
rotation	$x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu}$	$L_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)$
SCT	$x'^{\mu} = \frac{x^{\mu}-(x\cdot x)b^{\mu}}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)}$	$K_{\mu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - (x\cdot x)\partial_{\mu}\right)$

Let us also note that for finite Special Conformal Transformations, we can re-write the expression in Table 2.1 as follows:

$$\frac{x'^{\mu}}{x'\cdot x'} = \frac{x^{\mu}}{x\cdot x} - b^{\mu}\ .$$

From this relation, we see that the SCTs can be understood as an inversion of x^{μ}, followed by a translation b^{μ}, and followed again by an inversion. An illustration in two dimensions is shown in Fig. 2.2.

Finally, we observe that the finite Special Conformal Transformations given in Table 2.1 are not globally defined. In particular, for a given non-zero vector b^{μ}, there is a point $x^{\mu} = \frac{1}{b\cdot b}\, b^{\mu}$ such that

$$1 - 2\,(b\cdot x) + (b\cdot b)(x\cdot x) = 0\ .$$

Taking into account also the numerator, one finds that x^{μ} is mapped to infinity which does not belong to $\mathbb{R}^{d,0}$ or $\mathbb{R}^{d-1,1}$. Therefore, in order to define the finite conformal transformations globally, one considers the so-called conformal compactifications of $\mathbb{R}^{d,0}$ or $\mathbb{R}^{d-1,1}$, where additional points are included such that the conformal

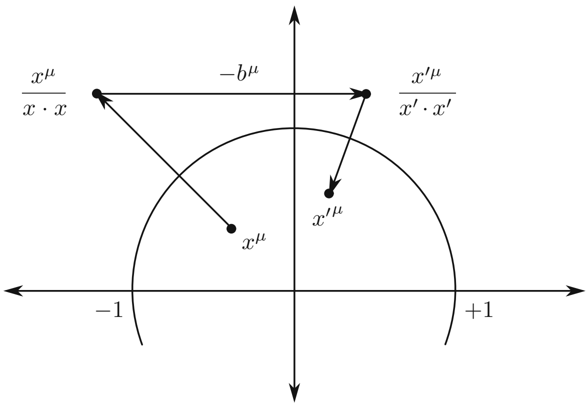


Fig. 2.2 Illustration of a finite Special Conformal Transformation

transformations are globally defined. We will not go into further detail here, but come back to this issue in Sect. 2.1.3.

The Conformal Group and Algebra

Before we identify the conformal group and the conformal algebra for the case of dimensions $d \geq 3$, let us first define these objects and point out a subtle difference.

Definition 1. *The conformal group is the group consisting of globally defined, invertible and finite conformal transformations (or more concretely, conformal diffeomorphisms).*

Definition 2. *The conformal algebra is the Lie algebra corresponding to the conformal group.*

Note that the algebra consisting of generators of infinitesimal conformal transformations contains the conformal algebra as a subalgebra, but it is larger in general. We will encounter an example of this fact in the case of two Euclidean dimensions.

Determining the Conformal Group

Let us finally determine the conformal group for dimensions $d \geq 3$. Since the group is closely related to its algebra, we will concentrate on the later. With the help of Table 2.1, we can fix the dimension of the algebra by counting the total number of generators. Keeping in mind that $L_{\mu\nu}$ is anti-symmetric, we find $N = d + 1 + \frac{d(d-1)}{2} + d = \frac{(d+2)(d+1)}{2}$. Guided by this result, we perform the definitions

$$\begin{aligned} J_{\mu,\nu} &= L_{\mu\nu} \,, & J_{-1,\mu} &= \tfrac{1}{2}\big(P_\mu - K_\mu\big) \,, \\ J_{-1,0} &= D \,, & J_{0,\mu} &= \tfrac{1}{2}\big(P_\mu + K_\mu\big) \,. \end{aligned}$$

One can then verify that $J_{m,n}$ with $m, n = -1, 0, 1, \ldots, (d-1)$ satisfy the following commutation relations:

$$\big[J_{mn}, J_{rs}\big] = i\big(\eta_{ms} J_{nr} + \eta_{nr} J_{ms} - \eta_{mr} J_{ns} - \eta_{ns} J_{mr}\big) \,. \tag{2.9}$$

For Euclidean d-dimensional space $\mathbb{R}^{d,0}$, the metric η_{mn} used above is $\eta_{mn} = \mathrm{diag}(-1, 1, \ldots, 1)$ and so we identify Eq. (2.9) as the commutation relations of the Lie algebra $\mathfrak{so}(d+1, 1)$. Similarly, in the case of $\mathbb{R}^{d-1,1}$, the metric is $\eta_{mn} = \mathrm{diag}(-1, -1, 1, \ldots, 1)$ for which Eq. (2.9) are the commutation relations of the Lie algebra $\mathfrak{so}(d, 2)$. These two examples are illustrations of the general result that

> For the case of dimensions $d = p + q \geq 3$, the conformal group of $\mathbb{R}^{p,q}$ is $SO(p+1, q+1)$.

2.1.3 Conformal Group in d = 2

Let us now study the conformal group for the special case of two dimensions. We will work with an Euclidean metric in a flat space but address the case of Lorentzian signature in the end.

Conformal Transformations

The condition (2.3) for invariance under infinitesimal conformal transformations in two dimensions reads as follows:

$$\partial_0 \epsilon_0 = +\partial_1 \epsilon_1 \ , \qquad\qquad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0 \ , \tag{2.10}$$

which we recognise as the familiar Cauchy–Riemann equations appearing in complex analysis. A complex function whose real and imaginary parts satisfy Eq. (2.10) is a holomorphic function (in some open set). We then introduce complex variables in the following way:

$$\begin{aligned} z &= x^0 + i x^1 \ , &\qquad \epsilon &= \epsilon^0 + i\epsilon^1 \ , &\qquad \partial_z &= \tfrac{1}{2}\big(\partial_0 - i\partial_1\big) \ , \\ \overline{z} &= x^0 - i x^1 \ , & \overline{\epsilon} &= \epsilon^0 - i\epsilon^1 & \partial_{\overline{z}} &= \tfrac{1}{2}\big(\partial_0 + i\partial_1\big) \ . \end{aligned}$$

Since $\epsilon(z)$ is holomorphic, so is $f(z) = z + \epsilon(z)$ from which we conclude that

> A holomorphic function $f(z) = z + \epsilon(z)$ gives rise to an infinitesimal two-dimensional conformal transformation $z \mapsto f(z)$.

This implies that the metric tensor transforms under $z \mapsto f(z)$ as follows:

$$ds^2 = dz\, d\overline{z} \ \mapsto \ \frac{\partial f}{\partial z} \frac{\partial \overline{f}}{\partial \overline{z}}\, dz\, d\overline{z} \ ,$$

from which we infer the scale factor as $\left|\frac{\partial f}{\partial z}\right|^2$.

The Witt Algebra

As we have observed above, for an infinitesimal conformal transformation in two dimensions the function $\epsilon(z)$ has to be holomorphic in some open set. However, it is reasonable to assume that $\epsilon(z)$ in general is a meromorphic function having isolated singularities outside this open set. We therefore perform a Laurent expansion of $\epsilon(z)$ around say $z = 0$. A general infinitesimal conformal transformation can then be written as

$$z' = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n\big(-z^{n+1}\big) ,$$
$$\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n\big(-\bar{z}^{n+1}\big) ,$$

where the infinitesimal parameters ϵ_n and $\bar{\epsilon}_n$ are constant. The generators corresponding to a transformation for a particular n are

$$l_n = -z^{n+1}\partial_z \qquad \text{and} \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} . \tag{2.11}$$

It is important to note that since $n \in \mathbb{Z}$, the number of independent infinitesimal conformal transformations is *infinite*. This observation is special to two dimensions and we will see that it has far-reaching consequences.

As a next step, let us compute the commutators of the generators (2.11) in order to determine the corresponding algebra. We calculate

$$\begin{aligned}
\big[l_m, l_n\big] &= z^{m+1}\partial_z\big(z^{n+1}\partial_z\big) - z^{n+1}\partial_z\big(z^{m+1}\partial_z\big) \\
&= (n+1)\, z^{m+n+1}\partial_z - (m+1)\, z^{m+n+1}\partial_z \\
&= -(m-n)\, z^{m+n+1}\partial_z \\
&= (m-n)\, l_{m+n} , \qquad\qquad (2.12)\\
\big[\bar{l}_m, \bar{l}_n\big] &= (m-n)\, \bar{l}_{m+n} , \\
\big[l_m, \bar{l}_n\big] &= 0 .
\end{aligned}$$

The first commutation relations define one copy of the so-called *Witt algebra*, and because of the other two relations, there is a second copy which commutes with the first one. We can then summarise our findings as follows:

> The algebra of infinitesimal conformal transformations in an Euclidean two-dimensional space is *infinite* dimensional.

Note that, since we can identify two independent copies of the Witt algebra generated by Eq. (2.11), it is customary to treat z and $\bar{z}$ as independent variables which means that we are effectively considering $\mathbb{C}^2$ instead of $\mathbb{C}$. We will come back to this point in Sect. 2.2.

Global Conformal Transformations

Let us now focus on the copy of the Witt algebra generated by $\{l_n\}$ and observe that on the Euclidean plane $\mathbb{R}^2 \simeq \mathbb{C}$, the generators l_n are not everywhere defined. In particular, there is an ambiguity at $z = 0$ and it turns out to be necessary not to work on $\mathbb{C}$ but on the Riemann sphere $S^2 \simeq \mathbb{C}\cup\{\infty\}$ being the conformal compactification of $\mathbb{R}^2$.

But even on the Riemann sphere, not all of the generators (2.11) are well defined. For $z = 0$, we find that

$$l_n = -z^{n+1}\partial_z, \quad \text{non-singular at } z = 0 \text{ only for } n \geq -1 \; .$$

The other ambiguous point is $z = \infty$ which is, however, part of the Riemann sphere S^2. To investigate the behaviour of l_n there, let us perform the change of variable $z = -\frac{1}{w}$ and study $w \to 0$. We then observe that

$$l_n = -\left(-\frac{1}{w}\right)^{n-1} \partial_w, \quad \text{non-singular at } w = 0 \text{ only for } n \leq +1 \; ,$$

where we employed that $\partial_z = (-w)^2\partial_w$. We therefore arrive at the conclusion that

> Globally defined conformal transformations on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$ are generated by $\{l_{-1}, l_0, l_{+1}\}$.

The Conformal Group

After having determined the operators generating global conformal transformations, we will now determine the conformal group.

- As it is clear from its definition, the operator l_{-1} generates translations $z \mapsto z+b$.
- For the operator l_0, let us recall that $l_0 = -z\,\partial_z$. Therefore, l_0 generates transformations $z \mapsto a\,z$ with $a \in \mathbb{C}$. In order to get a geometric intuition of such transformations, we perform the change of variables $z = re^{i\phi}$ to find

$$l_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\phi \; , \qquad \bar{l}_0 = -\frac{1}{2}r\partial_r - \frac{i}{2}\partial_\phi \; .$$

  Out of those, we can form the linear combinations

$$l_0 + \bar{l}_0 = -r\partial_r \qquad \text{and} \qquad i\big(l_0 - \bar{l}_0\big) = -\partial_\phi \; , \tag{2.13}$$

  and so we see that $l_0 + \bar{l}_0$ is the generator for two-dimensional dilations and that $i(l_0 - \bar{l}_0)$ is the generator of rotations.
- Finally, the operator l_{+1} corresponds to Special Conformal Transformations which are translations for the variable $w = -\frac{1}{z}$. Indeed, $c\,l_1 z = -c\,z^2$ is the infinitesimal version of the transformations $z \mapsto \frac{z}{c\,z+1}$ which corresponds to $w \mapsto w - c$.

In summary, we have argued that the operators $\{l_{-1}, l_0, l_{+1}\}$ generate transformations of the form

$$z \mapsto \frac{az+b}{cz+d} \qquad \text{with} \qquad a,b,c,d \in \mathbb{C}\,. \tag{2.14}$$

For this transformation to be invertible, we have to require that $ad - bc$ is non-zero. If this is the case, we can scale the constants a, b, c, d such that $ad - bc = 1$. Furthermore, note that the expression above is invariant under $(a, b, c, d) \mapsto (-a, -b, -c, -d)$. From the conformal transformations (2.14) together with these restrictions, we can then infer that

> The conformal group of the Riemann sphere $S^2 = \mathbb{C} \cup \infty$ is the Möbius group $SL(2, \mathbb{C})/\mathbb{Z}_2$.

Virasoro Algebra

Let us now come back to the Witt algebra of infinitesimal conformal transformations. As it turns out, this algebra admits a so-called central extension. Without providing a mathematically rigorous definition, we state that the central extension $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ of a Lie algebra $\mathfrak{g}$ by $\mathbb{C}$ is characterised by the commutation relations

$$\begin{aligned}
\big[\widetilde{x}, \widetilde{y}\big]_{\widetilde{\mathfrak{g}}} &= \big[x, y\big]_{\mathfrak{g}} + c\, p(x,y)\,, && \widetilde{x}, \widetilde{y} \in \widetilde{\mathfrak{g}}\,,\\
\big[\widetilde{x}, c\big]_{\widetilde{\mathfrak{g}}} &= 0\,, && x, y \in \mathfrak{g}\,,\\
\big[c, c\big]_{\widetilde{\mathfrak{g}}} &= 0\,, && c \in \mathbb{C}\,,
\end{aligned}$$

where $p : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is bilinear. Central extensions of algebras are closely related to projective representations which are common to Quantum Mechanics. In the following, we are going to allow for such additional structure.

More concretely, let us denote the elements of the central extension of the Witt algebra by L_n with $n \in \mathbb{Z}$ and write their commutation relations as

$$\big[L_m, L_n\big] = (m-n)\, L_{m+n} + cp(m,n)\,. \tag{2.15}$$

Of course, a similar analysis can be carried out for the generators $\bar{l}_n \leftrightarrow \overline{L}_n$. The precise form of $p(m, n)$ is determined in the following way:

- First, from Eq. (2.15) it is clear that $p(m, n) = -p(n, m)$ in order for $p(m, n)$ to be compatible with the anti-symmetry of the Lie bracket.
- We also observe that one can always arrange for $p(1, -1) = 0$ and $p(n, 0) = 0$ by a redefinition

$$\widehat{L}_n = L_n + \frac{cp(n,0)}{n} \qquad \text{for} \qquad n \neq 0\,, \qquad\qquad \widehat{L}_0 = L_0 + \frac{cp(1,-1)}{2}\,.$$

Indeed, for the modified generators we see that the $p(n, m)$ vanishes

$$\begin{aligned}\left[\widehat{L}_n, \widehat{L}_0\right] &= n\,L_n + cp(n,0) &= n\,\widehat{L}_n\ ,\\ \left[\widehat{L}_1, \widehat{L}_{-1}\right] &= 2\,L_0 + cp(1,-1) &= 2\,\widehat{L}_0\ .\end{aligned}$$

- Next, we compute the following particular Jacobi identity:

$$\begin{aligned}0 &= \big[\big[L_m, L_n\big], L_0\big] + \big[\big[L_n, L_0\big], L_m\big] + \big[\big[L_0, L_m\big], L_n\big]\\ 0 &= (m-n)\,cp(m+n,0) + n\,cp(n,m) - m\,cp(m,n)\\ 0 &= (m+n)\,p(n,m)\end{aligned}$$

  from which we infer that in the case $n \neq -m$, we have $p(n,m) = 0$. Therefore, the only non-vanishing central extensions are $p(n,-n)$ for $|n| \geq 2$.
- We finally calculate the following Jacobi identity:

$$\begin{aligned}0 &= \big[\big[L_{-n+1}, L_n\big], L_{-1}\big] + \big[\big[L_n, L_{-1}\big], L_{-n+1}\big] + \big[\big[L_{-1}, L_{-n+1}\big], L_n\big]\\ 0 &= (-2n+1)\,cp(1,-1) + (n+1)\,cp(n-1,-n+1) + (n-2)\,cp(-n,n)\end{aligned}$$

  which leads to a recursion relation of the form

$$\begin{aligned}p(n,-n) &= \frac{n+1}{n-2}\,p(n-1,-n+1) = \ldots\\ &= \frac{1}{2}\binom{n+1}{3} = \frac{1}{12}\,(n+1)n(n-1)\end{aligned}$$

  where we have normalised $p(2,-2) = \frac{1}{2}$. This normalisation is chosen such that the constant c has a particular value for the standard example of the free boson which we will study in Sect. 2.9.1.

The central extension of the Witt algebra is called the Virasoro algebra and the constant c is called the central charge. In summary,

> The Virasoro algebra Vir_c with central charge c has the commutation relations
>
> $$\big[L_m, L_n\big] = (m-n)\,L_{m+n} + \frac{c}{12}\,(m^3 - m)\,\delta_{m+n,0}\ . \qquad (2.16)$$

Remarks

- Without providing a rigorous mathematical definition, we note that above we have computed the second cohomology group H^2 of the Witt algebra. It is generally true that $H^2(\mathfrak{g}, \mathbb{C})$ classifies the central extensions of an algebra $\mathfrak{g}$ modulo redefinitions of the generators. However, for semi-simple finite dimensional Lie algebras, one finds that their second cohomology group vanishes and so in this case there do not exist any central extensions.

- Since $p(m,n) = 0$ for $m,n = -1, 0, +1$, it is still true that L_{-1} generates translations, L_0 generates dilations and rotations, and that L_{+1} generates Special Conformal Transformations. Therefore, also $\{L_{-1}, L_0, L_1\}$ are generators of $SL(2,\mathbb{C})/\mathbb{Z}_2$ transformations. This just reflects the above mentioned fact that the finite-dimensional Lie algebras do not have any non-trivial central extensions.
- By computing $||L_{-m}|\,0\rangle||^2 = \langle 0\,|L_{+m}L_{-m}|\,0\rangle = \frac{c}{12}\,(m^3 - m)$, one observes that only for $c \neq 0$ there exist non-trivial representations of the Virasoro algebra. We have not yet provided the necessary techniques to perform this calculation but we will do so in the rest of this chapter.
- In this section, we have determined the conformal transformations and the conformal algebra for two-dimensional Euclidean space. However, for a two-dimensional flat space–time with Lorentzian signature, one can perform a similar analysis. To do so, one defines light-cone coordinates $u = -t + x$ and $v = +t + x$ where t denotes the time direction and x the space direction. In these variables, we find

  $$ds^2 = -dt^2 + dx^2 = du\,dv\ ,$$

  and conformal transformations are given by $u \mapsto f(u)$ and $v \mapsto g(v)$ leading to $ds'^2 = \partial_u f\,\partial_v g\,du\,dv$. Therefore, the algebra of infinitesimal conformal transformations is again infinite dimensional.

2.2 Primary Fields

In this section, we will establish some basic definitions for two-dimensional conformal field theories in Euclidean space.

Complexification

Let us start with an Euclidean two-dimensional space $\mathbb{R}^2$ and perform the natural identification $\mathbb{R}^2 \simeq \mathbb{C}$ by introducing complex variables $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$. From Eq. (2.12), we have seen that we can identify two commuting copies of the Witt algebra which naturally extend to the Virasoro algebra. Since the generators of the Witt algebras are expressed in terms of z and $\bar{z}$, it turns out to be convenient to consider them as two independent complex variables. For the fields ϕ of our theory, this complexification $\mathbb{R}^2 \to \mathbb{C}^2$ means

$$\phi\big(x^0, x^1\big) \quad\longrightarrow\quad \phi\big(z, \bar{z}\big)\ ,$$

where $\{x^0, x^1\} \in \mathbb{R}^2$ and $\{z, \bar{z}\} \in \mathbb{C}^2$. However, note that at some point we have to identify $\bar{z}$ with the complex conjugate z^* of z.

Definition of Chiral and Primary Fields

Definition 3. *Fields only depending on z, i.e. $\phi(z)$, are called chiral fields and fields $\phi(\bar{z})$ only depending on $\bar{z}$ are called anti-chiral fields. It is also common to use the terminology holomorphic and anti-holomorphic in order to distinguish between chiral and anti-chiral quantities.*

Definition 4. *If a field $\phi(z,\bar{z})$ transforms under scalings $z \mapsto \lambda z$ according to*

$$\phi\big(z,\bar{z}\big) \mapsto \phi'\big(z,\bar{z}\big) = \lambda^h\,\bar{\lambda}^{\bar{h}}\,\phi\big(\lambda\,z,\bar{\lambda}\bar{z}\big)\;,$$

it is said to have conformal dimensions $(h,\bar{h})$.

Definition 5. *If a field transforms under conformal transformations $z \mapsto f(z)$ according to*

$$\phi\big(z,\bar{z}\big) \mapsto \phi'\big(z,\bar{z}\big) = \Big(\frac{\partial f}{\partial z}\Big)^h \Big(\frac{\partial \bar{f}}{\partial \bar{z}}\Big)^{\bar{h}} \phi\big(f(z),\bar{f}(\bar{z})\big)\;, \qquad (2.17)$$

it is called a primary field of conformal dimension $(h,\bar{h})$. If Eq. (2.17) *holds only for $f \in SL(2,\mathbb{C})/\mathbb{Z}_2$, i.e. only for global conformal transformations, then ϕ is called a quasi-primary field.*

Note that a primary field is always quasi-primary but the reverse is not true. Furthermore, not all fields in a CFT are primary or quasi-primary. Those fields are called secondary fields.

Infinitesimal Conformal Transformations of Primary Fields

Let us now investigate how a primary field $\phi(z,\bar{z})$ behaves under infinitesimal conformal transformations. To do so, we consider the map $f(z) = z + \epsilon(z)$ with $\epsilon(z) \ll 1$ and compute the following quantities up to first order in $\epsilon(z)$:

$$\Big(\frac{\partial f}{\partial z}\Big)^h = 1 + h\,\partial_z\epsilon(z) + \mathcal{O}(\epsilon^2)\;,$$

$$\phi\big(z+\epsilon(z),\bar{z}\big) = \phi(z) + \epsilon(z)\,\partial_z\phi(z,\bar{z}) + \mathcal{O}(\epsilon^2)\;.$$

Using these two expressions in the definition of a primary field (2.17), we obtain

$$\phi(z,\bar{z}) \mapsto \phi(z,\bar{z}) + \Big(h\,\partial_z\epsilon + \epsilon\partial_z + \bar{h}\,\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\Big)\phi(z,\bar{z})\;,$$

and so the transformation of a primary field under infinitesimal conformal transformations reads

$$\boxed{\delta_{\epsilon,\bar{\epsilon}}\,\phi(z,\bar{z}) = \Big(h\,\partial_z\epsilon + \epsilon\,\partial_z + \bar{h}\,\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\,\partial_{\bar{z}}\Big)\,\phi(z,\bar{z})}\;. \qquad (2.18)$$

2.3 The Energy–Momentum Tensor

Usually, a Field Theory is defined in terms of a Lagrangian action from which one can derive various objects and properties of the theory. In particular, the energy–momentum tensor can be deduced from the variation of the action with respect to the metric and so it encodes the behaviour of the theory under infinitesimal transformations $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$ with $\delta g_{\mu\nu} \ll 1$.

Since the algebra of infinitesimal conformal transformations in two dimensions is infinite dimensional, there are strong constraints on a conformal field theory. In particular, it turns out to be possible to study such a theory without knowing the explicit form of the action. The only information needed is the behaviour under conformal transformations which is encoded in the energy–momentum tensor.

Implication of Conformal Invariance

In order to study the energy–momentum tensor for CFTs, let us recall Noether's theorem which states that for every continuous symmetry in a Field Theory, there is a current j_μ which is conserved, i.e. $\partial^\mu j_\mu = 0$. Since we are interested in theories with a conformal symmetry $x^\mu \mapsto x^\mu + \epsilon^\mu(x)$, we have a conserved current which can be written as

$$j_\mu = T_{\mu\nu}\, \epsilon^\nu \, , \tag{2.19}$$

where the tensor $T_{\mu\nu}$ is symmetric and is called the energy–momentum tensor. Since this current is preserved, we obtain for the special case $\epsilon^\mu = \text{const.}$ that

$$0 = \partial^\mu j_\mu = \partial^\mu \left(T_{\mu\nu}\, \epsilon^\nu\right) = \left(\partial^\mu T_{\mu\nu}\right) \epsilon^\nu \qquad \Rightarrow \qquad \partial^\mu T_{\mu\nu} = 0 \, . \tag{2.20}$$

For more general transformations $\epsilon^\mu(x)$, the conservation of the current (2.19) implies the following relation:

$$\begin{aligned} 0 = \partial^\mu j_\mu &= \left(\partial^\mu T_{\mu\nu}\right)\epsilon^\nu + T_{\mu\nu}\left(\partial^\mu \epsilon^\nu\right) \\ &= 0 + \frac{1}{2}\, T_{\mu\nu}\left(\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu\right) = \frac{1}{2}\, T_{\mu\nu}\eta^{\mu\nu}\left(\partial \cdot \epsilon\right) \frac{2}{d} = \frac{1}{d}\, T_\mu{}^\mu \left(\partial \cdot \epsilon\right) \, , \end{aligned}$$

where we used Eq. (2.3) and the fact that $T_{\mu\nu}$ is symmetric. Since this equation has to be true for arbitrary infinitesimal transformations $\epsilon(z)$, we conclude

> In a conformal field theory, the energy–momentum tensor $T_{\mu\nu}$ is traceless, that is, $T_\mu{}^\mu = 0$.

Specialising to Two Euclidean Dimensions

Let us now investigate the consequences of a traceless energy–momentum tensor for two-dimensional CFTs in the case of Euclidean signature. To do so, we again perform the change of coordinates from the real to the complex ones shown on p. 12. Using then $T_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu} T_{\alpha\beta}$ for $x^0 = \frac{1}{2}(z+\bar{z})$ and $x^1 = \frac{1}{2i}(z-\bar{z})$, we find

$$\begin{aligned} T_{zz} &= \frac{1}{4}\big(T_{00} - 2i\,T_{10} - T_{11}\big)\,, \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}\big(T_{00} + 2i\,T_{10} - T_{11}\big)\,, \end{aligned} \qquad T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}\big(T_{00} + T_{11}\big) = \frac{1}{4}T_\mu{}^\mu = 0\,,$$

where for $T_{z\bar{z}}$ we used that $\eta_{\mu\nu} = \mathrm{diag}(+1,+1)$ together with $T_\mu{}^\mu = 0$. Employing the latter relation also for the left-hand side, we obtain

$$T_{zz} = \frac{1}{2}\big(T_{00} - i\,T_{10}\big)\,, \qquad T_{\bar{z}\bar{z}} = \frac{1}{2}\big(T_{00} + i\,T_{10}\big)\,. \tag{2.21}$$

Using finally the condition for translational invariance (2.20), we find

$$\partial_0 T_{00} + \partial_1 T_{10} = 0\,, \qquad \partial_0 T_{01} + \partial_1 T_{11} = 0\,, \tag{2.22}$$

from which it follows that

$$\partial_{\bar{z}} T_{zz} = \frac{1}{4}\big(\partial_0 + i\partial_1\big)\big(T_{00} - iT_{10}\big) = \frac{1}{4}\Big(\partial_0 T_{00} + \partial_1 T_{10} + i\partial_1 \underbrace{T_{00}}_{=-T_{11}} - i\partial_0 \underbrace{T_{10}}_{=T_{01}}\Big) = 0\,,$$

where we used Eq. (2.22) and $T_\mu{}^\mu = 0$. Similarly, one can show that $\partial_z T_{\bar{z}\bar{z}} = 0$ which leads us to the conclusion that

> The two non-vanishing components of the energy–momentum tensor are a *chiral* and an *anti-chiral* field
>
> $$T_{zz}(z,\bar{z}) =: T(z)\,, \qquad T_{\bar{z}\bar{z}}(z,\bar{z}) =: \bar{T}(\bar{z})\,.$$

2.4 Radial Quantisation

Motivation and Notation

In the following, we will focus our studies on conformal field theories defined on Euclidean two-dimensional flat space. Although this choice is arbitrary, for concreteness we denote the Euclidean time direction by x^0 and the Euclidean space direction by x^1. Furthermore, note that theories with a Lorentzian signature can be obtained from the Euclidean ones via a Wick rotation $x^0 \to ix^0$.

Next, we compactify the Euclidean space direction x^1 on a circle of radius R which we will mostly choose as $R = 1$. The CFT we obtain in this way is thus defined on a cylinder of infinite length for which we introduce the complex coordinate w defined as

$$w = x^0 + ix^1 , \qquad\qquad w \sim w + 2\pi i ,$$

where we also indicated the periodic identification. Let us emphasise that the theory on the cylinder is the starting point for our following analysis. This is also natural from a string theory point of view, since the world-sheet of a closed string in Euclidean coordinates is a cylinder.

Mapping the Cylinder to the Complex Plane

After having explained our initial theory, we now introduce the concept of radial quantisation of a two-dimensional Euclidean CFT. To do so, we perform a change of variables by mapping the cylinder to the complex plane in order to employ the power of complex analysis. In particular, this mapping is achieved by

$$z = e^w = e^{x^0} \cdot e^{ix^1} , \tag{2.23}$$

which is a map from an infinite cylinder described by x^0 and x^1 to the complex plane described by z (see Fig. 2.3). The former time translations $x^0 \mapsto x^0 + a$ are then mapped to complex dilation $z \mapsto e^a z$ and the space translations $x^1 \mapsto x^1 + b$ are mapped to rotations $z \mapsto e^{ib} z$.

As it is known from Quantum Mechanics, the generator of time translations is the Hamiltonian which in the present case corresponds to the dilation operator.

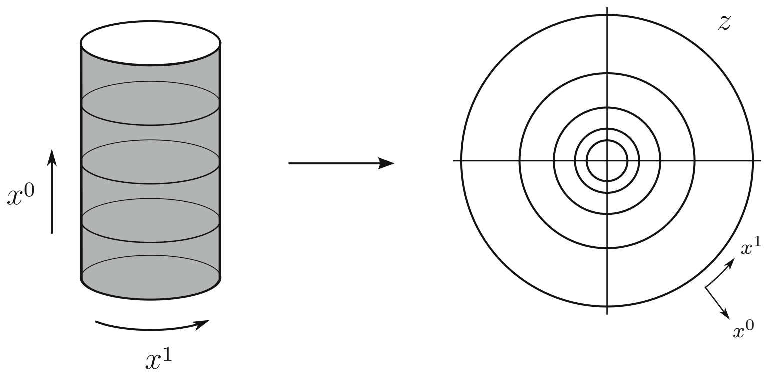


Fig. 2.3 Mapping the cylinder to the complex plane

Similarly, the generator for space translations is the momentum operator corresponding to rotations. Recalling Eq. (2.13) together with the observation that the central extension for L_0 and $\overline{L}_0$ vanishes, we find that

$$H = L_0 + \overline{L}_0 \,, \qquad\qquad P = i\big(L_0 - \overline{L}_0\big) \,. \tag{2.24}$$

Asymptotic States

We now consider a field $\phi(z, \overline{z})$ with conformal dimensions $(h, \overline{h})$ for which we can perform a Laurent expansion around $z_0 = \overline{z}_0 = 0$

$$\phi(z, \overline{z}) = \sum_{n,\overline{m}\in\mathbb{Z}} z^{-n-h}\, \overline{z}^{-\overline{m}-\overline{h}}\, \phi_{n,\overline{m}} \,, \tag{2.25}$$

where the additional factors of h and $\overline{h}$ in the exponents can be explained by the map (2.23), but also lead to scaling dimensions $(n, \overline{m})$ for $\phi_{n,\overline{m}}$. The quantisation of this field is achieved by promoting the Laurent modes $\phi_{n,\overline{m}}$ to operators. This procedure can be motivated by considering the theory on the cylinder and performing a Fourier expansion of $\phi(x^0, x^1)$. As usual, upon quantisation the Fourier modes are considered to be operators which, after mapping to the complex plane, agree with the approach above.

Next, we note that via Eq. (2.23) the infinite past on the cylinder $x^0 = -\infty$ is mapped to $z = \overline{z} = 0$. This motivates the definition of an asymptotic *in*-state $|\phi\rangle$ to be of the following form:

$$\big|\phi\big\rangle = \lim_{z,\overline{z}\to 0} \phi(z, \overline{z})\, \big|0\big\rangle \,. \tag{2.26}$$

However, in order for Eq. (2.26) to be non-singular at $z = 0$, that is, to be a well-defined asymptotic *in*-state, we require

$$\phi_{n,\overline{m}}\, \big|0\big\rangle = 0 \qquad \text{for} \qquad n > -h, \quad \overline{m} > -\overline{h} \,. \tag{2.27}$$

Using this restriction together with the mode expansion (2.25), we can simplify Eq. (2.26) in the following way:

$$\boxed{\big|\phi\big\rangle = \lim_{z,\overline{z}\to 0} \phi(z, \overline{z})\, \big|0\big\rangle = \phi_{-h,-\overline{h}}\, \big|0\big\rangle} \,. \tag{2.28}$$

Hermitian Conjugation

In order to obtain the hermitian conjugate $\phi^\dagger$ of a field ϕ, we note that in Euclidean space there is a non-trivial action on the Euclidean time $x^0 = it$ upon hermitian conjugation. Because of the complex conjugation, we find $x^0 \mapsto -x^0$ while the Euclidean space coordinate x^1 is left invariant. For the complex coordinate $z =$

$\exp(x^0+ix^1)$, hermitian conjugation thus translates to $z \mapsto 1/\bar{z}$, where we identified $\bar{z}$ with the complex conjugate z^* of z. We then define the hermitian conjugate of a field ϕ as

$$\phi^\dagger(z,\bar{z}) = \bar{z}^{-2h}\, z^{-2\bar{h}}\, \phi\left(\frac{1}{\bar{z}}\,,\, \frac{1}{z}\right) . \tag{2.29}$$

Performing a Laurent expansion of the hermitian conjugate field $\phi^\dagger$ gives us the following result:

$$\phi^\dagger(z,\bar{z}) = \bar{z}^{-2h}\, z^{-2\bar{h}} \sum_{n,\overline{m}\in\mathbb{Z}} \bar{z}^{+n+h}\, z^{+\overline{m}+\bar{h}}\, \phi_{n,\overline{m}} = \sum_{n,\overline{m}\in\mathbb{Z}} \bar{z}^{+n-h}\, z^{+\overline{m}-\bar{h}}\, \phi_{n,\overline{m}}\ , \tag{2.30}$$

and if we compare this expression with the hermitian conjugate of Eq. (2.25), we see that for the Laurent modes we find

$$\boxed{\left(\phi_{n,\overline{m}}\right)^\dagger = \phi_{-n,-\overline{m}}\ .} \tag{2.31}$$

Let us finally define a relation similar as Eq. (2.26) for an asymptotic *out*-states of a CFT. Naturally, this is achieved by using the hermitian conjugate field which reads

$$\langle\phi| = \lim_{z,\bar{z}\to 0} \langle 0|\, \phi^\dagger(z,\bar{z}) = \lim_{\bar{w},w\to\infty} w^{2h}\, \bar{w}^{2\bar{h}} \langle 0|\, \phi(w,\bar{w})\ ,$$

where we employed Eq. (2.29) together with $\bar{z} = w^{-1}$ and $z = \bar{w}^{-1}$. However, by the same reasoning as above, in order for the asymptotic *out*-state to be well defined, we require

$$\langle 0|\, \phi_{n,\overline{m}} = 0 \qquad \text{for} \qquad n < h, \quad \overline{m} < \bar{h}\ .$$

Recalling for instance Eq. (2.30), we can then simplify the definition of the *out*-state as follows:

$$\boxed{\langle\phi| = \lim_{\bar{w},w\to\infty} w^{2h}\, \bar{w}^{2\bar{h}} \langle 0|\, \phi(w,\bar{w}) = \langle 0|\, \phi_{+h,+\bar{h}}}\ . \tag{2.32}$$

2.5 The Operator Product Expansion

In this section, we will study in more detail the energy–momentum tensor and thereby introduce the operator formalism for two-dimensional conformal field theories.

Conserved Charges

To start, let us recall that since the current $j_\mu = T_{\mu\nu}\epsilon^\nu$ associated to the conformal symmetry is preserved, there exists a conserved charge which is expressed in the following way:

$$Q = \int dx^1 \, j_0 \qquad \text{at} \qquad x^0 = \text{const.} \, . \tag{2.33}$$

In Field Theory, this conserved charge is the generator of symmetry transformations for an operator A which can be written as

$$\delta A = \big[Q, A \big] \, ,$$

where the commutator is evaluated at equal times. From the change of coordinates (2.23), we infer that $x^0 = \text{const.}$ corresponds to $|z| = \text{const.}$ and so the integral over space $\int dx^1$ in Eq. (2.33) gets transformed into a contour integral. With the convention that contour integrals $\oint dz$ are always counter clockwise, the natural generalisation of the conserved charge (2.33) to complex coordinates reads

$$Q = \frac{1}{2\pi i} \oint_C \Big(dz \, T(z) \, \epsilon(z) + d\overline{z} \, \overline{T}(\overline{z}) \, \overline{\epsilon}(\overline{z}) \Big) \, . \tag{2.34}$$

This expression allows us now to determine the infinitesimal transformation of a field $\phi(z, \overline{z})$ generated by a conserved charge Q. To do so, we compute $\delta\phi = [Q, \phi]$ which, using Eq. (2.34), becomes

$$\delta_{\epsilon,\overline{\epsilon}} \, \phi(w, \overline{w}) = \frac{1}{2\pi i} \oint_C dz \left[T(z)\epsilon(z), \phi(w, \overline{w}) \right] + \frac{1}{2\pi i} \oint_C d\overline{z} \left[\overline{T}(\overline{z})\overline{\epsilon}(\overline{z}), \phi(w, \overline{w}) \right] . \tag{2.35}$$

Radial Ordering

Note that there is some ambiguity in Eq. (2.35) because we have to decide whether w and $\overline{w}$ are inside or outside the contour $\mathcal{C}$. However, from quantum field theory we know that correlation functions are only defined as a time ordered product. Considering the change of coordinates (2.23), in a CFT the time ordering becomes a radial ordering and thus the product $A(z)B(w)$ does only make sense for $|z| > |w|$. To this end, we define the radial ordering of two operators as

$$R\big(\, A(z) \, B(w) \, \big) := \begin{cases} A(z) \, B(w) \text{ for } |z| > |w| \, , \\ B(w) \, A(z) \text{ for } |w| > |z| \, . \end{cases}$$

With this definition, it is clear that we have to interpret an expression such as Eq. (2.35) in the following way:

$$\begin{aligned}\oint dz\big[A(z), B(w)\big] &= \oint_{|z|>|w|} dz\, A(z)\, B(w) - \oint_{|z|<|w|} dz\, B(w)\, A(z) \\ &= \oint_{\mathcal{C}(w)} dz\, R\Big(A(z)\, B(w)\Big)\,,\end{aligned} \tag{2.36}$$

where we employed the relation among contour integrals shown in Fig. 2.4. With the help of this observation, we can express Eq. (2.35) as

$$\delta_{\epsilon,\bar{\epsilon}}\, \phi(w,\overline{w}) = \frac{1}{2\pi i} \oint_{\mathcal{C}(w)} dz\, \epsilon(z)\, R\Big(T(z)\phi(w,\overline{w})\Big) + \text{anti-chiral}\,. \tag{2.37}$$

However, we have already computed this quantity for a primary field at the end of Sect. 2.2. By comparing with our previous result (2.18), that is,

$$\delta_{\epsilon,\bar{\epsilon}}\, \phi(w,\overline{w}) = h\, \big(\partial_w \epsilon(w)\big)\, \phi(w,\overline{w}) + \epsilon(w)\, \big(\partial_w\, \phi(w,\overline{w})\big) + \text{anti-chiral}\,,$$

we can deduce a relation for the radial ordering of the energy–momentum tensor and a primary field. In particular, employing the identities

$$\begin{aligned} h\, \big(\partial_w \epsilon(w)\big)\, \phi(w,\overline{w}) &= \frac{1}{2\pi i} \oint_{\mathcal{C}(w)} dz\, h\, \frac{\epsilon(z)}{(z-w)^2}\, \phi(w,\overline{w})\,, \\ \epsilon(w)\, \big(\partial_w \phi(w,\overline{w})\big) &= \frac{1}{2\pi i} \oint_{\mathcal{C}(w)} dz\, \frac{\epsilon(z)}{z-w}\, \partial_w \phi(w,\overline{w})\,, \end{aligned} \tag{2.38}$$

for a bi-holomorphic field $\phi(w,\overline{w})$, we obtain that

$$R\Big(T(z)\phi(w,\overline{w})\Big) = \frac{h}{(z-w)^2}\, \phi(w,\overline{w}) + \frac{1}{z-w}\, \partial_w \phi(w,\overline{w}) + \dots\,, \tag{2.39}$$

where the ellipsis denote non-singular terms. An expression like (2.39) is called an *operator product expansion* (OPE) which defines an algebraic product structure on the space of quantum fields.

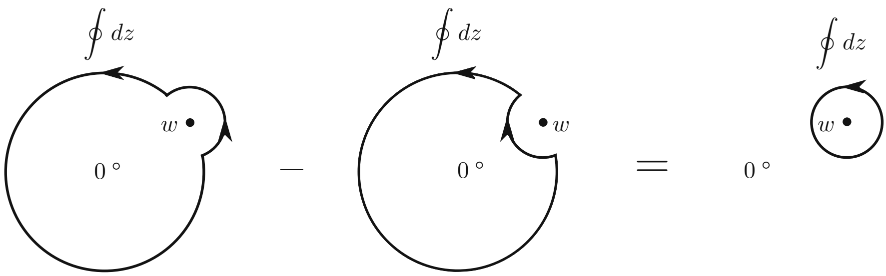


Fig. 2.4 Sum of contour integrals corresponding to Eq. (2.36)

To ease our notation, in the following, we will always assume radial ordering for a product of fields, i.e. we write $A(z)B(w)$ instead of $R(A(z)B(w))$. Furthermore, with the help of Eq. (2.39) we can give an alternative definition of a primary field:

Definition 6. *A field $\phi(z,\bar{z})$ is called primary with conformal dimensions $(h,\bar{h})$, if the operator product expansion between the energy–momentum tensors and $\phi(z,\bar{z})$ takes the following form:*

$$\begin{aligned} T(z)\,\phi(w,\bar{w}) &= \frac{h}{(z-w)^2}\,\phi(w,\bar{w}) + \frac{1}{z-w}\,\partial_w\phi(w,\bar{w}) + \ldots\;, \\ \bar{T}(\bar{z})\,\phi(w,\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\,\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\,\partial_{\bar{w}}\phi(w,\bar{w}) + \ldots\;, \end{aligned} \tag{2.40}$$

where the ellipsis denote non-singular terms.

Operator Product Expansion of the Energy–Momentum Tensor

After having defined the operator product expansion, let us now consider the example of the energy–momentum tensor. We first state that

The OPE of the chiral energy–momentum tensor with itself reads

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \ldots \tag{2.41}$$

where c denotes the central charge and $|z| > |w|$.

A similar result holds for the anti-chiral part $\bar{T}(\bar{z})$, and the OPE $T(z)\,\bar{T}(\bar{w})$ contains only non-singular terms.

Let us now prove the statement (2.41). To do so, we perform a Laurent expansion of $T(z)$ in the following way:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad \text{where} \qquad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}\, T(z)\,. \tag{2.42}$$

If we use this expansion for the conserved charge (2.34) and choose a particular conformal transformation $\epsilon(z) = -\epsilon_n z^{n+1}$, we find that

$$Q_n = \oint \frac{dz}{2\pi i}\, T(z)\left(-\epsilon_n z^{n+1}\right) = -\epsilon_n \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i}\, L_m\, z^{n-m-1} = -\epsilon_n\, L_n\;.$$

Referring to our discussion in Sect. 2.1.3, we can thus identify the Laurent modes L_m of the energy–momentum tensor (2.42) with the generators of infinitesimal conformal transformations. As such, they have to satisfy the Virasoro algebra for which we calculate

$$
\begin{aligned}
\big[L_m, L_n\big] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}\, z^{m+1}\, w^{n+1} \big[T(z), T(w)\big] \\
&= \oint_{C(0)} \frac{dw}{2\pi i}\, w^{n+1} \oint_{C(w)} \frac{dz}{2\pi i}\, z^{m+1} R\Big(T(z)T(w)\Big) \\
&= \oint_{C(0)} \frac{dw}{2\pi i}\, w^{n+1} \oint_{C(w)} \frac{dz}{2\pi i}\, z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \right) \\
&= \oint_{C(0)} \frac{dw}{2\pi i}\, w^{n+1} \Big((m+1)m(m-1)\, w^{m-2}\, \frac{c}{2 \cdot 3!} \\
&\quad + 2\,(m+1)\, w^m\, T(w) + w^{m+1} \partial_w T(w) \Big) \\
&= \oint \frac{dw}{2\pi i} \left(\frac{c}{12}\big(m^3 - m\big)\, w^{m+n-1} \right. \\
&\quad \left. + 2\,(m+1)\, w^{m+n+1}\, T(w) + w^{m+n+2}\, \partial_w T(w) \right) \\
&= \frac{c}{12}\big(m^3 - m\big)\, \delta_{m,-n} + 2\,(m+1)\, L_{m+n} \\
&\quad + 0 - \underbrace{\oint \frac{dw}{2\pi i}\, (m+n+2)\, T(w)\, w^{m+n+1}}_{=\,(m+n+2)\, L_{m+n}} \\
&= (m-n)\, L_{m+n} + \frac{c}{12}\big(m^3 - m\big)\, \delta_{m,-n} \;,
\end{aligned}
$$

where we performed an integration by parts to evaluate the $\partial_w T(w)$ term. Therefore, we have shown that Eq. (2.41) is the correct form of the OPE between two energy–momentum tensors.

Conformal Transformations of the Energy–Momentum Tensor

To end this section, we will investigate the behaviour of the energy–momentum tensor under conformal transformations. In particular, by comparing the OPE (2.41) with the definition (2.40), we see that for non-vanishing central charges, $T(z)$ is not a primary field. But, one can show that under conformal transformations $f(z)$, the energy–momentum tensor behaves as

$$
T'\big(z\big) = \left(\frac{\partial f}{\partial z}\right)^2 T\big(f(z)\big) + \frac{c}{12}\, S\big(f(z), z\big) \;, \tag{2.43}
$$

where $S(w, z)$ denotes the Schwarzian derivative

$$S(w,z)=\frac{1}{(\partial_z w)^2}\Big(\big(\partial_z w\big)\big(\partial_z^3 w\big)-\frac{3}{2}\,\big(\partial_z^2 w\big)^2\Big)\,.$$

We will not prove Eq. (2.43) in detail but verify it on the level of infinitesimal conformal transformations $f(z)=z+\epsilon(z)$. In order to do so, we first use Eq. (2.37) with the OPE of the energy–momentum tensor (2.41) to find

$$\begin{aligned}\delta_\epsilon\, T(z) &= \frac{1}{2\pi i}\oint_{C(z)} dw\,\epsilon(w)\,T(w)\,T(z)\\ &= \frac{1}{2\pi i}\oint_{C(z)} dw\,\epsilon(w)\,\left(\frac{c/2}{(w-z)^4}+\frac{2\,T(z)}{(w-z)^2}+\frac{\partial_z T(z)}{w-z}+\ldots\right)\\ &= \frac{c}{12}\,\partial_z^3\epsilon(z)+2\,T(z)\,\partial_z\epsilon(z)+\epsilon(z)\,\partial_z T(z)\,. \end{aligned} \tag{2.44}$$

Let us compare this expression with Eq. (2.43). For infinitesimal transformations $f(z)=z+\epsilon(z)$, the leading order contribution to the Schwarzian derivative reads

$$S\big(z+\epsilon(z),z\big)=\frac{1}{(1+\partial_z\epsilon)^2}\left(\big(1+\partial_z\epsilon\big)\,\partial_z^3\epsilon-\frac{3}{2}\,\big(\partial_z^2\epsilon\big)^2\right)\simeq\partial_z^3\epsilon(z)\,.$$

The variation of the energy–momentum tensor can then be computed as

$$\begin{aligned}\delta_\epsilon\, T(z) &= T'(z)-T(z)\\ &= \Big(1+\partial_z\epsilon(z)\Big)^2\Big(T(z)+\epsilon(z)\,\partial_z T(z)\Big)+\frac{c}{12}\,\partial_z^3\epsilon(z)-T(z)\\ &= \frac{c}{12}\,\partial_z^3\epsilon(z)+2\,T(z)\,\partial_z\epsilon(z)+\epsilon(z)\,\partial_z T(z)\,,\end{aligned}$$

which is the same as in Eq. (2.44). We have thus verified Eq. (2.43) at the level of infinitesimal conformal transformations.

Remarks

- The calculation on p. 27 shows that the *singular* part of the OPE of the chiral energy–momentum tensors $T(z)$ is equivalent to the Virasoro algebra for the modes L_m.
- Performing a computation along the same lines as on p. 27, one finds that for a chiral primary field $\phi(z)$, the holomorphic part of the OPE (2.40) is equivalent to

$$\boxed{\big[L_m,\phi_n\big]=\big(\,(h-1)m-n\,\big)\,\phi_{m+n}} \tag{2.45}$$

for all $m,n\in\mathbb{Z}$. If relations (2.40) and (2.45) hold only for $m=-1,0,+1$, then $\phi(z)$ is called a quasi-primary field.

- Applying the relation (2.45) to the Virasoro algebra (2.16) for values $m = -1, 0, +1$, we see that the chiral energy–momentum tensor is a quasi-primary field of conformal dimension $(h, \overline{h}) = (2, 0)$. In view of Eq. (2.25), this observation also explains the form of the Laurent expansion (2.42).
- It is worth to note that the Schwarzian derivative $S(w, z)$ vanishes for $SL(2, \mathbb{C})$ transformations $w = f(z)$ in agreement with the fact that $T(z)$ is a quasi-primary field.

2.6 Operator Algebra of Chiral Quasi-Primary Fields

The objects of interest in quantum field theories are n-point correlation functions which are usually computed in a perturbative approach via either canonical quantisation or the path integral method. In this section, we will see that the *exact* two- and three-point functions for certain fields in a conformal field theory are already determined by the symmetries. This will allow us to derive a general formula for the OPE among quasi-primary fields.

2.6.1 Conformal Ward Identity

In quantum field theory, so-called Ward identities are the quantum manifestation of symmetries. We will now derive such an identity for the conformal symmetry of two-dimensional CFTs on general grounds. For primary fields ϕ_i, we calculate

$$\begin{aligned} &\Big\langle \oint \frac{dz}{2\pi i} \epsilon(z)\, T(z)\, \phi_1(w_1, \overline{w}_1) \ldots \phi_N(w_N, \overline{w}_N) \Big\rangle \\ &\quad = \sum_{i=1}^{N} \Big\langle \phi_1(w_1, \overline{w}_1) \ldots \Big(\oint_{C(w_i)} \frac{dz}{2\pi i} \epsilon(z)\, T(z)\, \phi_i(w_i, \overline{w}_i) \Big) \ldots \phi_N(w_N, \overline{w}_N) \Big\rangle \\ &\quad = \sum_{i=1}^{N} \Big\langle \phi_1(w_1, \overline{w}_1) \ldots \Big(h_i\, \partial\epsilon(w_i) + \epsilon(w_i)\, \partial_{w_i} \Big) \phi_i(w_i, \overline{w}_i) \ldots \phi_N(w_N, \overline{w}_N) \Big\rangle \end{aligned} \tag{2.46}$$

where we have applied the deformation of the contour integrals illustrated in Fig. 2.5 and used Eq. (2.37). Employing then the two relations shown in Eq. (2.38), we can write

$$\begin{aligned} 0 = \oint \frac{dz}{2\pi i} \epsilon(z) \Big[&\Big\langle T(z)\, \phi_1(w_1, \overline{w}_1) \ldots \phi_N(w_N, \overline{w}_N) \Big\rangle \\ &- \sum_{i=1}^{N} \Big(\frac{h_i}{(z - w_i)^2} + \frac{1}{z - w_i}\, \partial_{w_i} \Big) \Big\langle \phi_1(w_1, \overline{w}_1) \ldots \phi_N(w_N, \overline{w}_N) \Big\rangle \Big] \end{aligned} \tag{2.47}$$

Since this must hold for all $\epsilon(z)$ of the form $\epsilon(z) = -z^{n+1}$ with $n \in \mathbb{Z}$, the integrand must already vanish and we arrive at the Conformal Ward identity

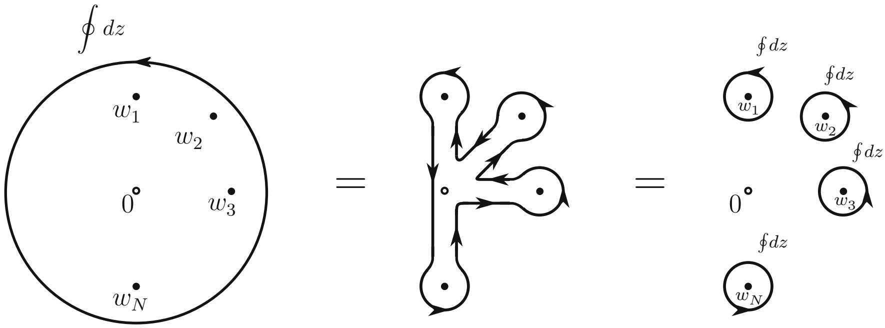


Fig. 2.5 Deformation of contour integrals

$$\boxed{\begin{aligned}&\Big\langle T(z)\,\phi_1(w_1,\overline{w}_1)\ldots\phi_N(w_N,\overline{w}_N)\Big\rangle\\&\qquad=\sum_{i=1}^{N}\bigg(\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\,\partial_{w_i}\bigg)\Big\langle\phi_1(w_1,\overline{w}_1)\ldots\phi_N(w_N,\overline{w}_N)\Big\rangle\end{aligned}}\;. \tag{2.48}$$

2.6.2 Two- and Three-Point Functions

In this subsection, we will employ the global conformal $SL(2,\mathbb{C})/\mathbb{Z}_2$ symmetry to determine the two- and three-point function for chiral quasi-primary fields.

The Two-Point Function

We start with the two-point function of two chiral quasi-primary fields

$$\big\langle\phi_1(z)\,\phi_2(w)\big\rangle = g\big(z,w\big)\;.$$

The invariance under translations $f(z) = z + a$ generated by L_{-1} requires g to be of the form $g(z,w) = g(z-w)$. The invariance under L_0, i.e. rescalings of the form $f(z) = \lambda z$, implies that

$$\big\langle\phi_1(z)\,\phi_2(w)\big\rangle \to \big\langle\lambda^{h_1}\phi_1(\lambda z)\,\lambda^{h_2}\phi_2(\lambda w)\big\rangle = \lambda^{h_1+h_2} g\big(\lambda(z-w)\big) \overset{!}{=} g\big(z-w\big)\;,$$

from which we conclude

$$g\big(z-w\big) = \frac{d_{12}}{(z-w)^{h_1+h_2}}\;,$$

where d_{12} is called a structure constant. Finally, the invariance of the two-point function under L_1 essentially implies the invariance under transformations $f(z) = -\frac{1}{z}$ for which we find

$$\begin{aligned}\big\langle \phi_1(z)\,\phi_2(w)\big\rangle \quad &\to \quad \Big\langle \frac{1}{z^{2h_1}}\,\frac{1}{w^{2h_2}}\,\phi_1\big(-\frac{1}{z}\big)\,\phi_2\big(-\frac{1}{w}\big)\Big\rangle \\ &= \frac{1}{z^{2h_1}w^{2h_2}}\,\frac{d_{12}}{\big(-\frac{1}{z}+\frac{1}{w}\big)^{h_1+h_2}} \\ &\stackrel{!}{=} \frac{d_{12}}{(z-w)^{h_1+h_2}}\end{aligned}$$

which can only be satisfied if $h_1 = h_2$. We therefore arrive at the result that

> The $SL(2,\mathbb{C})/\mathbb{Z}_2$ conformal symmetry fixes the two-point function of two chiral quasi-primary fields to be of the form
>
> $$\big\langle \phi_i(z)\,\phi_j(w)\big\rangle = \frac{d_{ij}\ \delta_{h_i,h_j}}{(z-w)^{2h_i}}\ . \tag{2.49}$$

As an example, let us consider the energy–momentum tensor $T(z)$. From the OPE shown in Eq. (2.41) (and using the fact that one-point functions of conformal fields on the sphere vanish), we find that

$$\big\langle\, T(z)\,T(w)\,\big\rangle = \frac{c/2}{(z-w)^4}\ .$$

The Three-Point Function

After having determined the two-point function of two chiral quasi-primary fields up to a constant, let us now consider the three-point function. From the invariance under translations, we can infer that

$$\big\langle \phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)\big\rangle = f(z_{12}, z_{23}, z_{13})\ ,$$

where we introduced $z_{ij} = z_i - z_j$. The requirement of invariance under dilation can be expressed as

$$\begin{aligned}\big\langle \phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)\big\rangle \quad &\to \quad \big\langle \lambda^{h_1}\phi_1(\lambda z_1)\,\lambda^{h_2}\phi_2(\lambda z_2)\,\lambda^{h_3}\phi_3(\lambda z_3)\big\rangle \\ &= \lambda^{h_1+h_2+h_3}\ f(\,\lambda z_{12}, \lambda z_{23}, \lambda z_{13}\,) \\ &\stackrel{!}{=} f(\,z_{12}, z_{23}, z_{13}\,)\end{aligned}$$

from which it follows that

$$f(z_{12}, z_{23}, z_{13}) = \frac{C_{123}}{z_{12}^a\, z_{23}^b\, z_{13}^c} \,,$$

with $a + b + c = h_1 + h_2 + h_3$ and C_{123} some structure constant. Finally, from the Special Conformal Transformations, we obtain the condition

$$\frac{1}{z_1^{2h_1} z_2^{2h_2} z_3^{2h_3}} \; \frac{(z_1 z_2)^a\, (z_2 z_3)^b\, (z_1 z_3)^c}{z_{12}^a\, z_{23}^b\, z_{13}^c} = \frac{1}{z_{12}^a\, z_{23}^b\, z_{13}^c} \,.$$

Solving this expression for a, b, c leads to

$$a = h_1 + h_2 - h_3 \,, \qquad b = h_2 + h_3 - h_1 \,, \qquad c = h_1 + h_3 - h_2 \,,$$

and so we have shown that

> The $SL(2, \mathbb{C})/\mathbb{Z}_2$ conformal symmetry fixes the three-point function of chiral quasi-primary fields up to a constant to
>
> $$\big\langle \phi_1(z_1)\, \phi_2(z_2)\, \phi_3(z_3) \big\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\; z_{23}^{h_2+h_3-h_1}\; z_{13}^{h_1+h_3-h_2}} \,. \qquad (2.50)$$

Remarks

- Using the $SL(2, \mathbb{C})/\mathbb{Z}_2$ global symmetry, it is possible to map any three points $\{z_1, z_2, z_3\}$ on the Riemann sphere to $\{0, 1, \infty\}$.
- The results for the two- and three-point function have been derived using only the $SL(2, \mathbb{C})/\mathbb{Z}_2 \simeq SO(3, 1)$ symmetry. As we have mentioned before, the conformal group $SO(3, 1)$ extends to higher dimensions $\mathbb{R}^{d,0}$ as $SO(d + 1, 1)$. By analogous reasoning, the two- and three-point functions for CFTs in dimensions $d > 2$ then have the same form as for $d = 2$.
- In order for the two-point function (2.49) to be single-valued on the complex plane, that is, to be invariant under rotations $z \mapsto e^{2\pi i} z$, we see that the conformal dimension of a chiral quasi-primary field has to be integer or half-integer.

2.6.3 General Form of the OPE for Chiral Quasi-Primary Fields

General Expression for the OPE

The generic form of the two- and three-point functions allows us to extract the general form of the OPE between two chiral quasi-primary fields in terms of other quasi-primary fields and their derivatives[1]. To this end we make the ansatz

$$\phi_i(z)\,\phi_j(w) = \sum_{k,n\geq 0} C_{ij}^k \, \frac{a_{ijk}^n}{n!} \, \frac{1}{(z-w)^{h_i+h_j-h_k-n}} \, \partial^n \phi_k(w) \,, \qquad (2.51)$$

where the $(z-w)$ part is fixed by the scaling behaviour under dilations $z \mapsto \lambda z$. Note that we have chosen our ansatz such that a_{ijk}^n only depends on the conformal weights h_i, h_j, h_k of the fields i, j, k (and on n), while C_{ij}^k contains further information about the fields.

Let us now take $w = 1$ in Eq. (2.51) and consider it as part of the following three-point function:

$$\Big\langle \Big(\phi_i(z)\,\phi_j(1)\Big)\,\phi_k(0) \Big\rangle = \sum_{l,n\geq 0} C_{ij}^l \, \frac{a_{ijl}^n}{n!} \, \frac{1}{(z-1)^{h_i+h_j-h_l-n}} \, \big\langle \partial^n \phi_l(1)\,\phi_k(0)\big\rangle \,.$$

Using then the general formula for the two-point function (2.49), we find for the correlator on the right-hand side that

$$\big\langle \partial_z^n \phi_l(z)\phi_k(0)\big\rangle\Big|_{z=1} = \partial_z^n \left(\frac{d_{lk}\;\delta_{h_l,h_k}}{z^{2h_k}} \right)\Bigg|_{z=1} = (-1)^n\, n! \binom{2h_k+n-1}{n} d_{lk}\;\delta_{h_l,h_k} \,.$$

We therefore obtain

$$\big\langle \phi_i(z)\,\phi_j(1)\,\phi_k(0)\big\rangle = \sum_{l,n\geq 0} C_{ij}^l \; d_{lk}\; a_{ijk}^n \binom{2h_k+n-1}{n} \frac{(-1)^n}{(z-1)^{h_i+h_j-h_k-n}} \,. \qquad (2.52)$$

However, we can also use the general expression for the three-point function (2.50) with values $z_1 = z$, $z_2 = 1$ and $z_3 = 0$. Combining then Eq. (2.50) with Eq. (2.52), we find

$$\sum_{l,n\geq 0} C_{ij}^l \; d_{lk}\; a_{ijk}^n \binom{2h_k+n-1}{n} \frac{(-1)^n}{(z-1)^{h_i+h_j-h_k-n}} \stackrel{!}{=} \frac{C_{ijk}}{(z-1)^{h_i+h_j-h_k}\; z^{h_i+h_k-h_j}} \,,$$

$$\sum_{l,n\geq 0} C_{ij}^l \; d_{lk}\; a_{ijk}^n \binom{2h_k+n-1}{n} (-1)^n\;(z-1)^n \stackrel{!}{=} \frac{C_{ijk}}{\big(1+(z-1)\big)^{h_i+h_k-h_j}} \,.$$

Finally, we use the following relation with $x = z-1$ for the term on the right-hand side of the last formula:

[1] The proof that the OPE of two quasi-primary fields involves indeed *just* other quasi-primary fields and their derivatives is non-trivial and will not be presented.

$$\frac{1}{(1+x)^H} = \sum_{n=0}^{\infty}(-1)^n \binom{H+n-1}{n} x^n \, .$$

Comparing coefficients in front of the $(z-1)$ terms, we can fix the constants C_{ij}^l and a_{ijk}^n and arrive at the result that

The OPE of two chiral quasi-primary fields has the general form

$$\phi_i(z)\,\phi_j(w) = \sum_{k,n\geq 0} C_{ij}^k \,\frac{a_{ijk}^n}{n!}\, \frac{1}{(z-w)^{h_i+h_j-h_k-n}}\, \partial^n \phi_k(w) \tag{2.53}$$

with coefficients

$$a_{ijk}^n = \binom{2h_k+n-1}{n}^{-1} \binom{h_k+h_i-h_j+n-1}{n} \, ,$$

$$C_{ijk} = C_{ij}^l \; d_{lk} \, .$$

General Expression for the Commutation Relations

The final expression (2.53) gives a general form for the OPE of chiral quasi-primary fields. However, as we have seen previously, the same information is encoded in the commutation relations of the Laurent modes of the fields. We will not derive these commutators but just summarise the result. To do so, we recall the Laurent expansion of chiral fields $\phi_i(z)$

$$\phi_i(z) = \sum_m \phi_{(i)m}\, z^{-m-h_i} \, ,$$

where the conformal dimensions h_i of a chiral quasi-primary field are always integer or half-integer. Note that here i is a label for the fields and $m \in \mathbb{Z}$ or $m \in \mathbb{Z}+\frac{1}{2}$ denotes a particular Laurent mode. After expressing the modes $\phi_{(i)m}$ as contour integrals over $\phi_i(z)$ and performing a tedious evaluation of the commutator, one arrives at the following compact expression for the algebra:

$$\left[\phi_{(i)m}, \phi_{(j)n}\right] = \sum_k C_{ij}^k \; p_{ijk}(m,n)\; \phi_{(k)m+n} + d_{ij}\; \delta_{m,-n} \binom{m+h_i-1}{2h_i-1} \tag{2.54}$$

with the polynomials

$$p_{ijk}(m,n) = \sum_{\substack{r,s \in \mathbb{Z}_0^+ \\ r+s=h_i+h_j-h_k-1}} C_{r,s}^{ijk} \cdot \binom{-m+h_i-1}{r} \cdot \binom{-n+h_j-1}{s},$$

$$C_{r,s}^{ijk} = (-1)^r \frac{(2h_k-1)!}{(h_i+h_j+h_k-2)!} \prod_{t=0}^{s-1}(2h_i-2-r-t) \prod_{u=0}^{r-1}(2h_j-2-s-u)\,. \tag{2.55}$$

Remarks

- Note that on the right-hand side of Eq. (2.54), only fields with conformal dimension $h_k < h_i + h_j$ can appear. This can be seen by studying the coefficients (2.55).
- Furthermore, because the polynomials p_{ijk} depend only on the conformal dimensions h_i, h_j, h_j of the fields i, j, k, it is also common to use the conformal dimensions as subscripts, that is, $p_{h_i h_j h_j}$.
- The generic structure of the chiral algebra of quasi-primary fields (2.53) is extremely helpful for the construction of extended symmetry algebras. We will consider such so-called $\mathcal{W}$ algebras in Sect. 3.7.
- Clearly, not all fields in a conformal field theory are quasi-primary and so the formulas above do not apply for all fields! For instance, the derivatives $\partial^n \phi_k(z)$ of a quasi-primary field $\phi(z)$ are not quasi-primary.

Applications I: Two-Point Function Revisited

Let us now consider four applications of the results obtained in this section. First, with the help of Eq. (2.54), we can compute the norm of a state $\phi_{(i)-n}|0\rangle$. Assuming $n \geq h$, we obtain

$$\begin{aligned}
\big|\big|\, \phi_{(i)-n}\big|0\big\rangle \,\big|\big|^2 &= \big\langle 0\big|\, \phi^{\dagger}_{(i)-n}\, \phi_{(i)-n}\, \big|0\big\rangle \\
&= \big\langle 0\big|\, \phi_{(i)+n}\, \phi_{(i)-n}\, \big|0\big\rangle \\
&= \big\langle 0\big|\big[\phi_{(i)+n}, \phi_{(i)-n}\big]\big|0\big\rangle \\
&= C_{ii}^{j}\, p_{h_i h_i h_j}(n,-n)\, \big\langle 0\big|\, \phi_{(j)0}\, \big|0\big\rangle + d_{ii} \binom{n+h_i-1}{2h_i-1} \\
&= d_{ii} \binom{n+h_i-1}{2h_i-1}\,.
\end{aligned}$$

Employing then Eq. (2.28) as well as Eq. (2.32), we see that the norm of a state $|\phi\rangle = \phi_{-h}|0\rangle$ is equal to the structure constant of the two-point function

$$\big\langle \phi \,\big|\, \phi \big\rangle = \big\langle 0\big|\, \phi_{+h}\, \phi_{-h}\, \big|0\big\rangle = d_{\phi\phi}\,. \tag{2.56}$$

Applications II: Three-Point Function Revisited

Let us also determine the structure constant of the three-point function between chiral quasi-primary fields. To do so, we write Eq. (2.50) as

$$C_{123} = z_{12}^{h_1+h_2-h_3}\, z_{23}^{h_2+h_3-h_1}\, z_{13}^{h_1+h_3-h_2} \big\langle \phi_1(z_1)\, \phi_2(z_2)\, \phi_3(z_3)\big\rangle \,,$$

and perform the limits $z_1 \to \infty$, $z_3 \to 0$ while keeping z_2 finite. Using then again (2.28) as well as (2.32), we find

$$\begin{aligned} C_{123} &= \lim_{z_1\to\infty} \lim_{z_3\to 0} z_{12}^{h_1+h_2-h_3}\, z_{23}^{h_2+h_3-h_1}\, z_{13}^{h_1+h_3-h_2} \big\langle \phi_1(z_1)\, \phi_2(z_2)\, \phi_3(z_3)\big\rangle \\ &= \lim_{z_1\to\infty} \lim_{z_3\to 0} z_1^{2h_1}\, z_2^{h_2+h_3-h_1} \big\langle \phi_1(z_1)\, \phi_2(z_2)\, \phi_3(z_3)\big\rangle \\ &= z_2^{h_2+h_3-h_1} \big\langle 0\big|\, \phi_{(1)+h_1}\, \phi_2(z_2)\, \phi_{(3)-h_3} \big|0\big\rangle \,. \end{aligned}$$

Because the left-hand side of this equation is a constant, the right-hand side cannot depend on z_2 and so only the z_2^0 term does give a non-trivial contribution. We therefore conclude that

$$C_{123} = \big\langle 0\big|\, \phi_{(1)+h_1}\, \phi_{(2)h_3-h_1}\, \phi_{(3)-h_3} \big|0\big\rangle \,. \tag{2.57}$$

Applications III: Virasoro Algebra

After studying the two- and three-point function, let us now turn to the Virasoro algebra and determine the structure constants C_{ij}^k and d_{ij}. From the general expression (2.54), we infer the commutation relations between the Laurent modes of the energy–momentum tensor to be of the following form:

$$\big[L_m, L_n\big] = C_{LL}^{L}\; p_{222}(m,n)\; L_{m+n} + d_{LL}\; \delta_{m,-n} \binom{m+1}{3} \,,$$

where in view of the final result, we identified $C_{LL}^k = 0$ for $k \neq L$. Note also that the subscripts of p_{ijk} denote the conformal weight of the chiral fields involved. Using the explicit expression (2.55) for p_{ijk} and recalling from p. 29 that the conformal dimension of $T(z)$ is $h = 2$, we find

$$p_{222}(m,n) = C_{1,0}^{222} \binom{-m+1}{1} + C_{0,1}^{222} \binom{-n+1}{1}$$

with coefficients $C_{r,s}^{222}$ of the form

$$C_{1,0}^{222} = (-1)^1 \cdot \frac{3!}{4!} \cdot 2 = -\frac{1}{2} \qquad \text{and} \qquad C_{0,1}^{222} = (-1)^0 \cdot \frac{3!}{4!} \cdot 2 = +\frac{1}{2} \,.$$

Putting these results together, we obtain for the Virasoro algebra

$$\big[L_m, L_n\big] = C_{LL}^{L}\, \frac{m-n}{2}\, L_{m+n} + d_{LL}\, \delta_{m,-n}\, \frac{m^3-m}{6}\ .$$

If we compare with Eq. (2.16), we can fix the two unknown constants as

$$d_{LL} = \frac{c}{2}\ , \qquad\qquad C_{LL}^{L} = 2\ . \tag{2.58}$$

Applications IV: Current Algebras

Finally, let us study the so-called current algebras. The definition of a current in a two-dimensional conformal field theory is the following:

Definition 7. *A chiral field $j(z)$ with conformal dimension $h = 1$ is called a current. A similar definition holds for the anti-chiral sector.*

Let us assume we have a theory with N quasi-primary currents $j_i(z)$ where $i = 1, \ldots, N$. We can express these fields as a Laurent series $j_i(z) = \sum_{n\in\mathbb{Z}} z^{-n-1} j_{(i)n}$ in the usual way and determine the algebra of the Laurent modes $j_{(i)n}$ using Eq. (2.54)

$$\big[j_{(i)m}, j_{(j)n}\big] = \sum_k C_{ij}^{k}\ p_{111}(m,n)\ j_{(k)m+n} + d_{ij}\ m\ \delta_{m,-n}\ . \tag{2.59}$$

From Eq. (2.55), we compute $p_{111}(m,n) = 1$ and so it follows that $C_{ij}^{k} = -C_{ji}^{k}$ due to the anti-symmetry of the commutator.

Next, we perform a rotation among the fields such that the matrix d_{ij} is diagonalised, and by a rescaling of the fields we can achieve $d_{ij} \to k\,\delta_{ij}$ where k is some constant. Changing then the labels of the fields from subscript to superscript and denoting the constants C_{ij}^{k} in the new basis by f^{ijk}, we can express the algebra Eq. (2.59) as

$$\big[j_m^i, j_n^j\big] = \sum_l f^{ijl}\ j_{m+n}^{l} + k\ m\ \delta^{ij}\ \delta_{m,-n}\ , \tag{2.60}$$

where f^{ijl} are called structure constants and k is called the level. As it turns out, the algebra (2.60) is a generalisation of a Lie algebra called a *Kač–Moody algebra* which is infinite dimensional. conformal field theories based on such algebras provide many examples of abstract CFTs which we will study in much more detail in Chap. 3.

2.7 Normal Ordered Products

Operations on the field space of a theory are provided by the action of derivatives $\partial\phi_i, \partial^2\phi_i, \ldots$ and by taking products of fields at the same point in space–time. As known from quantum field theory, since the ϕ_i are operators, we need to give an ordering prescription for such products. This will be *normal ordering* which in QFT language means "creation operators to the left".

In this section, we will illustrate that the regular part of an OPE provides a notion of normal ordering for the product fields.

Normal Ordering Prescription

Let us start by investigating what are the creation and what are the annihilation operators in a CFT. To do so, we recall Eq. (2.27) which reads

$$\phi_{n,\overline{m}} \left|0\right\rangle = 0 \qquad \text{for} \qquad n > -h, \quad \overline{m} > -\overline{h} \,. \tag{2.61}$$

From here we can see already that we can interpret operators $\phi_{n,\overline{m}}$ with $n > -h$ or $\overline{m} > -\overline{h}$ as annihilation operators.

However, in order to explore this point further, let us recall from Eq. (2.24) that the Hamiltonian is expressed in terms of L_0 and $\overline{L}_0$ as $H = L_0 + \overline{L}_0$, which motivates the notion of "chiral energy" for the L_0 eigenvalue of a state. For the special case of a chiral primary, let us calculate

$$L_0\, \phi_n \left|0\right\rangle = \big(L_0\, \phi_n - \phi_n\, L_0\big) \left|0\right\rangle = \big[L_0, \phi_n\big] \left|0\right\rangle = -n\, \phi_n \left|0\right\rangle , \tag{2.62}$$

where we employed Eq. (2.45) as well as $L_0 \left|0\right\rangle = 0$. Taking into account (2.61), we see that the chiral energy is bounded from below, i.e. only values $(h + m)$ with $m \geq 0$ are allowed. Requiring that creation operators should create states with positive energy, we conclude that

$$\begin{array}{llll} \phi_n & \text{with} & n > -h & \text{are annihilation operators ,} \\ \phi_n & \text{with} & n \leq -h & \text{are creation operators .} \end{array}$$

The anti-chiral sector can be included by following the same arguments for $\overline{L}_0$. Coming then back to the subject of this paragraph, the normal ordering prescription is to put all creation operators to the left.

Normal Ordered Products and OPEs

After having discussed the normal ordering prescription for operators in a conformal field theory, let us state that

> The regular part of an OPE naturally gives rise to normal ordered products (NOPs) which can be written in the following way:
>
> $$\phi(z)\,\chi(w) = \text{sing.} + \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!}\, N\big(\chi\,\partial^n\phi\big)(w) \,. \tag{2.63}$$

The notation for normal ordering we will mainly employ is $N(\chi\phi)$, however, it is also common to use $:\ \phi\chi\ :$, $(\phi\chi)$ or $[\phi\chi]_0$. In the following, we will verify the statement above for the case $n = 0$.

Let us first use Eq. (2.63) to obtain an expression for the normal ordered product of two operators. To do so, we apply $\frac{1}{2\pi i}\oint dz(z-w)^{-1}$ to both sides of Eq. (2.63) which picks out the $n = 0$ term on the right-hand side leading to

$$N\big(\chi\,\phi\big)(w) = \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i}\,\frac{\phi(z)\chi(w)}{z-w}\,. \tag{2.64}$$

However, we can also perform a Laurent expansion of $N(\chi\phi)$ in the usual way which gives us

$$N\big(\chi\,\phi\big)(w) = \sum_{n\in\mathbb{Z}} w^{-n-h^{\phi}-h^{\chi}}\; N\big(\chi\,\phi\big)_n\,,$$

$$N\big(\chi\,\phi\big)_n = \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i}\; w^{n+h^{\phi}+h^{\chi}-1} N\big(\chi\,\phi\big)(w)\,, \tag{2.65}$$

where we also include the expression for the Laurent modes $N(\chi\,\phi)_n$. Let us now employ the relation (2.64) in Eq. (2.65) for which we find

$$\begin{aligned} N\big(\chi\,\phi\big)_n &= \oint_{\mathcal{C}(0)} \frac{dw}{2\pi i}\; w^{n+h^{\phi}+h^{\chi}-1} \oint_{\mathcal{C}(w)} \frac{dz}{2\pi i}\,\frac{\phi(z)\chi(w)}{z-w} \\ &= \underbrace{\oint \frac{dw}{2\pi i}\; w^{n+h^{\phi}+h^{\chi}-1}\Bigg(\underbrace{\oint_{|z|>|w|} \frac{dz}{2\pi i}\,\frac{\phi(z)\chi(w)}{z-w}}_{\mathcal{I}_1}}_{\mathcal{I}_2} - \oint_{|z|<|w|} \frac{dz}{2\pi i}\,\frac{\chi(w)\phi(z)}{z-w}\Bigg) \end{aligned} \tag{2.66}$$

where we applied the deformation of contour integrals formulated in Eq. (2.36). Expressing ϕ and χ as a Laurent series, the term $\mathcal{I}_1$ can be evaluated as

$$\begin{aligned} \mathcal{I}_1 &= \oint_{|z|>|w|} \frac{dz}{2\pi i}\,\frac{1}{z-w} \sum_{r,s} z^{-r-h^{\phi}}\, w^{-s-h^{\chi}}\;\phi_r\,\chi_s \\ &= \oint_{|z|>|w|} \frac{dz}{2\pi i}\,\frac{1}{z} \sum_{p\geq 0}\left(\frac{w}{z}\right)^p \sum_{r,s} z^{-r-h^{\phi}}\, w^{-s-h^{\chi}}\;\phi_r\,\chi_s \\ &= \oint_{|z|>|w|} \frac{dz}{2\pi i}\, \sum_{p\geq 0}\sum_{r,s} z^{-r-h^{\phi}-p-1}\, w^{-s-h^{\chi}+p}\;\phi_r\,\chi_s\,. \end{aligned}$$

Note that we employed $\frac{1}{z-w} = \frac{1}{z(1-w/z)}$ as well as the geometric series to go from the first to the second line and that only the z^{-1} term gives a non-zero contribution. Thus, performing the integral over dz leads to a δ-function setting $r = -h^{\phi} - p$ and so the integral $\mathcal{I}_2$ reads

$$\mathcal{I}_2 = \oint \frac{dw}{2\pi i} \sum_{p\geq 0} \sum_{s} w^{-s-h^\chi+p+n+h^\phi+h^\chi-1} \phi_{-h^\phi-p}\, \chi_s \ ,$$

for which again only the w^{-1} term contributes and therefore $s = p + n + h^\phi$. We then arrive at the final expression for the first term in $N(\chi\phi)_n$

$$\mathcal{I}_2 = \sum_{p\geq 0} \phi_{-h^\phi-p}\, \chi_{h^\phi+n+p} = \sum_{k\leq -h^\phi} \phi_k\, \chi_{n-k} \ .$$

For the second term, we perform a similar calculation to find $\sum_{k>-h^\phi} \chi_{n-k}\phi_k$, which we combine into the final result for the Laurent modes of normal ordered products

$$\boxed{N\left(\chi\, \phi\right)_n = \sum_{k>-h^\phi} \chi_{n-k}\, \phi_k + \sum_{k\leq -h^\phi} \phi_k\, \chi_{n-k}} \ . \tag{2.67}$$

Here we see that indeed the ϕ_k in the first term are annihilation operators at the right and that the ϕ_k in the second term are creation operators at the left. Therefore, the regular part of an OPE contains normal ordered products.

Useful Formulas

For later reference, let us now consider a special case of a normal ordered product. In particular, let us compute $N(\chi\partial\phi)_n$ and $N(\partial\chi\phi)_n$ for which we note that the Laurent expansion of say $\partial\phi$ can be inferred from ϕ in the following way:

$$\partial\phi(z) = \partial \sum_n z^{-n-h}\, \phi_n = \sum_n (-n-h)\, z^{-n-(h+1)}\, \phi_n \ . \tag{2.68}$$

Replacing $\phi \to \partial\phi$ in Eq. (2.66), using the Laurent expansion (2.68) and performing the same steps as above, one arrives at the following results:

$$\begin{aligned} N\left(\chi\, \partial\phi\right)_n &= \sum_{k>-h^\phi-1} \left(-h^\phi - k\right) \chi_{n-k}\, \phi_k + \sum_{k\leq -h^\phi-1} \left(-h^\phi - k\right) \phi_k\, \chi_{n-k} \ , \\ N\left(\partial\chi\, \phi\right)_n &= \sum_{k>-h^\phi} (-h^\chi - n + k)\, \chi_{n-k}\, \phi_k + \sum_{k\leq -h^\phi} (-h^\chi - n + k)\, \phi_k\, \chi_{n-k} \ . \end{aligned} \tag{2.69}$$

Normal Ordered Products of Quasi-Primary Fields

Let us also note that normal ordered products of quasi-primary fields are in general not quasi-primary, but can be projected to such. To illustrate this statement, we consider the example of the energy–momentum tensor. Recalling the OPE (2.41) together with Eq. (2.63), we can write

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + N\big(TT\big)(w) + \ldots\ . \tag{2.70}$$

However, using the general expression for the OPE of two quasi-primary fields shown in Eq. (2.53), we observe that there is a $\partial^2 T$ term at $(z-w)^0$ with coefficient

$$C_{TT}^{T}\,\frac{a_{222}^2}{2!} \qquad \text{where} \qquad a_{222}^2 = \binom{5}{2}^{-1}\binom{3}{2} = \frac{3}{10} \qquad \text{and} \qquad C_{TT}^{T} = 2\ . \tag{2.71}$$

But, since the index k in Eq. (2.53) runs over all quasi-primary fields of the theory, we expect also other terms at order $(z-w)^0$. If we denote these by $\mathcal{N}(TT)$, we find from Eq. (2.70) that

$$N\big(TT\big) = \mathcal{N}\big(TT\big) + \frac{3}{10}\,\partial^2 T\ .$$

One can easily check that $\partial^2 T$ is not a quasi-primary field, and by computing for instance $[L_m, N(TT)_n]$ and comparing with Eq. (2.45), one arrives at the same conclusion for $N(TT)$. However, we note that

$$\boxed{\mathcal{N}\big(TT\big) = N\big(TT\big) - \frac{3}{10}\,\partial^2 T} \tag{2.72}$$

actually is a quasi-primary normal ordered product. Moreover, it turns out that this procedure can be iterated which allows one to write the entire field space in terms of quasi-primary fields and derivatives thereof.

2.8 The CFT Hilbert Space

In this section, we are going summarise some general properties of the Hilbert space of a conformal field theory.

The Verma Module

Let us consider again the chiral energy–momentum tensor. For the Laurent expansions of $T(z)$ and $\partial T(z)$ as well as for the corresponding asymptotic *in*-states, we find

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}\, L_n \qquad \longleftrightarrow \qquad L_{-2}\,\big|0\big\rangle\,,$$
$$\partial T(z) = \sum_{n\in\mathbb{Z}} (-n-2)\, z^{-n-3}\, L_n \qquad \longleftrightarrow \qquad L_{-3}\,\big|0\big\rangle\,,$$

where we employed the relation (2.28). The state corresponding to the normal ordered product of two energy–momentum tensors can be determined as follows. From the Laurent expansion

$$N\big(TT\big) = \sum_{n\in\mathbb{Z}} z^{-n-4}\, N\big(TT\big)_n\,,$$

we see that only the mode with $n = -4$ gives a well-defined contribution in the limit $z \to 0$. But from the general expression for the normal ordered product (2.67), we obtain

$$N\big(TT\big)_n = \sum_{k>-2} L_{n-k}\, L_k + \sum_{k\leq -2} L_k\, L_{n-k}\,,$$

where the first sum vanishes when applied to $|0\rangle$ and the second sum acting on the vacuum only contributes for $n-k \leq -2$. Taking into account that $n = -4$ from above, we find

$$N\big(TT\big)_{-4}\,\big|0\big\rangle = L_{-2}\, L_{-2}\,\big|0\big\rangle \qquad \text{and} \qquad N\big(TT\big) \quad \longleftrightarrow \quad L_{-2}\, L_{-2}\big|0\big\rangle\,.$$

Finally, we note that using Eq. (2.69), one can similarly show $N\big(T\,\partial T\big) \leftrightarrow L_{-3}L_{-2}|0\rangle$. These examples motivate the following statement:

For each state $|\Phi\rangle$ in the so-called *Verma module*

$$\big\{\, L_{k_1}\ldots L_{k_n}\big|0\big\rangle \;:\; k_i \leq -2 \,\big\}\,,$$

we can find a field $F \in \big\{T, \partial T, \ldots, N(\ldots)\big\}$ with the property that $\lim_{z\to 0} F(z)\,|0\rangle = |\Phi\rangle$.

Conformal Family

Let us consider now a general (chiral) primary field $\phi(z)$ of conformal dimension h. This field gives rise to the state $|\phi\rangle = |h\rangle = \phi_{-h}|0\rangle$ which, due to the definition of a primary field (2.45), satisfies

$$L_n\,\big|\phi\big\rangle = \big[L_n, \phi_{-h}\big]\,\big|0\big\rangle = \Big(h\,(n+1) - n\Big)\phi_{-h+n}\,\big|0\big\rangle = 0\,, \tag{2.73}$$

for $n > 0$. Without providing detailed computations, we note that the modes of the energy–momentum tensor with $n < 0$ acting on a state $|h\rangle$ correspond to the following fields:

Field	State	Level
$\phi(z)$	$\phi_{-h}\|0\rangle = \|h\rangle$	0
$\partial\phi$	$L_{-1}\phi_{-h}\,\|0\rangle$	1
$\partial^2\phi$	$L_{-1}L_{-1}\phi_{-h}\,\|0\rangle$	2
$N\big(T\phi\big)$	$L_{-2}\phi_{-h}\,\|0\rangle$	2
$\partial^3\phi$	$L_{-1}L_{-1}L_{-1}\phi_{-h}\,\|0\rangle$	3
$N\big(T\partial\phi\big)$	$L_{-2}L_{-1}\phi_{-h}\,\|0\rangle$	3
$N\big(\partial T\phi\big)$	$L_{-3}\phi_{-h}\,\|0\rangle$	3
...	...	...

(2.74)

The lowest lying state in such a tower of states, that is $|h\rangle$, is called a highest weight state. From Eq. (2.74), we conclude furthermore that

Each primary field $\phi(z)$ gives rise to an infinite set of *descendant fields* by taking derivatives ∂^k and taking normal ordered products with T. The set of fields

$$\big[\phi(z)\big] := \Big\{\, \phi\,,\ \partial\phi\,,\ \partial^2\phi\,,\ \ldots\,,\ N\big(T\phi\big)\,,\ \ldots \Big\}$$

is called a *conformal family* which is also denoted by $\big\{\hat{L}_{k_1}\ldots\hat{L}_{k_n}\,\phi(z) : k_i \leq -1\big\}$.

Remarks

- Referring to the table in Eq. (2.74), note that there are $P(n)$ different states at level n where $P(n)$ is the number of partitions of n. The generating function for $P(n)$ will be important later and reads

$$\prod_{n=1}^{\infty}\frac{1}{1-q^n} = \sum_{N=0}^{\infty} P(N)\,q^N\ .$$

- For unitary theories, we know that the norm of all states has to be non-negative. In particular, recalling Eq. (2.56), for the structure constant of a two-point function this implies $d_{\phi\phi} \geq 0$, where ϕ is a primary field. Let us then consider the norm of the state $L_{-1}|\phi\rangle$ for which we compute

$$\big|\big|\, L_{-1}|\phi\rangle\, \big|\big|^2 = \langle\phi|\, L_{+1}\, L_{-1}\, |\phi\rangle = \langle\phi|\big[L_{+1}, L_{-1}\big]|\phi\rangle = \langle\phi|\, 2L_0|\phi\rangle = 2\, h\, d_{\phi\phi},$$

where we employed Eq. (2.73) as well as the Virasoro algebra (2.16). In order for this state to have non-negative norm, we see that $h \geq 0$. Therefore, a necessary condition for a theory to be unitary is that the conformal weights of all primary fields are non-negative.

2.9 Simple Examples of CFTs

So far, we have outlined part of the generic structure of conformal field theories without any reference to a Lagrangian formulation. In particular, we introduced CFTs via OPEs, respectively, operator algebras which, as we will see later, is extremely powerful for studying CFTs and in certain cases leads to a complete solution of the dynamics. However, to make contact with the usual approach to quantum field theories and because they naturally appear in string theory, let us consider three simple examples of conformal field theories given in terms of a Lagrangian action.

2.9.1 The Free Boson

Motivation

Let us start with a real massless scalar field $X(x^0, x^1)$ defined on a cylinder given by $x^0 \in \mathbb{R}$ and $x^1 \in \mathbb{R}$ subject to the identification $x^1 \simeq x^1 + 2\pi$. The action for such a theory takes the following form:

$$\begin{aligned} \mathcal{S} &= \frac{1}{4\pi\kappa} \int dx^0\, dx^1\, \sqrt{|h|}\, h^{\alpha\beta}\, \partial_\alpha X \partial_\beta X \\ &= \frac{1}{4\pi\kappa} \int dx^0\, dx^1\, \Big(\big(\partial_{x^0} X\big)^2 + \big(\partial_{x^1} X\big)^2\Big) \end{aligned} \tag{2.75}$$

where $h = \det h_{\alpha\beta}$ with $h_{\alpha\beta} = \mathrm{diag}(+1, +1)$, and κ is some normalisation constant. This is the (Euclideanised) world-sheet action (in conformal gauge) of a string moving in a flat background with coordinate X. Since in this theory there is no mass term setting a scale, we expect this action to be conformally invariant.

In order to study the action (2.75) in more detail, as we have seen in Sect. 2.4, it is convenient to map the cylinder to the complex plane which is achieved by

$$z = e^{x^0} \cdot e^{ix^1}\,. \tag{2.76}$$

Performing this change of variables for the action (2.75) and denoting the new fields by $X(z, \bar{z})$, we find

$$\mathcal{S} = \frac{1}{4\pi\kappa} \int dz\, d\bar{z} \sqrt{|g|}\, g^{ab}\, \partial_a X \partial_b X \tag{2.77}$$

$$= \frac{1}{4\pi\kappa} \int dz\, d\bar{z}\;\, \partial X \cdot \bar{\partial} X \tag{2.78}$$

with a, b standing for z and $\bar{z}$. Note that here and in the following we will use the notation $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$ interchangeably. Furthermore, in going from Eq. (2.77) to Eq. (2.78), we employed the explicit form of the metric g_{ab} which we obtained via $g_{ab} = \frac{\partial x^\alpha}{\partial x^a}\frac{\partial x^\beta}{\partial x^b} h_{\alpha\beta}$ and which reads

$$g_{ab} = \begin{bmatrix} 0 & \frac{1}{2z\bar{z}} \\ \frac{1}{2z\bar{z}} & 0 \end{bmatrix}, \qquad\qquad g^{ab} = \begin{bmatrix} 0 & 2z\bar{z} \\ 2z\bar{z} & 0 \end{bmatrix}.$$

Basic Properties

In the last paragraph, we have provided a connection between the string theory naturally defined on a cylinder and the example of the free boson defined on the complex plane. From a conformal field theory point of view, however, we do not need this discussion but can simply start from the action (2.78)

$$\boxed{\mathcal{S} = \frac{1}{4\pi\kappa} \int dz\, d\bar{z}\;\, \partial X \cdot \bar{\partial} X}\;. \tag{2.79}$$

The equation of motion for this action is derived by varying $\mathcal{S}$ with respect to X. We thus calculate

$$\begin{aligned}
0 &= \delta_X \mathcal{S} \\
&= \frac{1}{4\pi\kappa} \int dz\, d\bar{z} \left(\partial\delta X \cdot \bar{\partial} X + \partial X \cdot \bar{\partial}\delta X \right) \\
&= \frac{1}{4\pi\kappa} \int dz\, d\bar{z} \left(\partial\Big(\delta X \cdot \bar{\partial} X\Big) - \delta X \cdot \partial\bar{\partial} X + \bar{\partial}\Big(\partial X \cdot \delta X\Big) - \bar{\partial}\partial X \cdot \delta X \right) \\
&= -\frac{1}{2\pi\kappa} \int dz\, d\bar{z}\, \delta X \left(\partial\bar{\partial} X \right)
\end{aligned}$$

which has to be satisfied for all variations δX. Therefore, we obtain the equation of motion as

$$\partial\,\bar{\partial}\, X(z, \bar{z}) = 0\;,$$

from which we conclude that

$$\begin{aligned}
j(z) &= i\, \partial X(z, \bar{z}) &\qquad& \text{is a chiral field,} \\
\bar{j}(\bar{z}) &= i\, \bar{\partial} X(z, \bar{z}) && \text{is an anti-chiral field.}
\end{aligned} \tag{2.80}$$

Next, we are going to determine the conformal properties of the fields in the action (2.79). In particular, this action is invariant under conformal transformations if the field $X(z,\overline{z})$ has vanishing conformal dimensions, that is, $X'(z,\overline{z}) = X(w,\overline{w})$. Let us then compute

$$
\begin{aligned}
\mathcal{S} \quad \longrightarrow \quad & \frac{1}{4\pi\kappa}\int dz\, d\overline{z}\;\partial_z X'(z,\overline{z})\cdot\partial_{\overline{z}}X'(z,\overline{z}) \\
&= \frac{1}{4\pi\kappa}\int \frac{\partial z}{\partial w}\,dw\,\frac{\partial \overline{z}}{\partial \overline{w}}\,d\overline{w}\,\frac{\partial w}{\partial z}\,\partial_w X(w,\overline{w})\cdot\frac{\partial \overline{w}}{\partial \overline{z}}\,\partial_{\overline{w}}X(w,\overline{w}) \\
&= \frac{1}{4\pi\kappa}\int dw\, d\overline{w}\;\partial_w X(w,\overline{w})\cdot\partial_{\overline{w}}X(w,\overline{w})
\end{aligned}
$$

which indeed shows the invariance of the action (2.79) under conformal transformations if $X(z,\overline{z})$ has conformal dimensions $(h,\overline{h}) = (0,0)$. Moreover, by considering again the calculation above, we can conclude that the fields (2.80) are primary with dimensions $(h,\overline{h}) = (1,0)$ and $(h,\overline{h}) = (0,1)$ respectively. Finally, because we are considering a free theory, the engineering dimensions of the fields argued for above is the same as the dimension after quantisation.

Two-Point Function and Laurent Mode Algebra

Let us proceed and determine the propagator $K(z,\overline{z},w,\overline{w}) = \langle X(z,\overline{z})X(w,\overline{w})\rangle$ of the free boson $X(z,\overline{z})$ from the action (2.79). To do so, we note that $K(z,\overline{z},w,\overline{w})$ in the present case has to satisfy

$$
\partial_z\,\partial_{\overline{z}}\,K(z,\overline{z},w,\overline{w}) = -2\pi\kappa\,\delta^{(2)}(z-w)\;.
$$

Using the representation of the δ-function $2\pi\delta^{(2)} = \partial_z\overline{z}^{-1}$, one can then check that the following expression is a solution to this equation:

$$
K(z,\overline{z},w,\overline{w}) = \big\langle X(z,\overline{z})\,X(w,\overline{w})\big\rangle = -\kappa\log\big|z-w\big|^2\;, \tag{2.81}
$$

which gives the result for the two-point function of $X(z,\overline{z})$. In particular, by comparing with Eq. (2.49), we see again that the free boson itself is not a quasi-primary field. However, from the propagator above we can deduce the two-point function of say two chiral fields $j(z)$ by applying derivatives ∂_z and ∂_w to Eq. (2.81)

$$
\begin{aligned}
-\big\langle\partial_z X(z,\overline{z})\,\partial_w X(w,\overline{w})\big\rangle &= -\kappa\,\partial_z\partial_w\Big(-\log\big(z-w\big)-\log\big(\overline{z}-\overline{w}\big)\Big) \\
\big\langle j(z)\,j(w)\big\rangle &= \frac{\kappa}{(z-w)^2}\;,
\end{aligned} \tag{2.82}
$$

and along similar lines we obtain the result in the anti-chiral sector as well as $\langle j(z)\,\overline{j}(\overline{w})\rangle = 0$. We can then summarise that the normalisation constant of the two-point function is $d_{jj} = \kappa$.

Let us finally recall our discussion on p. 37 about current algebras and determine the algebra of the Laurent modes of $j(z)$. Since we only have one such (chiral) current in our theory, the anti-symmetry of $C_{ij}^{k} = -C_{ji}^{k}$ implies $C_{jj}^{j} = 0$ which leads us to

$$\boxed{\big[j_m, j_n\big] = \kappa\, m\, \delta_{m+n,0}}\ . \tag{2.83}$$

The Energy–Momentum Tensor

We will now turn to the energy–momentum tensor for the theory of the free boson. Since we have an action for our theory, we can actually derive this quantity. We define the energy–momentum tensor for the action (2.77) in the following way:

$$T_{ab} = 4\pi\kappa\,\gamma\,\frac{1}{\sqrt{|g|}}\,\frac{\delta\mathcal{S}}{\delta g^{ab}}\ , \tag{2.84}$$

where we have allowed for a to be determined normalisation constant γ and a, b stand for z and $\bar{z}$ respectively. Performing the variation of the action (2.77) using

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}\,g_{ab}\,\delta g^{ab}\ ,$$

we find the following result:

$$T_{zz} = \gamma\,\partial X\partial X\ , \qquad T_{z\bar{z}} = T_{\bar{z}z} = 0\ , \qquad T_{\bar{z}\bar{z}} = \gamma\,\bar{\partial} X\bar{\partial} X\ .$$

However, for a quantum theory we want the expectation value of the energy–momentum tensor to vanish and so we take the normal ordered expression. Focussing only on the chiral part, this reads

$$T(z) = \gamma\ N\big(\partial X\partial X\big)(z) = \gamma\ N\big(jj\big)(z)\ .$$

The constant γ can be fixed via the requirement that $j(z)$ is a primary field of conformal dimension $h = 1$ with respect to $T(z)$. To do so, we expand the energy–momentum tensor as $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$ and find for the Laurent modes

$$L_n = \gamma\ N\big(jj\big)_n = \gamma\sum_{k>-1} j_{n-k}\, j_k + \gamma\sum_{k\le -1} j_k\, j_{n-k}\ , \tag{2.85}$$

where we used the expression for normal ordered products derived in Sect. 2.7. Recalling the Laurent expansion (2.84) of $j(z)$, we can compute the following commutator:

$$
\begin{aligned}
\big[L_m, j_n\big] &= \gamma\, \big[N\big(jj\big)_m, j_n\big] \\
&= \gamma \sum_{k>-1} \Big(j_{m-k}\big[j_k, j_n\big] + \big[j_{m-k}, j_n\big] j_k \Big) \\
&\quad + \gamma \sum_{k\leq -1} \Big(j_k\big[j_{m-k}, j_n\big] + \big[j_k, j_n\big] j_{m-k} \Big) \\
&= \gamma\, \kappa\, \Big(\sum_{k>-1} j_{m-k}\, k\, \delta_{k,-n} + (m-k)\, \delta_{m-k,-n}\; j_k \\
&\quad + \sum_{k\leq -1} j_k\, (m-k)\, \delta_{m-k,-n} + k\, \delta_{k,-n}\; j_{m-k} \Big) \\
&= -\,2\gamma\kappa n\;\, j_{m+n}\ ,
\end{aligned}
$$

where we employed Eq. (2.83). If we compare this expression with Eq. (2.45), we see that for $2\gamma\kappa = 1$ this is the commutator of L_m with a primary field of conformal dimension $h = 1$. Therefore, we can conclude that the energy–momentum tensor reads

$$
T(z) = \frac{1}{2\kappa}\; N\big(jj\big)(z)\ . \tag{2.86}
$$

The Central Charge

After having determined the energy–momentum tensor $T(z)$ up to a constant, we can now ask what is the central charge of the free boson conformal field theory. To determine c we employ the Virasoro algebra to compute

$$
\big\langle 0\big| L_{+2} L_{-2} \big|0\big\rangle = \big\langle 0\big|\big[L_2, L_{-2}\big]\big|0\big\rangle = \frac{c}{2}\,, \tag{2.87}
$$

where we used that $L_n|0\rangle = 0$ for $n > -2$. Next, we recall $L_{\mp 2} = \frac{1}{2\kappa} N(jj)_{\mp 2}$ from which we find

$$
L_{-2}\,\big|0\big\rangle = \frac{1}{2\kappa}\; j_{-1} j_{-1} \big|0\big\rangle\,, \qquad \big\langle 0\big| L_{+2} = \frac{1}{2\kappa}\,\big\langle 0\big|\Big(j_2\, j_0 + j_1\, j_1 \Big) = \frac{1}{2\kappa}\,\big\langle 0\big|\; j_1\, j_1\ ,
$$

where we used that $\big\langle 0\big|\; j_2\, j_0 = -\big\langle 0\big|\; j_0\, j_2 = 0$. Employing these expressions in Eq. (2.87), we obtain

$$\begin{aligned}\frac{c}{2} &= \frac{1}{4\kappa^2}\,\langle 0|\; j_1\; j_1\; j_{-1}\; j_{-1}\;|0\rangle \\ &= \frac{1}{4\kappa^2}\,\Big(\langle 0|\; j_1\; j_{-1}\; j_1\; j_{-1}\;|0\rangle + \langle 0|\; j_1\,\underbrace{[j_1, j_{-1}]}_{\kappa}\,j_{-1}\;|0\rangle\Big) \\ &= \frac{1}{4\kappa^2}\,\Big(\langle 0|[j_1, j_{-1}][j_1, j_{-1}]|0\rangle + \kappa\,\langle 0|[j_1, j_{-1}]|0\rangle\Big) \\ &= \frac{1}{4\kappa^2}\,2\kappa^2 = \frac{1}{2}\end{aligned}$$

where we employed $[j_1, j_{-1}] = \kappa$ as well as that $j_k|0\rangle = 0$ for $k > -1$ and that $\langle 0|j_k = 0$ for $k < 1$. We have therefore shown

> The conformal field theory of a free boson has central charge $c = 1$.

Remarks

- Let us make contact with our discussion in Sect. 2.6.3 and compare our results to Eq. (2.58). First, by recalling equation (2.56) we obtain from Eq. (2.87) that $d_{LL} = \frac{c}{2}$. Second, with the help of Eq. (2.57) we can determine C_{LLL} as follows:

$$C_{LLL} = \langle 0\big|L_2\,L_0\,L_{-2}\big|0\rangle = \langle 0\big|L_2\,\big[L_0, L_{-2}\big]\big|0\rangle = 2\,\langle 0\big|L_2\,L_{-2}\big|0\rangle = 2\,\frac{c}{2} = c$$

  where we also employed Eq. (2.87). Finally, referring to p. 34, we use $C_{ij}^{k} = C_{ijl}\,(d_{lk})^{-1}$ to find $C_{LL}^{L} = 2$ which is in agreement with Eq. (2.58). Therefore, as expected, the Laurent modes of the energy–momentum tensor (2.86) of the free boson satisfy the Virasoro algebra.
- Note that on p. 16, we have chosen the normalisation of the central extension $p(2, -2)$ in such a way that we obtain $c = 1$ for the free boson.
- Usually, one chooses the normalisation constant κ to be $\kappa = 1$ in order to simplify various expressions such as

$$\big\langle\, j(z)\, j(w)\,\big\rangle = \frac{1}{(z-w)^2} \qquad \text{and} \qquad T(z) = \frac{1}{2}\,N\big(jj\big)(z)\,. \tag{2.88}$$

Centre of Mass Position and Momentum

Let us come back to the chiral and anti-chiral currents from Eq. (2.80) and recall once more their Laurent expansion

$$j(z) = i\,\partial X(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1}\,, \qquad \bar{j}(\bar{z}) = i\,\bar{\partial} X(z) = \sum_{n\in\mathbb{Z}} \bar{j}_n z^{-n-1}\,,$$

which we can integrate to find

$$X(z,\bar{z}) = x_0 - i\left(j_0 \ln z + \bar{j}_0 \ln \bar{z}\right) + i \sum_{n\neq 0} \frac{1}{n}\left(j_n\, z^{-n} + \bar{j}_n\, \bar{z}^{-n}\right). \tag{2.89}$$

If we identify $\bar{z}$ with the complex conjugate z^* of z, then the field X is defined on the complex plane. As such, it has to be invariant under rotations $z \mapsto e^{2\pi i} z$ which, referring to Eq. (2.89), implies that

$$j_0 = \bar{j}_0\ . \tag{2.90}$$

However, as we will see in Sect. 4.2.2, for the free boson $X(z,\bar{z})$ compactified on a circle, this relation will be modified.

Let us furthermore recall from the beginning of this section that the example of the free boson can be related to string theory. In particular, using the mapping (2.76), we can express Eq. (2.89) in terms of coordinates x^0 and x^1 on the cylinder

$$\begin{aligned} X\left(x^0, x^1\right) &= x_0 - i\left(j_0 + \bar{j}_0\right)x^0 + \left(j_0 - \bar{j}_0\right)x^1 \\ &\quad + i \sum_{n\neq 0} \frac{1}{n}\left(j_n\, e^{-n\left(x^0 + i x^1\right)} + \bar{j}_n\, e^{-n\left(x^0 - i x^1\right)}\right). \end{aligned}$$

Computing the centre of mass momentum π_0 of a string, we obtain

$$\pi_0 = \frac{1}{4\pi} \int_0^{2\pi} dx^1\, \frac{\partial X\left(x^0, x^1\right)}{\partial\left(-i x^0\right)} = \frac{j_0 + \bar{j}_0}{2} = j_0\ , \tag{2.91}$$

where the additional factor of $(-i)$ is due to the fact that we are working with Euclidean signature. Similarly, the following expression

$$\frac{1}{2\pi} \int_0^{2\pi} dx^1\, X\left(x^0 = 0, x^1\right) = x_0$$

shows that x_0 is the centre of mass coordinate of a string. Performing then the usual quantisation, not only j_n and $\bar{j}_n$ are promoted to operators but also x_0. Since x_0 and π_0 are the position, and momentum operator, respectively, we naturally impose the commutation relation

$$\left[x_0\,,\,\pi_0\right] = i\ . \tag{2.92}$$

Vertex Operator

As we have mentioned earlier, the free boson $X(z,\bar{z})$ is not a conformal field since its conformal dimensions vanish $(h,\bar{h}) = (0,0)$. However, using $X(z,\bar{z})$ we can define the so-called vertex operators which have non-vanishing conformal weights.

The vertex operator $V(z,\overline{z}) =: e^{i\alpha X(z,\overline{z})}$: is a primary field of conformal dimension $(h,\overline{h}) = (\frac{\alpha^2}{2}, \frac{\alpha^2}{2})$ with respect to the energy–momentum tensors $T(z) = \frac{1}{2}N(jj)(z)$ and $\overline{T}(\overline{z}) = \frac{1}{2}N(\overline{j}\,\overline{j})(\overline{z})$.

Here we have used the notation : ... : to denote the normal ordering which is also common in the literature. In the following, we will verify that this vertex operator has indeed the conformal dimensions stated above.

To do so, we start by making the expression for the vertex operator more concrete using Eq. (2.89). Keeping in mind that j_n for $n > -1$ are annihilation operators, we can perform the normal ordering to obtain

$$V(z,\overline{z}) = \exp\left(i\alpha x_0 - \alpha \sum_{n\leq -1} \frac{j_n}{n}\, z^{-n}\right) \cdot \exp\left(\alpha\pi_0 \ln z - \alpha \sum_{n\geq 1} \frac{j_n}{n} z^{-n}\right) \cdot \overline{v}(\overline{z})\ , \tag{2.93}$$

where for convenience we have put all the anti-holomorphic dependence into $\overline{v}(\overline{z})$. Next, let us compute the j_0 eigenvalue of this vertex operator which we can infer from $[j_0, V]$. Because of $[j_m, j_n] = m\,\delta_{m,-n}$, we see that j_0 commutes with all j_n and so we only need to evaluate

$$\left[j_0, e^{i\alpha x_0}\right] = \sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!} \underbrace{\left[j_0, x_0^k\right]}_{=\,k\,(-i)\,x_0^{k-1}} = -\sum_{k=1}^{\infty} \frac{i\,(i\alpha)^k}{(k-1)!}\, x_0^{k-1} = -i\,(i\alpha)\, e^{i\alpha x_0}, \tag{2.94}$$

where we employed Eq. (2.92). We therefore find that $[j_0, V] = \alpha V$. Recalling then our definition Eq. (2.28) of an asymptotic state and computing

$$j_0\,\big|\alpha\big\rangle = \lim_{z,\overline{z}\to 0} \left[j_0, V_\alpha(z,\overline{z})\right] \big|0\big\rangle = \lim_{z,\overline{z}\to 0} \alpha\; V(z,\overline{z})\,\big|0\big\rangle = \alpha\,\big|\alpha\big\rangle\ ,$$

we see that the j_0 eigenvalue of the vertex operator (2.93) is α.

Now, we are going to determine the conformal dimension of the vertex operator (2.93) by computing the commutator $[L_0, V]$. To do so, let us first recall from Eq. (2.88) the explicit form of L_0

$$L_0 = \frac{1}{2}\, j_0\, j_0 + \frac{1}{2} \sum_{k\geq 1} j_{-k}\, j_k + \frac{1}{2} \sum_{k\leq -1} j_k\, j_{-k}\ .$$

Next, we note again that only j_0 has non-trivial commutation relations with x_0 so we find for the first factor in Eq. (2.93)

$$\left[L_0, e^{i\alpha x_0}\right] = \frac{1}{2}\left[j_0\, j_0, e^{i\alpha x_0}\right] = \frac{\alpha}{2}\Big(j_0\, e^{i\alpha x_0} + e^{i\alpha x_0}\, j_0\Big)\ . \tag{2.95}$$

For the terms in Eq. (2.93) involving the modes j_n with $n \neq 0$, let us define

$$J^- = -\sum_{n\leq -1} \frac{j_n}{n}\, z^{-n}\,, \qquad J^+ = -\sum_{n\geq 1} \frac{j_n}{n}\, z^{-n}\,,$$

for which we calculate using $[L_0, j_n] = -n\, j_n$

$$\big[L_0, J^-\big] = \sum_{n\leq -1} j_n\, z^{-n} = z\,\partial_z J^-\,, \qquad \big[L_0, J^+\big] = z\,\partial_z J^+\,.$$

Performing then the series expansion of the exponential and employing our findings from above, we can evaluate

$$\begin{aligned}\big[L_0, e^{\alpha J^-}\big] &= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \Big[L_0, \big(J^-\big)^k\Big] = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}\, k\, z\,\big(\partial J^-\big)\big(J^-\big)^{k-1} \\ &= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}\, z\,\partial\Big(\big(J^-\big)^k\Big) = z\,\partial\big(e^{\alpha J^-}\big)\,,\end{aligned}$$

and we find a similar result for J^+. Observing finally that $[L_0, \overline{v}(\overline{z})] = 0$, we are now in the position to calculate the full commutator of L_0 and $V(z, \overline{z})$ to determine the conformal dimension of $V(z, \overline{z})$

$$\begin{aligned}\big[L_0, V(z,\overline{z})\big] &= \big[L_0, e^{i\alpha x_0}\big] e^{\alpha J^-} z^{\alpha\pi_0} e^{\alpha J^+}\overline{v} + e^{i\alpha x_0}\big[L_0, e^{\alpha J^-}\big] z^{\alpha\pi_0} e^{\alpha J^+}\overline{v} \\ &\qquad + e^{i\alpha x_0} e^{\alpha J^-}\big[L_0, z^{\alpha\pi_0}\big] e^{\alpha J^+}\overline{v} + e^{i\alpha x_0} e^{\alpha J^-} z^{\alpha\pi_0}\big[L_0, e^{\alpha J^+}\big]\overline{v} \\ &= \frac{\alpha}{2}\big(j_0 e^{i\alpha x_0} + e^{i\alpha x_0} j_0\big) e^{\alpha J^-} z^{\alpha\pi_0} e^{\alpha J^+}\overline{v} + e^{i\alpha x_0}\big(z\partial_z e^{\alpha J^-}\big) z^{\alpha\pi_0} e^{\alpha J^+}\overline{v} \\ &\qquad + 0 + e^{i\alpha x_0} e^{\alpha J^-} z^{\alpha\pi_0}\big(z\partial_z e^{\alpha J^+}\big)\overline{v} \\ &= \frac{\alpha}{2}\big(j_0 e^{i\alpha x_0} + e^{i\alpha x_0} j_0\big) e^{\alpha J^-} z^{\alpha\pi_0} e^{\alpha J^+}\overline{v} + z\partial V(z,\overline{z}) \\ &\qquad - e^{i\alpha x_0} e^{\alpha J^-}\big(z\partial_z z^{\alpha\pi_0}\big) e^{\alpha J^+}\overline{v} \\ &= \frac{\alpha}{2}\big(j_0 e^{i\alpha x_0} + e^{i\alpha x_0} j_0\big) e^{\alpha J^-} z^{\alpha\pi_0} e^{\alpha J^+}\overline{v} + z\partial V(z,\overline{z}) \\ &\qquad - \alpha\, e^{i\alpha x_0} j_0\, e^{\alpha J^-} z^{\alpha\pi_0} e^{\alpha J^+}\overline{v} \\ &= \frac{\alpha}{2}\big[j_0, e^{i\alpha x_0}\big] e^{\alpha J^-} z^{\alpha\pi_0} e^{\alpha J^+}\overline{v} + z\partial V(z,\overline{z}) \\ &= \frac{\alpha^2}{2}\, V(z,\overline{z}) + z\,\partial V(z,\overline{z})\end{aligned}$$

where we used Eqs. (2.94) and (2.95). Let us then again recall our definition (2.28) of an asymptotic state in terms of an operator and compute

$$L_0 \left|\alpha\right\rangle = \lim_{z,\bar{z}\to 0} \left[L_0, V_\alpha(z,\bar{z})\right] \left|0\right\rangle = \lim_{z,\bar{z}\to 0} \left(\frac{\alpha^2}{2} V(z,\bar{z}) + z\, \partial V(z,\bar{z})\right) \left|0\right\rangle$$
$$= \frac{\alpha^2}{2} \lim_{z,\bar{z}\to 0} V(z,\bar{z}) \left|0\right\rangle = \frac{\alpha^2}{2} \left|\alpha\right\rangle .$$

The conformal weight of a vertex operator $V_\alpha(z,\bar{z})$ therefore is $h = \frac{\alpha^2}{2}$, and a similar result is obtained in the anti-chiral sector. This verifies our statement from the beginning of this paragraph regarding the conformal weights of the vertex operator. In order to show that $V(z,\bar{z})$ is a primary field, one can compute along similar lines the commutator $[L_m, V(z,\bar{z})]\,|0\rangle$ and compare with the definition of a primary field given in Eq. (2.45).

Next, let us note that the action of a free boson (2.79) is invariant under transformations $X(z,\bar{z}) \mapsto X(z,\bar{z}) + a$ where a is an arbitrary constant. In order for the correlator of two vertex operators $\langle\, V_\alpha\, V_\beta\, \rangle$ to respect this symmetry, we infer from the definition $V_\alpha =: e^{i\alpha X} :$ the condition $\alpha + \beta = 0$. Recalling then our discussion in Sect. 2.6.2 and keeping in mind that $V_\alpha(z,\bar{z})$ is a primary field, for the two-point function of two vertex operators we find

$$\left\langle V_{-\alpha}(z,\bar{z})\, V_\alpha(w,\bar{w})\right\rangle = \frac{1}{(z-w)^{\alpha^2}\,(\bar{z}-\bar{w})^{\alpha^2}} .$$

Here we have included the result for the anti-holomorphic sector, which can be obtained in a similar fashion as the holomorphic part. However, we will study non-holomorphic OPEs in much more detail in Sect. 2.12. Since $\alpha + \beta = 0$, for the two-point function of vertex operators with equal j_0 eigenvalues, we have

$$\left\langle V_{+\alpha}\, V_{+\alpha}\right\rangle = \left\langle V_{-\alpha}\, V_{-\alpha}\right\rangle = 0 .$$

Let us also mention that the current $j(z) = i\partial X(z,\bar{z})$ is conserved and that the conserved charge is $Q = \oint \frac{dz}{2\pi i}\, j(z) = j_0$. We can thus interpret α as the charge of an vertex operator $V_\alpha(z,\bar{z})$ and the requirement $\alpha + \beta = 0$ as charge conservation.

In passing, we note that vertex operators play a very important role in string theory, where the charge α is interpreted as the space–time momentum along the space–time direction X. The condition of charge conservation then corresponds to momentum conservation in space–time.

Current Algebra

After having verified the conformal dimension of the vertex operator $V_\alpha =: e^{i\alpha X} :$ to be $(h,\bar{h}) = (\frac{\alpha^2}{2}, \frac{\alpha^2}{2})$, let us now turn to the special case $\alpha = \pm\sqrt{2}$ for which $V_{\pm\sqrt{2}}(z,\bar{z})$ has conformal dimension $(h,\bar{h}) = (1,1)$. Therefore, following our definition from p. 37, these fields are currents. In order to simplify our discussion in this paragraph, let us focus only on the holomorphic part of the vertex operator (including the position operator x_0) which we write as

$$j^{\pm}(z) = : e^{\pm i\sqrt{2}X} : ,$$

and study the current algebra of $j^{\pm}(z)$ and $j(z) = i(\partial X)(z)$. On p. 37, we have given the general form of a current algebra of quasi-primary fields which we recall for convenience

$$\left[j_{(i)m}, j_{(j)n} \right] = \sum_k C_{ij}^k \; j_{(k)m+n} + d_{ij} \; m \; \delta_{m,-n} \; , \qquad\qquad C_{ij}^k = -C_{ji}^k \; .$$

Let us then determine the normalisation constants of the two-point function d_{ij}. By the argument that the overall j_0 charge in a correlation function should vanish, we see that

$$d_{jj} = d_{+-} = d_{-+} = 1 \; , \qquad\qquad d_{j\pm} = d_{\pm j} = 0 \; ,$$

where we use subscripts $\pm$ for $j^{\pm}(z)$ and employed the usual normalisation. Next, using Eq. (2.57) as well as relations (2.26) and (2.28), we can compute

$$\begin{aligned} C_{+j-} = \langle 0 | \, j_1^+ \, j_0 \, j_{-1}^- \, | 0 \rangle &= \lim_{z\to 0} \langle 0 | \, j_1^+ \, j_0 \, V_{-\sqrt{2}}(z) \, | 0 \rangle \\ &= -\sqrt{2} \, \langle 0 | \, j_1^+ \, j_{-1}^- \, | 0 \rangle \\ &= -\sqrt{2} \, d_{+-} = -\sqrt{2} \; , \end{aligned}$$

and a similar computation leads to $C_{-j+} = +\sqrt{2}$. Noting that also for the three-point function the overall j_0 charge has to vanish and using the relation $C_{ijk} = C_{ij}^l d_{lk}$ together with the anti-symmetry of C_{ij}^k, we can determine the non-vanishing structure constants to be

$$C_{j+}^+ = -C_{+j}^+ = +\sqrt{2} \; , \qquad C_{j-}^- = -C_{-j}^- = -\sqrt{2} \; , \qquad C_{+-}^j = -C_{-+}^j \; .$$

The only unknown constant C_{+-}^j can be fixed using the relation $C_{ijk} = C_{ij}^l d_{lk}$ from p. 34 in the following way:

$$C_{+-}^j = C_{+-j} \, d^{jj} = C_{-j}^- \, d_{-+} \, d^{jj} = \sqrt{2} \cdot 1 \cdot \frac{1}{1} = \sqrt{2} \; .$$

Combining all these results, we can finally write down the current algebra of $j^{\pm}(z)$ and $j(z)$ which reads

$$\begin{aligned} \left[j_m, j_n \right] &= m \, \delta_{m+n,0} \; , & \left[j_m^{\pm}, j_n^{\pm} \right] &= 0 \; , \\ \left[j_m, j_n^{\pm} \right] &= \pm\sqrt{2} \; j_{m+n}^{\pm} \; , & \left[j_m^+, j_n^- \right] &= \sqrt{2} \; j_{m+n} + m \, \delta_{m+n,0} \; . \end{aligned} \tag{2.96}$$

However, in order to highlight the underlying structure, let us make the following definitions:

$$j^1 = \frac{1}{\sqrt{2}}\left(j^+ + j^-\right), \qquad j^2 = \frac{1}{\sqrt{2}i}\left(j^+ - j^-\right), \qquad j^3 = j\,.$$

The commutation relations for the Laurent modes of the new currents are determined using the relations in Eq. (2.96). We calculate for instance

$$\begin{aligned}
\left[j^1_m, j^2_n\right] &= \frac{1}{2i}\left(-\left[j^+_m, j^-_n\right] + \left[j^-_m, j^+_n\right]\right) = \frac{1}{2i}\left(-2\sqrt{2}\, j_{m+n}\right) &&= +i\sqrt{2}\, j^3_{m+n}\,, \\
\left[j^3_m, j^1_n\right] &= \frac{1}{\sqrt{2}}\left[j^3_m, j^+_n + j^-_n\right] = \frac{1}{\sqrt{2}}\left(\sqrt{2} j^+_{m+n} - \sqrt{2} j^-_{m+n}\right) &&= +i\sqrt{2}\, j^2_{m+n}\,, \\
\left[j^1_m, j^1_n\right] &= \frac{1}{2}\left(\left[j^+_m, j^-_n\right] + \left[j^-_m, j^+_n\right]\right) = \frac{1}{2}(m-n)\,\delta_{m+n,0} &&= \quad m\,\delta_{m+n,0}\,,
\end{aligned}$$

from which we infer the general expression

$$\boxed{\left[j^i_m, j^j_n\right] = +i\sqrt{2}\sum_k \epsilon^{ijk}\, j^k_{m+n} + m\,\delta^{ij}\,\delta_{m,-n}}$$

where ϵ^{ijk} is the totally anti-symmetric tensor. These commutation relations define the $\mathfrak{su}(2)$ Kač–Moody algebra at level $k = 1$, which is usually denoted as $\widehat{\mathfrak{su}}(2)_1$. These algebras are discussed in Chap. 3 in more generality. Furthermore, this current algebra is related to the theory of the free boson X compactified on a radius $R = \frac{1}{\sqrt{2}}$ which we will study in Sect. 4.2.2.

Hilbert Space

Let us note that the Hilbert space of the free boson theory contains for instance the following chiral states:

$$\begin{array}{llll}
\text{level } 1: & j_{-1}\,|0\rangle\,, & & \\
\text{level } 2: & j_{-2}\,|0\rangle\,, & j_{-1}\,j_{-1}\,|0\rangle\,, & \\
\text{level } 3: & j_{-3}\,|0\rangle\,, & j_{-2}\,j_{-1}\,|0\rangle\,, & j_{-1}\,j_{-1}\,j_{-1}\,|0\rangle\,, \\
\ldots & \ldots & &
\end{array} \tag{2.97}$$

where we have used that $[j_{-m}, j_{-n}] = 0$ for $m, n \geq 0$. Taking also the anti-chiral sector as well as $[j_m, \overline{j}_n] = 0$ into account, we can conclude that the Hilbert space of the free boson theory is

$$\mathcal{H} = \left\{\text{Fock space freely generated by } j_{-n}, \overline{j}_{-m} \text{ for } n, m \geq 1\right\}.$$

The number of states at each level N is given by the number of partitions $P(N)$ of N whose generating function we have already encountered at the end of Sect. 2.8. The generating function for the degeneration of states at each level N in the chiral sector therefore is

$$\mathcal{Z}(q) = \prod_{n\geq 1} \frac{1}{1-q^n} = \sum_{N=0}^{\infty} P(N)\, q^N . \tag{2.98}$$

Combining now the chiral and anti-chiral sectors, we obtain

$$\mathcal{Z}(q,\overline{q}) = \prod_{n\geq 1} \frac{1}{(1-q^n)(1-\overline{q}^n)} .$$

In Chap. 4, we will identify such expressions with partition functions of conformal field theories and relate them to modular forms.

2.9.2 The Free Fermion

Motivation

As a second example for a conformal field theory, we will study the action of a free Majorana fermion in two-dimensional Minkowski space with metric $h_{\alpha\beta} = \text{diag}(+1,-1)$

$$\mathcal{S} = \frac{1}{4\pi\kappa} \int dx^0\, dx^1 \sqrt{|h|}\, (-i)\, \overline{\Psi}\, \gamma^\alpha\, \partial_\alpha\, \Psi , \tag{2.99}$$

where κ is a normalisation constant. Here, $\overline{\Psi}$ is defined as $\overline{\Psi} = \Psi^\dagger \gamma^0$ where $\dagger$ denotes hermitian conjugation and the $\{\gamma^\alpha\}$ are two-by-two matrices satisfying the Clifford algebra

$$\gamma^\alpha\, \gamma^\beta + \gamma^\beta\, \gamma^\alpha = 2\, h^{\alpha\beta}\, \mathbb{1}_2 ,$$

with $\mathbb{1}_2$ the two-by-two unit matrix. There are various representations of γ-matrices satisfying this algebra which are, however, equivalent. We make the following choice:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

for which the Majorana condition becomes that the components $\psi, \overline{\psi}$ of the spinor Ψ are real. We then perform a Wick rotation $x_1 \to i x_1$ under which the partial derivative transforms as $\partial_1 \to -i\partial_1$. Effectively, the Wick rotation means choosing the γ-matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ,$$

and we furthermore note that the Wick rotation introduces an additional factor of i for Eq. (2.99). We can simplify the action (2.99) by observing

$$\gamma^0 \gamma^\mu \partial_\mu = \gamma^0 \left(\gamma^0 \partial_0 + \gamma^1 \partial_1\right) = \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix} = 2 \begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix} ,$$

where we have defined $z = x^0 + ix^1$. As we have seen before, it is convenient to work with fields depending on complex variables and so we write the Majorana spinors as

$$\Psi = \begin{pmatrix} \psi(z, \bar z) \\ \overline{\psi}(z, \bar z) \end{pmatrix} .$$

Note that $\psi(z, \bar z)$ and $\overline{\psi}(z, \bar z)$ are still real fields, in particular $\psi^\dagger = \psi$ and $\overline{\psi}^\dagger = \overline{\psi}$. Employing then the various arguments above, we can write the action (2.99) after a Wick rotation in the following way:

$$\begin{aligned} \mathcal{S} &= \frac{1}{4\pi\kappa} \int dz\, d\bar z \sqrt{|g|}\, 2\, \Psi^\dagger \begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix} \Psi \\ &= \frac{1}{4\pi\kappa} \int dz\, d\bar z \left(\psi\, \bar\partial\, \psi + \overline{\psi}\, \partial\, \overline{\psi} \right) \end{aligned} \tag{2.100}$$

where we used that the components of the metric g, obtained from $h_{\alpha\beta}$ via the change of coordinates $z = x^0 + ix^1$, read

$$g_{ab} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} , \qquad g^{ab} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} . \tag{2.101}$$

Basic Properties

From a conformal field theory point of view, we do not necessarily need the derivation above but can simply start from the action (2.100)

$$\boxed{\mathcal{S} = \frac{1}{4\pi\kappa} \int dz\, d\bar z \left(\psi\, \bar\partial\, \psi + \overline{\psi}\, \partial\, \overline{\psi} \right)} . \tag{2.102}$$

The equations of motion for this theory are obtained by varying the action with respect to the fields ψ and $\overline{\psi}$ which reads

$$
\begin{aligned}
0 = \delta_\psi \mathcal{S} &= \frac{1}{4\pi\kappa}\int d^2z \left(\delta\psi\,\overline{\partial}\psi + \psi\overline{\partial}(\delta\psi)\right) \\
&= \frac{1}{4\pi\kappa}\int d^2z \left(\delta\psi\,\overline{\partial}\psi + \overline{\partial}(\psi\delta\psi) - (\overline{\partial}\psi)\,\delta\psi\right) \\
&= \frac{1}{2\pi\kappa}\int d^2z\,\delta\psi\,\overline{\partial}\psi
\end{aligned}
$$

where we performed a partial integration on the second term and noted that fermionic fields anti-commute. Since the equation above has to be satisfied for all variations $\delta\psi$, we find for the equations of motion

$$
\partial\overline{\psi} = \overline{\partial}\psi = 0\,, \tag{2.103}
$$

where we also included the result for the variation with respect to $\delta\overline{\psi}$ obtained along similar lines. We can then conclude that $\psi = \psi(z)$ is a chiral field and $\overline{\psi} = \overline{\psi}(\overline{z})$ is an anti-chiral field.

Next, we will determine the conformal properties of the fields $\psi(z)$ and $\overline{\psi}(z)$. By performing similar steps as for the example of the free boson, we see that the action (2.102) is invariant under conformal transformations if the fields ψ and $\overline{\psi}$ are primary with conformal dimensions $(h,\overline{h}) = (\frac{1}{2},0)$, and $(h,\overline{h}) = (0,\frac{1}{2})$ respectively. Let us verify this observation by computing

$$
\begin{aligned}
\mathcal{S} \;\longrightarrow\;& \frac{1}{4\pi\kappa}\int dz\,d\overline{z}\left(\psi'(z,\overline{z})\,\partial_{\overline{z}}\,\psi'(z,\overline{z}) + \overline{\psi}'(z,\overline{z})\,\partial_z\,\overline{\psi}'(z,\overline{z})\right) \\
&= \frac{1}{4\pi\kappa}\int \frac{\partial z}{\partial w}\,dw\,\frac{\partial\overline{z}}{\partial\overline{w}}\,d\overline{w}\left(\left(\frac{\partial w}{\partial z}\right)^{\frac{1}{2}}\psi(w,\overline{w})\,\frac{\partial\overline{w}}{\partial\overline{z}}\,\partial_{\overline{w}}\left(\frac{\partial w}{\partial z}\right)^{\frac{1}{2}}\psi(w,\overline{w})\right. \\
&\quad \left. + \left(\frac{\partial\overline{w}}{\partial\overline{z}}\right)^{\frac{1}{2}}\overline{\psi}(w,\overline{w})\,\frac{\partial w}{\partial z}\,\partial_w\left(\frac{\partial\overline{w}}{\partial\overline{z}}\right)^{\frac{1}{2}}\overline{\psi}(w,\overline{w})\right) \\
&= \frac{1}{4\pi\kappa}\int dw\,d\overline{w}\left(\psi(w,\overline{w})\,\partial_{\overline{w}}\,\psi(w,\overline{w}) + \overline{\psi}(w,\overline{w})\,\partial_w\,\overline{\psi}(w,\overline{w})\right)
\end{aligned}
$$

which shows that the action is indeed invariant under conformal transformations if ψ and $\overline{\psi}$ are primary fields of conformal dimension $\frac{1}{2}$. Furthermore, because we are studying a free theory, the engineering dimension $\frac{1}{2}$ agrees with the dimension in the quantum theory.

Finally, let us note that due to the fermionic nature of fields in our theory, there are two different possibilities for their behaviour under rotations by 2π. In particular, focussing on the chiral sector, on the complex plane we have

$$
\begin{aligned}
\psi(e^{2\pi i}z) &= +\psi(z) \qquad \text{Neveu–Schwarz sector (NS),} \\
\psi(e^{2\pi i}z) &= -\psi(z) \qquad \text{Ramond sector (R).}
\end{aligned} \tag{2.104}
$$

Radial Ordering and Laurent Expansion

Let us recall that our theory of the free fermion is defined on the complex plane with coordinate $z = x^0 + ix^1$, where x^0 and x^1 are coordinates of $\mathbb{R}^2$. However, we have seen in Sect. 2.4 and explicitly for the example of the free boson, the quantum theory is usually defined on a cylinder of infinite length. Without providing a detailed derivation, let us just assume we started on a cylinder and have performed the mapping to the complex plane giving us our present theory.

This allows us in particular to introduce the concept of radial ordering also for the fermions. Taking into account the fermionic nature of the fields, we define

$$R\big(\,\Psi(z)\Theta(w)\,\big) := \begin{cases} +\Psi(z)\,\Theta(w) & \text{for} \quad |z| > |w|\,, \\ -\Theta(w)\,\Psi(z) & \text{for} \quad |w| > |z|\,. \end{cases} \tag{2.105}$$

Next, keeping in mind the conformal weight $\frac{1}{2}$ of our fields, we can perform a Laurent expansion of $\psi(z)$ in the usual way

$$\psi(z) = \sum_r \psi_r\, z^{-r-\frac{1}{2}}\;, \tag{2.106}$$

and similarly for the anti-chiral field. However, due to the two possibilities shown in Eq. (2.104), the values for r differ between the Neveu–Schwarz and Ramond sectors. It is easy to see that the following choice is consistent with Eq. (2.104):

$$\begin{aligned} &r \in \mathbb{Z} + \tfrac{1}{2} && \text{Neveu–Schwarz sector (NS)}, \\ &r \in \mathbb{Z} && \text{Ramond sector (R)}. \end{aligned}$$

OPE and Laurent Mode Algebra

Recalling our discussion on p. 34 together with the observation that $\psi(z)$ is a primary field of conformal dimension $\frac{1}{2}$, we can determine the following OPE:

$$\psi(z)\,\psi(w) = \frac{\kappa}{z-w} + \cdots\;, \tag{2.107}$$

where the ellipsis denote non-singular terms. The normalisation constant of the two-point function κ is the same as in the action (2.102) which can be verified by computing the propagator of Eq. (2.102). Note furthermore that this OPE respects the fermionic property of ψ since interchanging $z \leftrightarrow w$ leads to a minus sign on the right-hand side which is obtained on the left-hand side by interchanging fermions. This fact also explains why there is no single fermion $\psi(z)$ on the right-hand side because $(z-w)^{-\frac{1}{2}}\,\psi(z)$ would not respect the fermionic nature of the OPE.

Let us now determine the algebra of the Laurent modes of $\psi(z)$. To do so, we recall that the modes in Eq. (2.106) can be expressed in the following way:

$$\psi_r = \oint \frac{dz}{2\pi i}\, \psi(z)\, z^{r-\frac{1}{2}} \; .$$

Because the fields under consideration are fermions, we are going to evaluate anti-commutators between the modes ψ_r and not commutators. Knowing the OPE (2.107), keeping in mind the radial ordering (2.105) and the deformation of contour integrals illustrated in Fig. 2.4, we calculate

$$\begin{aligned}
\big\{\psi_r, \psi_s\big\} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i}\, \big\{\psi(z), \psi(w)\big\}\, z^{r-\frac{1}{2}}\, w^{s-\frac{1}{2}} \\
&= \oint \frac{dw}{2\pi i}\, w^{s-\frac{1}{2}} \bigg(\oint_{|z|>|w|} \frac{dz}{2\pi i}\, \psi(z)\psi(w)\, z^{r-\frac{1}{2}} \\
&\quad - \oint_{|z|<|w|} \frac{dz}{2\pi i}\, - \psi(w)\psi(z)\, z^{r-\frac{1}{2}} \bigg) \\
&= \oint \frac{dw}{2\pi i}\, w^{s-\frac{1}{2}} \oint_{C(w)} \frac{dz}{2\pi i}\, \underbrace{R\big(\psi(z)\psi(w)\big)}_{\frac{\kappa}{z-w}}\, z^{r-\frac{1}{2}} \\
&= \kappa \oint \frac{dw}{2\pi i}\, w^{r+s-1} \\
&= \kappa\, \delta_{r+s,0} \; .
\end{aligned} \tag{2.108}$$

Energy–Momentum Tensor

Let us also determine the energy–momentum tensor from the action (2.102). For fermionic theories, the definition of T_{ab} differs from the bosonic expression (2.84) and we would have to introduce additional structure to state the explicit form. Let us therefore provide a different but equivalent way to obtain the energy–momentum tensor. The *canonical* energy–momentum tensor for a theory with fields ϕ_i and Lagrangian $\mathcal{L}$ is defined as

$$T^c_{\mu\nu} = 8\pi\kappa\,\gamma \left(-\eta_{\mu\nu}\, \mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial\big(\partial^\mu \phi_i\big)} \partial_\nu \phi_i \right) ,$$

where we allowed for a to be determined normalisation constant γ. However, in general T^c is not symmetric but can be made so using the equations of motion.

For the action (2.102), we can compute the canonical energy–momentum tensor using the metric (2.101) together with the observation that $\partial^z = 2\,\partial_{\bar{z}}$ as well as $\partial^{\bar{z}} = 2\,\partial_z$. We then find

$$T_{zz} = \gamma\, \psi\, \partial\psi \; , \qquad T_{z\bar{z}} = -\gamma\, \overline{\psi}\, \partial\overline{\psi} \; , \qquad T_{\bar{z}z} = -\gamma\, \psi\, \bar{\partial}\psi \; , \qquad T_{\bar{z}\bar{z}} = \gamma\, \overline{\psi}\, \bar{\partial}\overline{\psi} \; .$$

We see that so far, the energy–momentum tensor is not symmetric, but using Eq. (2.103) shows that $T_{z\bar{z}} = T_{\bar{z}z} = 0$. In this way, we have arrived at the result which we would have obtained using a modified form of Eq. (2.84). Focussing then only on the chiral part $T(z) = T_{zz}$ and using at the quantum level the normal ordered expression, we find

$$T(z) = \gamma \; N\big(\psi \, \partial\psi \, \big) \, . \tag{2.109}$$

To compute the Laurent modes $L_m = \gamma N(\psi\partial\psi)_m$ of $T(z)$, we note that the derivation in Sect. 2.7 leading to expressions for normal ordered products was done for bosonic fields. For fermionic fields, we need to take into account the modified radial ordering prescription (2.105). Performing then the same analysis, we find for Eq. (2.67) in the case of fermionic fields

$$N\left(\psi \; \theta\right)_r = - \sum_{s>-h^\theta} \psi_{r-s} \; \theta_s + \sum_{s\leq -h^\phi} \theta_s \; \psi_{r-s} \, . \tag{2.110}$$

However, employing the same reasoning also for our expression involving derivatives (2.69), we can express the Laurent modes of the energy–momentum tensor (2.109) in the following way:

$$L_m = \gamma \sum_{s>-\frac{3}{2}} \psi_{m-s} \, \psi_s \left(s + \frac{1}{2}\right) - \gamma \sum_{s\leq -\frac{3}{2}} \psi_s \, \psi_{m-s} \left(s + \frac{1}{2}\right) \, . \tag{2.111}$$

Let us now fix the constant γ by requiring $\psi(z)$ to be a primary field of conformal dimension $\frac{1}{2}$ with respect to the energy–momentum tensor (2.109). We therefore calculate the commutator between the modes L_m and the Laurent modes of the field $\psi(z)$

$$\begin{aligned}
\big[L_m, \psi_r\big] &= + \gamma \sum_{s>-\frac{3}{2}} \big[\psi_{m-s}\psi_s, \psi_r\big] \left(s + \frac{1}{2}\right) \\
&\quad - \gamma \sum_{s\leq -\frac{3}{2}} \big[\psi_s\psi_{m-s}, \psi_r\big] \left(s + \frac{1}{2}\right) \\
&= + \gamma \sum_{s>-\frac{3}{2}} \left(s + \frac{1}{2}\right) \Big(\psi_{m-s}\big\{\psi_s, \psi_r\big\} - \big\{\psi_{m-s}, \psi_r\big\}\psi_s\Big) \\
&\quad - \gamma \sum_{s\leq -\frac{3}{2}} \left(s + \frac{1}{2}\right) \Big(\psi_s\big\{\psi_{m-s}, \psi_r\big\} - \big\{\psi_s, \psi_r\big\}\psi_{m-s}\Big) \\
&= + \gamma \, \kappa \sum_{s>-\frac{3}{2}} \left(s + \frac{1}{2}\right) \Big(\psi_{m-s} \, \delta_{s,-r} - \psi_s \, \delta_{m-s,-r}\Big) \\
&\quad - \gamma \, \kappa \sum_{s\leq -\frac{3}{2}} \left(s + \frac{1}{2}\right) \Big(\psi_s \, \delta_{m-s,-r} - \psi_{m-s} \, \delta_{s,-r}\Big)
\end{aligned}$$

$$= +\gamma\,\kappa\,\left(\left(-r+\frac{1}{2}\right)\,\psi_{m+r} - \left(m+r+\frac{1}{2}\right)\,\psi_{m+r}\right)$$
$$= +\gamma\,\kappa\,\bigl(-m-2r\bigr)\,\psi_{m+r}\ .$$

If we then choose $\gamma\kappa = \frac{1}{2}$, we find

$$\bigl[L_m, \psi_r\bigr] = \left(-\frac{m}{2} - r\right)\psi_{m+r}\ ,$$

and by comparing with Eq. (2.45), we indeed see that $\psi(z)$ is a primary field of conformal dimension $h = \frac{1}{2}$ with respect to the energy–momentum tensor (2.109). In order to simplify the following formulas, let us in the following choose the usual normalisation:

$$\kappa = +1\ .$$

The Central Charge

We will now compute the central charge c of the free fermion theory and start by recalling a general expression derived previously from the Virasoro algebra

$$\langle 0|\,L_2\,L_{-2}\,|0\rangle = \frac{c}{2}\ .$$

From Eq. (2.111), we then infer that

$$L_{-2}\,|0\rangle = \frac{1}{2}\,\psi_{-\frac{3}{2}}\,\psi_{-\frac{1}{2}}\,|0\rangle\ , \qquad \langle 0|\,L_2 = \frac{1}{2}\,\langle 0|\Bigl(\psi_{\frac{3}{2}}\,\psi_{\frac{1}{2}} + 2\,\psi_{\frac{1}{2}}\,\psi_{\frac{3}{2}}\Bigr)$$
$$= \frac{1}{2}\,\langle 0|\Bigl(\bigl\{\psi_{\frac{3}{2}}\,,\,\psi_{\frac{1}{2}}\bigr\} + \psi_{\frac{1}{2}}\,\psi_{\frac{3}{2}}\Bigr)$$
$$= \frac{1}{2}\,\langle 0|\,\psi_{\frac{1}{2}}\,\psi_{\frac{3}{2}}\ ,$$

and so we calculate using the anti-commutation relations (2.108)

$$\frac{c}{2} = \langle 0|\,L_2\,L_{-2}\,|0\rangle = \frac{1}{4}\,\langle 0|\,\psi_{\frac{1}{2}}\,\psi_{\frac{3}{2}}\,\psi_{-\frac{3}{2}}\,\psi_{-\frac{1}{2}}\,|0\rangle$$
$$= \frac{1}{4}\,\langle 0|\,\psi_{\frac{1}{2}}\bigl\{\psi_{\frac{3}{2}}, \psi_{-\frac{3}{2}}\bigr\}\psi_{-\frac{1}{2}}\,|0\rangle - 0$$
$$= \frac{1}{4}\,\langle 0|\bigl\{\psi_{\frac{1}{2}}, \psi_{-\frac{1}{2}}\bigr\}|0\rangle - 0 = \frac{1}{4}\ .$$

We therefore conclude that

> The central charge of the conformal field theory given by a real free fermion is $c = \frac{1}{2}$.

Complex Fermions and Bosonisation

Let us now turn to a system with two real chiral fermions $\psi^{(1)}(z)$ and $\psi^{(2)}(z)$ which we combine into a complex chiral fermion in the following way:

$$\Psi(z) = \frac{1}{\sqrt{2}}\Big(\psi^{(1)}(z) + i\,\psi^{(2)}(z)\Big) . \tag{2.112}$$

Note that we will always work with chiral quantities so we can denote the complex conjugate of $\Psi(z)$ by $\overline{\Psi}(z)$ which has, however, no relation with the anti-chiral part or the notation in Eq. (2.99). Furthermore, similarly to the real case, we can expand $\Psi(z)$ in a Laurent series as $\Psi(z) = \sum_r \Psi_r z^{-r-\frac{1}{2}}$ and we can easily check that the modes satisfy

$$\big\{\Psi_r, \Psi_s\big\} = \big\{\overline{\Psi}_r, \overline{\Psi}_s\big\} = 0 , \qquad \big\{\Psi_r, \overline{\Psi}_s\big\} = \delta_{r+s,0} ,$$

where we applied the same expansion also for $\overline{\Psi}(z)$. However, besides these two chiral fields of conformal dimension $h = \frac{1}{2}$, we find now an additional field in the theory which is expressed as

$$j(z) = N\big(\Psi\,\overline{\Psi}\big)(z) = -i\;N\big(\,\psi^{(1)}\psi^{(2)}\,\big)(z) . \tag{2.113}$$

In order to write $j(z)$ in terms of the real fermions $\psi^{(1,2)}(z)$, we have used that $N(\psi^{(a)}\psi^{(b)}) = -N(\psi^{(b)}\psi^{(a)})$ with $a, b = 1, 2$. To verify this relation at the level of Laurent modes, we write out the normal ordered product (2.110) and perform the change $s \to -s + r$ in the summation index

$$\begin{aligned}
N\big(\,\psi^{(a)}\psi^{(b)}\big)_r &= -\sum_{s>-\frac{1}{2}} \psi^{(a)}_{r-s}\psi^{(b)}_s + \sum_{s\leq-\frac{1}{2}} \psi^{(b)}_s\psi^{(a)}_{r-s} \\
&= -\sum_{s<r+\frac{1}{2}} \psi^{(a)}_{s}\psi^{(b)}_{r-s} + \sum_{s\geq r+\frac{1}{2}} \psi^{(b)}_{r-s}\psi^{(a)}_{s} \\
&= -\sum_{s\leq-\frac{1}{2}} \psi^{(a)}_{s}\psi^{(b)}_{r-s} + \sum_{s>-\frac{1}{2}} \psi^{(b)}_{r-s}\psi^{(a)}_{s} - \sum_{s=\frac{1}{2}}^{r-\frac{1}{2}}\big\{\psi^{(a)}_{s}, \psi^{(b)}_{r-s}\big\} \\
&= -N\big(\,\psi^{(b)}\psi^{(a)}\big)_r .
\end{aligned}$$

Note that the sum over $\{\psi^{(a)}, \psi^{(b)}\} = \delta^{ab}\,\delta_{r,0}$ in the last line gives no contribution since the summand is only non-zero for $r = 0$ for which the sum disappears. Furthermore, because the modes in general do not (anti-)commute, one has to be careful when changing the summation index.

Let us proceed and perform a series expansion of Eq. (2.113) as $j(z) = \sum_n j_n z^{-n-1}$ and investigate the algebra of the modes j_n. Writing out the normal ordered product and noting that $\psi_r^{(1)}$ and $\psi_s^{(2)}$ anti-commute, we find

$$j_m = -i \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_{m-r}^{(1)} \psi_r^{(2)} \,.$$

Here and in the rest of this paragraph, we concentrate on fermions in the Neveu–Schwarz sector with half-integer modes but the result in the Ramond sector is obtained in a similar fashion. Employing then that the energy–momentum tensor of $\Psi(z)$ is a sum of the individual ones, we calculate

$$\begin{aligned}
\big[L_m, j_n\big] &= \sum_{s\in\mathbb{Z}+\frac{1}{2}} \big[L_m^{(1)} + L_m^{(2)}, -i\, \psi_{n-s}^{(1)}\, \psi_s^{(2)}\big] \\
&= \sum_{s\in\mathbb{Z}+\frac{1}{2}} \Big(-i\big[L_m^{(1)}, \psi_{n-s}^{(1)}\big]\, \psi_s^{(2)} - i\, \psi_{n-s}^{(1)} \big[L_m^{(2)}, \psi_s^{(2)}\big]\Big) \\
&= \sum_{s\in\mathbb{Z}+\frac{1}{2}} \bigg(-\Big(-\frac{m}{2} - n + s\Big)\, i\, \psi_{m+n-s}^{(1)}\, \psi_s^{(2)} - i\, \psi_{n-s}^{(1)} \Big(-\frac{m}{2} - s\Big)\, \psi_{m+s}^{(2)}\bigg) \\
&= -\,n\, j_{m+n} \,,
\end{aligned}$$

where in going from the third to the last line we performed a redefinition $s \to s - m$ in the last summand. Note that this equation is the statement that $j(z)$ defined above is a primary field of conformal dimension $h = 1$ and thus a current. Let us then move on and determine the current algebra

$$\begin{aligned}
\big[j_m, j_n\big] &= \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} -\big[\psi_{m-s}^{(1)}\, \psi_s^{(2)}, \psi_{n-r}^{(1)}\, \psi_r^{(2)}\big] \\
&= \sum_{r,s\in\mathbb{Z}+\frac{1}{2}} \Big(\psi_{m-s}^{(1)} \psi_{n-r}^{(1)} \big\{\psi_s^{(2)}, \psi_r^{(2)}\big\} - \psi_r^{(2)} \psi_s^{(2)} \big\{\psi_{n-r}^{(1)}, \psi_{m-s}^{(1)}\big\}\Big) \\
&= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \Big(\psi_{m+r}^{(1)} \psi_{n-r}^{(1)} - \psi_r^{(2)} \psi_{n+m-r}^{(2)}\Big) \,.
\end{aligned}$$

In order to proceed, we have to perform a careful analysis of the first term in the sum above. As mentioned above, because of the fermions do in general not anti-commute, we cannot simply shift the summation index but have to also take care of the normal ordering. We find

$$\begin{aligned}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_{m+r}\psi_{n-r} &= \sum_{r\leq-\frac{1}{2}+n} \psi_{m+r}\,\psi_{n-r} + \sum_{r\geq\frac{1}{2}+n} \psi_{m+r}\,\psi_{n-r} \\ &= \sum_{r\leq-\frac{1}{2}+n} \psi_{m+r}\,\psi_{n-r} - \sum_{r\geq\frac{1}{2}+n} \psi_{n-r}\,\psi_{m+r} + \sum_{r\geq\frac{1}{2}+n} \left\{\psi_{m+r},\psi_{n-r}\right\} \\ &= \sum_{r=\frac{1}{2}-m}^{-\frac{1}{2}+n} \psi_{m+r}\,\psi_{n-r} + \sum_{r\geq\frac{1}{2}+n} \delta_{m+n,0} = \sum_{r\geq\frac{1}{2}+n} \delta_{m+n,0}\ ,\end{aligned}$$

where the step from the second to the third line can be understood by relabelling $r \to -r + n - m$ in the second sum. Furthermore, the last step is easily verified using the anti-commutation relation for fermions. With this result, we arrive at

$$\left[j_m, j_n\right] = \sum_{r\geq\frac{1}{2}+n} \delta_{m+n,0} - \sum_{r\geq\frac{1}{2}+m+n} \delta_{m+n,0} = \sum_{r=\frac{1}{2}+n}^{-\frac{1}{2}+m+n} \delta_{m+n,0} = m\,\delta_{m+n,0}\ .$$

A current satisfying this algebra is called a $U(1)$ current and we will study such theories in more detail in Chap. 3. However, let us finally determine the $U(1)$ charge of the complex fermion $\Psi(z)$ and its complex conjugate $\overline{\Psi}(z)$. To do so, we calculate

$$\begin{aligned}\left[j_m, \Psi_s\right] &= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \left[-i\,\psi^{(1)}_{m-r}\psi^{(2)}_r,\ \frac{1}{\sqrt{2}}\left(\psi^{(1)}_s + i\,\psi^{(2)}_s\right)\right] \\ &= \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \left(i\left\{\psi^{(1)}_{m-r},\psi^{(1)}_s\right\}\psi^{(2)}_r + \psi^{(1)}_{m-r}\left\{\psi^{(2)}_r,\psi^{(2)}_s\right\}\right) \\ &= +\Psi_{m+s}\ ,\end{aligned}$$

and for $\overline{\Psi}(z)$, we find along the same lines that $[j_m, \overline{\Psi}_s] = -\overline{\Psi}_{m+s}$. Therefore, the complex fields carry charge ± 1 under the current $j(z)$. Let us now summarise the algebra generated by the complex fermions $\Psi(z)$ and $\overline{\Psi}(z)$ defined in Eq. (2.112) and the current $j(z)$ defined in Eq. (2.113) as

$$\boxed{\begin{aligned} \left\{\Psi_m, \overline{\Psi}_n\right\} &= \delta_{m+n,0}\ , & \left\{\Psi_m, \Psi_n\right\} &= \left\{\overline{\Psi}_m, \overline{\Psi}_n\right\} = 0\ , \\ \left[j_m, j_n\right] &= m\,\delta_{m+n,0}\ , & \left[L_m, j_n\right] &= -n\,j_{m+n}\ , \\ \left[j_m, \Psi_s\right] &= +\Psi_{m+s}\ , & \left[j_m, \overline{\Psi}_s\right] &= -\overline{\Psi}_{m+s}\ . \end{aligned}} \tag{2.114}$$

Note that this algebra can also be realised by a free boson $X(z,\bar{z})$ compactified on a circle of radius $R = 1$ for which, focussing only on the holomorphic part, we have the following fields:

$$j(z) = i\,\partial X(z,\bar{z})\;, \qquad\qquad j^{\pm}(z) = V_{\pm 1}(z) =: e^{\pm i X} :\;.$$

As we have seen before, the chiral field $j(z)$ has conformal dimension $h = 1$ and for the vertex operators $j^{\pm}(z)$, we find $h = \frac{1}{2}$ using $h = \frac{\alpha^2}{2}$ with $\alpha = \pm 1$. The algebra of $j(z)$ and $j^{\pm}(z)$ can be determined using the general expressions (2.54) and (2.55) for the commutation relations of quasi-primary fields. In particular, the only non-vanishing constants $p_{ijk}(m,n)$ of Eq. (2.55) are found to be

$$p_{\frac{1}{2}1\frac{1}{2}}(m,n) = 1\;, \qquad\qquad p_{111}(m,n) = 1\;,$$

where the subscripts label the conformal dimension of the fields. By the same argument as before, since the overall charge in a correlation function has to vanish, we find for the two-point functions that

$$d_{jj} = d_{+-} = d_{-+} = 1\;, \qquad\qquad d_{j\pm} = d_{\pm\pm} = 0\;,$$

where we employed the usual normalisation. Applying the same argument to the three-point function, we find that $C_{-j+} = -C_{+j-} = 1$. Together with the antisymmetry $C_{ij}^{k} = -C_{ji}^{k}$ and the relation $C_{ij}^{l} d_{lk} = C_{ij}^{k}$, we conclude that the non-vanishing structure constants are

$$C_{j+}^{+} = -C_{+j}^{+} = +1\;, \qquad\qquad C_{j-}^{-} = -C_{-j}^{-} = -1\;.$$

Using these results in the general expression (2.54), we can easily write the down the algebra of Laurent modes of the fields $j(z)$ and $j^{\pm}(z)$

$$\begin{aligned} \big[j_m, j_n\big] &= m\,\delta_{m+n}\;, & \big[j_m^{+}, j_n^{-}\big] &= \delta_{m+n,0}\;, \\ \big[j_m, j_n^{\pm}\big] &= \pm\, j_{m+n}^{\pm}\;, & \big[j_m^{\pm}, j_n^{\pm}\big] &= 0\;. \end{aligned}$$

By comparing with Eq. (2.114), we see that the algebra of a complex fermion can indeed be realised in terms of a free boson.

It is not surprising that a theory of a boson can equivalently be expressed as a theory of two fermions, but it is very special to conformal field theories that we can also express the fermions in terms of the boson. This is called the bosonisation of a complex fermion. We note in passing that this intriguing relation has been used in the so-called covariant lattice approach to string theory model building.

Hilbert Space

Let us finally turn to the Hilbert space $\mathcal{H}$ of the free fermion theory. For the Neveu–Schwarz sector, we find the following chiral states:

$$|0\rangle\,, \qquad \psi_{-\frac{1}{2}}\,|0\rangle\,, \qquad \underbrace{\psi_{-\frac{1}{2}}\psi_{-\frac{1}{2}}}_{=0}|0\rangle\,, \qquad \psi_{-\frac{3}{2}}\,|0\rangle\,, \quad \psi_{-\frac{3}{2}}\,\psi_{-\frac{1}{2}}\,|0\rangle\,, \quad \ldots\,,$$

where we have used that $\{\psi_r, \psi_s\} = \delta_{r,-s}$. In particular, due to Fermi-statistics, each mode ψ_r can only appear once. Taking also the anti-chiral sector into account, we can conclude that in the Neveu–Schwarz sector, the Hilbert space is

$$\mathcal{H}_{\text{NS}} = \left\{\text{Fock space freely generated by } \psi_{-r}, \overline{\psi}_{-s} \text{ for } r, s \geq \tfrac{1}{2}\right\}.$$

The generating function for the degeneration of states at each level N in the chiral sector can be shown to be of the form

$$\mathcal{Z}_{\text{NS}}(q) = \prod_{r\geq 0}\left(1 + q^{r+\frac{1}{2}}\right) = \sum_{N\in\frac{1}{2}\mathbb{Z}} P(N)\, q^N\,.$$

We will perform a detailed study of these expressions in Chap. 4. In the Ramond sector, that is, for $r \in \mathbb{Z}$, there will be fermionic zero modes ψ_0 which deserve a special treatment. This issue will be discussed in Sect. 4.2.4.

2.9.3 The (b,c) Ghost Systems

After having studied the free boson and the free fermion conformal field theories, we will now briefly consider the (b, c) ghost system which plays an important role in the covariant quantisation of the bosonic string.

Basic Properties

Let us start on the complex plane for which we have the metric shown in Eq. (2.101). The action of the (b, c) ghost system reads

$$\mathcal{S} = \frac{1}{4\pi}\int d^2z\,\left(\,b_{zz}\,\partial^z c^z + b_{\overline{z}\overline{z}}\,\partial^{\overline{z}} c^{\overline{z}}\,\right)\,, \tag{2.115}$$

where $\partial^z = 2\partial_{\overline{z}}$ and $\partial^{\overline{z}} = 2\partial_z$ due to the metric (2.101). Here the fields b and c are primary free bosonic fields of conformal dimension $h^b = 2$ and $h^c = -1$ satisfying the wrong spin-statistics relation, that is, they are anti-commuting bosons.

The equation of motion is obtained in the usual way by varying the action above with respect to b and c. Since the calculation is similar to the ones we have presented previously, we just state the results

$$\begin{aligned}
\partial_{\bar{z}}\, b_{zz} &= 0 \qquad \Rightarrow \qquad b_{zz} = b(z)\;, \\
\partial_{z}\, b_{\bar{z}\bar{z}} &= 0 \qquad \Rightarrow \qquad b_{\bar{z}\bar{z}} = \overline{b}(\bar{z})\;, \\
\partial_{\bar{z}}\, c^{z} &= 0 \qquad \Rightarrow \qquad c^{z} = c(z)\;, \\
\partial_{z}\, c^{\bar{z}} &= 0 \qquad \Rightarrow \qquad c^{\bar{z}} = \overline{c}(\bar{z})\;,
\end{aligned}$$

where we also indicated a new notation for the holomorphic and anti-holomorphic fields. Taking into account the conformal dimensions of these fields, we perform a Laurent expansion in the following way:

$$b(z) = \sum_{n\in\mathbb{Z}} b_n\, z^{-n-2}\;, \qquad\qquad c(z) = \sum_{n\in\mathbb{Z}} c_n\, z^{-n+1}\;,$$

and similarly for the anti-chiral fields. Note also that the modes satisfy $b_n\,|0\rangle = 0$ for $n > -2$ and $c_n\,|0\rangle = 0$ for $n > 1$.

Let us finally determine the propagator of this theory given by the action (2.115). The condition to be satisfied for the propagator as well as its solution read

$$\partial^{z}\big\langle b(z)c(w)\big\rangle = 4\pi\,\delta^{(2)}(z-w) \qquad \Rightarrow \qquad \big\langle b(z)c(w)\big\rangle = \frac{1}{z-w}\;.$$

Recalling Eq. (2.53) and taking into account the anti-commuting property of the fields, we can conclude that the OPE has the form

$$b(z)c(w) = \frac{1}{z-w} + \ldots\;,$$

from which we can determine the anti-commutation relation of the Laurent modes by employing $b_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} b(z)$ as well as $c_n = \frac{1}{2\pi i}\oint dz\, z^{n-2} c(z)$. We then find

$$\big\{b_m, c_n\big\} = \delta_{n+m,0}\;, \qquad \big\{b_m, b_n\big\} = 0\;, \qquad \big\{c_m, c_n\big\} = 0\;. \tag{2.116}$$

Energy–Momentum Tensor and Central Charge

As we have argued on general grounds, the energy–momentum tensor has conformal dimension $h = 2$ which of course also applies to the (b, c) ghost system. Without providing a detailed derivation for the energy–momentum tensor, we simply make an ansatz for $T(z)$ similar to the free fermion

$$T(z) = \alpha\, N\big(b\,\partial c\big) + \beta\, N\big(\partial b\; c\big)\;,$$

and fix the constants α and β by requiring $h^b = 2$ and $h^c = 1$. Using our formula for the Laurent modes of normal ordered products involving derivatives (2.69) and keeping in mind the anti-commuting property of the fields, we find

$$L_m = +\alpha\Big(-\sum_{k>0} b_{m-k}\, c_k\,(+1-k) + \sum_{k\leq 0}(+1-k)\, c_k\, b_{m-k}\Big)$$
$$+\beta\Big(-\sum_{k>1} b_{m-k}\, c_k\,(-2-m+k) + \sum_{k\leq 1} c_k\, b_{m-k}\,(-2-m+k)\Big)\,. \tag{2.117}$$

With the help of the anti-commutation relations (2.116), we can then compute

$$\begin{aligned} L_0\,\big|b\big\rangle &= L_0\, b_{-2}\,\big|0\big\rangle = \big\{L_0, b_{-2}\big\}\big|0\big\rangle = \alpha\, b_{-2}\big\{c_2, b_{-2}\big\}\big|0\big\rangle = \alpha\,\big|b\big\rangle \\ L_0\,\big|c\big\rangle &= L_0\, c_1\,\big|0\big\rangle = \big\{L_0, c_1\big\}\big|0\big\rangle = -\beta\, c_1\big\{b_{-1}, c_1\big\}\big|0\big\rangle = -\beta\,\big|c\big\rangle \end{aligned}$$

from which we conclude that $\alpha = 2$ and $\beta = 1$. Knowing the energy–momentum tensor, we can finally compute the central charge of the (b, c) ghost system. As in the previous examples, we first note that $L_2 L_{-2}|0\rangle = \frac{c}{2}\,|0\rangle$. From Eq. (2.117), we find

$$L_{-2}\,\big|0\big\rangle = 2\, c_0\, b_{-2}\,\big|0\big\rangle + c_1\, b_{-3}\,\big|0\big\rangle\,.$$

Furthermore, we have

$$L_2 = 2\,\Big(-\sum_{k>0} b_{2-k}\, c_k\,(-k+1) + \sum_{k\leq 0}(-k+1)\, c_k\, b_{2-k}\Big)$$
$$+\Big(-\sum_{k>1}(-(2-k)-2)\, b_{2-k}\, c_k + \sum_{k\leq 1}(-(2-k)-2)\, c_k\, b_{2-k}\Big)\,.$$

The only terms in L_2 which do not commute with L_{-2} and therefore do not annihilate $|0\rangle$ are those involving (c_2, b_0) as well as those with (c_3, b_{-1}). We therefore extract the relevant expressions from L_2 in the following way:

$$L_2 = 2\,\big(b_0 c_2 + 2\, b_{-1} c_3\big) + \big(2\, b_0 c_2 + b_{-1} c_3\big) + \ldots = 4\, b_0\, c_2 + 5\, b_{-1}\, c_3 + \ldots$$

from which we calculate

$$L_2\, L_{-2}\,\big|0\big\rangle = 8\, b_0\, c_2\, c_0\, b_{-2}\,\big|0\big\rangle + 5\, b_{-1}\, c_3\, c_1\, b_{-3}\,\big|0\big\rangle = (-8-5)\,\big|0\big\rangle = -13\,\big|0\big\rangle\,.$$

Comparing now with $L_2 L_{-2}|0\rangle = \frac{c}{2}\,|0\rangle$, we arrive at the result that

> The central charge of the (b, c) ghost system conformal field theory is $c = -26$.

Remark

The result that the (b, c) ghost system has central charge $c = -26$ is the reason for the statement that the bosonic string is free of anomalies only in $D = 26$ flat space–time dimensions, because the CFT of a single free boson has central charge $c = 1$. For Superstring Theory, each free boson X^μ is paired with one free fermion ψ^μ, where $\mu = 0, \ldots, D$. In this case, in addition to the (b, c) ghost system, also commuting fermionic ghosts (β, γ) of conformal dimensions $(3/2, -1/2)$ have to be included. The central charge of this sector is determined as $c = 11$, so that the superstring can be consistently quantised in $D = \frac{2}{3}(26-11) = 10$ dimensions where the factor $\frac{2}{3}$ comes from the central charge $c = 1 + \frac{1}{2} = \frac{3}{2}$ of a single boson–fermion pair.

2.10 Highest Weight Representations of the Virasoro Algebra

After having studied in detail three examples of conformal field theories, our aim in the present section is to study the representation theory of the symmetry algebra $\mathcal{A} \oplus \overline{\mathcal{A}}$ on general grounds. Here, $\mathcal{A}$ denotes the chiral sector, while $\overline{\mathcal{A}}$ stands for the anti-chiral part. In particular, we will focus on the minimal case with $\mathcal{A}$ and $\overline{\mathcal{A}}$ being the Virasoro algebra generated by $T(z)$ and $\overline{T}(\overline{z})$, respectively. However, it is possible for CFTs to have larger symmetry algebras, for instance the so-called $\mathcal{W}$ algebras, which we will consider in Sect. 3.7.

Highest Weight Representations and Verma Module

Analogously to the $\mathfrak{su}(2)$ spin algebra in Quantum Mechanics, we want to construct highest weight representations (HWR) of the Virasoro algebra. As we have seen in Eqs. (2.62) and (2.73), a highest weight state $|h\rangle$ corresponding to a primary field of conformal dimension h has the property

$$\begin{aligned} L_n \left|h\right\rangle &= 0 \qquad \text{for} \qquad n > 0\ , \\ L_0 \left|h\right\rangle &= h \left|h\right\rangle , \end{aligned} \tag{2.118}$$

so that the action of L_n for $n < 0$ on the state $|h\rangle$ creates new states. The set of all these states is called the *Verma module* $V_{h,c}$, where h stands for the highest weight state $|h\rangle$ and c is the central charge of the Virasoro algebra. The lowest level states in the Verma module $V_{h,c}$ are

$$L_{-1}\left|h\right\rangle , \quad L_{-2}\left|h\right\rangle , \quad L_{-1}L_{-1}\left|h\right\rangle , \quad L_{-3}\left|h\right\rangle , \quad \ldots\ .$$

Roughly speaking, the Verma module $V_{h,c}$ is the set of states corresponding to the conformal family $\left[\phi(z)\right]$ of a primary field $\phi(z)$ with conformal dimension h.

Depending on the combination (h, c), there can be states of vanishing or even of negative norm in a Verma module. For unitary theories, the later should be absent and vanishing norm states should be removed from $V_{h,c}$. However, it turns out that states of vanishing norm generate an independent Verma module which is "orthogonal" to the parent one. We will not go into further detail but refer to the existing literature.

Null States and the Kač-Determinant

To illustrate how to determine zero-norm states in a Verma module, we are going to consider a simple example from linear algebra. Suppose we have a vector $|v\rangle$ in a real n-dimensional vector space with basis vectors $|a\rangle$. Note that in particular, this basis does not need to be orthonormal. We then express our vector as $|v\rangle = \sum_{a=1}^{n} \lambda_a \, |a\rangle$ where not all λ_a are zero. The condition for $|v\rangle$ to have vanishing norm is

$$0 = ||\, v \,||^2 = \sum_{a,b=1}^{n} \lambda_a \, \langle\, a \,|\, b \,\rangle \, \lambda_b = \sum_{a,b=1}^{n} \lambda_a \, M_{ab} \, \lambda_b = \vec{\lambda}^T \; M \, \vec{\lambda}$$

where we defined the elements of the matrix M as $M_{ab} = \langle a|b\rangle$. This expression is zero if $\vec{\lambda}$ is an eigenvector of M with eigenvalue zero. The number of such (linearly independent) eigenvectors is given by the number of roots of the equation $\det M = 0$.

Let us now come back to the null states in the Verma module. Analogously to the example above, to decide whether there exist zero-norm states, we are going to compute the so-called *Kač-determinant* at level N. We denote the corresponding matrix as $M_N(h, c)$ where the entries are defined as the product of states in the Verma module

$$\big\langle h \big| \prod_i L_{+k_i} \, \prod_j L_{-m_j} \, \big| h \big\rangle \, , \qquad\qquad \sum_i k_i = \sum_j m_j = N \, ,$$

with all $k_i, m_j \geq 0$. Note that the condition on the right-hand side guarantees that we are considering only states at level N. For two states at different level, the corresponding matrix element vanishes because the net number of operators $\sum_i k_i - \sum_j m_j$ is non-zero and so only creation or only annihilation operators survive.

Let us illustrate this procedure for the first and second level. For $N = 1$, we have only one state in the Verma module and so we find

$$M_1(h, c) = \big\langle h \big| \, L_1 \, L_{-1} \, \big| h \big\rangle = 2 \, \big\langle h \big| \, L_0 \, \big| h \big\rangle = 2 \, h \; .$$

The Kač-determinant is therefore trivially

$$\det M_1(h,c) = 2\,h\ .$$

Here we see that for $h = 0$, we have one null state at level $N = 1$. At level $N = 2$, there are the two states $L_{-2}\,|h\rangle$ and $L_{-1}\,L_{-1}\,|h\rangle$ in the Verma module. For the elements of the two-by-two matrix $M_2(h,c)$, we use Eq. (2.118) to calculate

$$\begin{aligned}
\big\langle h\big|\, L_2\, L_{-2}\,\big|h\big\rangle &= \big\langle h\big|\, \frac{c}{2} + 4\,L_0\,\big|h\big\rangle \;= 4\,h + \frac{c}{2}\ ,\\
\big\langle h\big|\, L_1\, L_1\, L_{-2}\big|h\big\rangle &= \big\langle h\big|\, L_1 \cdot 3\,L_{-1}\,\big|h\big\rangle = 6\,h\ ,\\
\big\langle h\big|\, L_2\, L_{-1}\, L_{-1}\big|h\big\rangle &= \big\langle h\big|\, 3\,L_1 \cdot L_{-1}\,\big|h\big\rangle = 6\,h\ ,\\
\big\langle h\big|\, L_1\, L_1\, L_{-1}\, L_{-1}\,\big|h\big\rangle &= \big\langle h\big|\, L_1\,\big[L_1, L_{-1}\big]\, L_{-1}\,\big|h\big\rangle + \big\langle h\big|\, L_1\, L_{-1}\, L_1\, L_{-1}\,\big|h\big\rangle\\
&= \big\langle h\big|\, L_1\, 2\,L_0\, L_{-1}\,\big|h\big\rangle + \big\langle h\big|\,\big[L_1, L_{-1}\big]\,\big[L_1, L_{-1}\big]\,\big|h\big\rangle\\
&= 2\,\big\langle h\big|\, L_1\,\big[L_0, L_{-1}\big]\,\big|h\big\rangle + 4\,h^2 + 4\,h^2\\
&= 4\,h + 8\,h^2\ .
\end{aligned}$$

For the Kač-determinant at level $N = 2$, we then find

$$\det M_2(c,h) = \det\begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1)\end{pmatrix} = 32\,h\,\left(h^2 - \frac{5}{8}h + \frac{1}{8}hc + \frac{1}{16}c\right)\ .$$

The roots of $\det M_2(c,h)$ are the following:

$$h_{1,1} = 0\ , \qquad \begin{aligned} h_{1,2} &= \frac{5-c}{16} - \frac{1}{16}\sqrt{(1-c)(25-c)}\ ,\\ h_{2,1} &= \frac{5-c}{16} + \frac{1}{16}\sqrt{(1-c)(25-c)}\ ,\end{aligned}$$

where our notation will become clear in the following. We can then write the Kač-determinant as

$$\det M_2(c,h) = 32\,\Big(h - h_{1,1}(c)\Big)\,\Big(h - h_{1,2}(c)\Big)\,\Big(h - h_{2,1}(c)\Big)\ .$$

In summary, at level $N = 2$, we found three states of vanishing norm where the root $h_{1,1} = 0$ is due to the null state at level 1. This is a general feature: if a null state $|h+n\rangle$ occurs at level n, then at level $N > n$ there are $P(N-n)$ resulting null states. Here $P(n)$ is again the number of partitions of n.

V. Kač found and proved the general formula for the determinant $\det M_N(c,h)$ at arbitrary level N.

The so-called Kač-determinant at level N reads

$$\det M_N(c,h) = \alpha_N \prod_{\substack{p,q\le N\\ p,q>0}} \Big(h - h_{p,q}(c)\Big)^{P(N-pq)}$$

with

$$h_{p,q}(m) = \frac{\big((m+1)\,p - mq\,\big)^2 - 1}{4\,m\,(m+1)}\,, \qquad m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}\,.$$

Here, α_N is a positive constant and we note that in general, m is not an integer but complex. For $c < 1$, one conventionally chooses the branch $m \in (0,\infty)$; however, $h_{p,q}$ possesses the symmetry $\{p \to m-p, q \to m+1-q\}$ so that $\det M_N$ is independent of the choice of branch in m as it can be compensated by $p \leftrightarrow q$.

Unitary Representations

So far, we have focused on the null states in a Verma module. However, to find unitary representations we have to exclude also the regions in the (h,c)-plane where states of negative norm appear. We will not discuss all the details but just summarise the results.

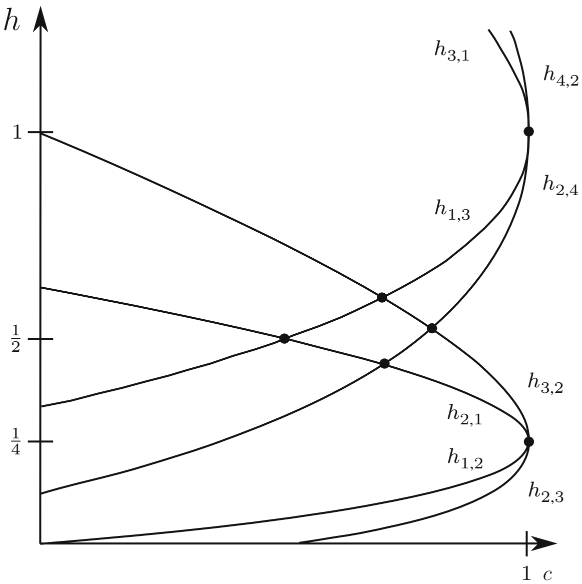


Fig. 2.6 Some curves $h_{p,q}(c)$ of vanishing Kač-determinant. Unitary representations are labelled by a *dot*

- For $c > 1$ and $h \geq 0$, there are no zeros and all eigenvalues of M_N are positive. Therefore, unitary representations can exist.
- In the case of $c = 1$, one finds $\det M_N = 0$ for $h = \frac{n^2}{4}$ where $n \in \mathbb{Z}$.
- The region $c < 1$ and $h \geq 0$ is much more complicated. It can be shown that all points which do not lie on a curve $h_{p,q}(c)$ where $\det M_N = 0$ are non-unitary. A more careful analysis reveals that in fact non-negative states are absent only on certain intersection points of such vanishing curves. This is illustrated in Fig. 2.6 where the dots stand for unitary representations.

We can summarise the results as follows:

For the case of $c < 1$ and $h \geq 0$, the discrete set of points where unitary representations are not excluded occur at values of c

$$c = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, \ldots \,.$$

To each c there are only $\binom{m}{2}$ allowed values of h

$$h_{p,q}(m) = \frac{\big((m+1)\,p - mq\,\big)^2 - 1}{4\,m\,(m+1)} \tag{2.119}$$

with $1 \leq p \leq m-1$ and $1 \leq q \leq m$.

So far, this condition is only necessary. But we will see later that concrete conformal field theories can indeed be found.
Because of the severe constraints on unitary CFTs, only a discrete set of values c with a finite number of highest weight representations survive. conformal field theories with this latter property are known to exist only for rational values of the central charge and are therefore called Rational CFTs or RCFTs.

Examples

To close this section, let us consider some examples for the case $c < 1$ and $h \geq 0$. For $m = 3$, we find $c = \frac{1}{2}$ as well as $1 \leq p \leq 2$ and $1 \leq q \leq 3$. The possible values of $h_{p,q}$ are then conveniently organised in the so-called conformal grid

$q \uparrow$

$\frac{1}{2}$	0
$\frac{1}{16}$	$\frac{1}{16}$
0	$\frac{1}{2}$

$\rightarrow p$

which in the present case describes the critical point of the second-order phase transition of the Ising model. For the case of $m = 4$, we find $c = \frac{7}{10}$ which is the tri-critical Ising model. The conformal grid is displayed below.

$\frac{3}{2}$	$\frac{7}{10}$	0
$\frac{3}{5}$	$\frac{3}{80}$	$\frac{1}{10}$
$\frac{1}{10}$	$\frac{3}{18}$	$\frac{3}{5}$
0	$\frac{7}{16}$	$\frac{3}{2}$

Finally, for $m = 5$, we get $c = \frac{4}{5}$ which is the three states Potts model.

Remarks

- Note that for deriving the results in this section, we have not referred at any stage to a concrete realisation of a CFT, but have solely exploited the consequences of conformal symmetry. This shows the far-reaching consequences of the conformal symmetry in two dimensions.
- For Euclidean CFTs and their application to statistical models, unitarity, that is, reflection positivity, is not a necessary condition. Weakening this constraint and allowing for states with negative norm, the representation theory of the Virasoro algebra contains a more general discrete series of RCFTs. These are given by the central charges

$$c = 1 - 6\,\frac{(p-q)^2}{p\,q} \tag{2.120}$$

  with the $p, q \geq 2$ and p, q relatively coprime. The finite set of highest weights is given by

$$h_{r,s}(p,q) = \frac{(p\,r - q\,s)^2 - (p-q)^2}{4\,p\,q}$$

  with $1 \leq r \leq q-1$ and $1 \leq s \leq p-1$. Unless $|p-q| = 1$, there always exist highest weight states of negative norm, and the unitary series is precisely given by $p = m+2$, $q = m+3$ with $m \geq 1$. As an example, note that the model $(p,q) = (5,2)$ with central charge $c = -22/5$ describes the Yang–Lee edge singularity.
- Note that the theories defined by Eq. (2.120) are called *minimal models*. In the case of $p = m+2$, $q = m+3$ with $m \geq 1$, i.e. if the theory is unitary, they are called unitary (minimal) models.

2.11 Correlation Functions and Fusion Rules

In this section, we are going to discuss one of the most powerful results of the boot-strap approach to CFTs. We will see that the appearance of null states in the unitary models with $0 < c < 1$ severely restricts the form of the OPEs between the primary fields $\phi_{(p,q)}$.

Null States at Level Two

We start by discussing the null states at level $N = 2$. Note that in the previous section, we did not determine their precise form, however, in general such a state is a linear combination written in the following way:

$$L_{-2} \left|h\right\rangle + a\, L_{-1}\, L_{-1} \left|h\right\rangle = 0\,. \tag{2.121}$$

If we apply L_1 to this equation, we can fix the constant a as

$$\begin{aligned} 0 &= \left[L_1, L_{-2}\right] \left|h\right\rangle + a \left[L_1, L_{-1}L_{-1}\right] \left|h\right\rangle \\ &= 3\, L_{-1} \left|h\right\rangle + a \left(2\, L_0\, L_{-1} + 2\, L_{-1}\, L_0\right)\left|h\right\rangle \\ &= \left(3 + 2a\left(2h+1\right)\right) L_{-1} \left|h\right\rangle \qquad\qquad \Rightarrow \qquad a = -\frac{3}{2(2h+1)} \end{aligned}$$

where we used that $L_{-1}|h\rangle \neq 0$ for $h \neq 0$. Next, we apply L_2 to Eq. (2.121) in order to determine the allowed values for h

$$\begin{aligned} 0 &= \left[L_2, L_{-2}\right] \left|h\right\rangle + a \left[L_2, L_{-1}\, L_{-1}\right] \left|h\right\rangle \\ &= \left(4\, L_0 + \frac{c}{2}\right) \left|h\right\rangle + a\, L_{-1} \left[L_2, L_{-1}\right] \left|h\right\rangle + a \left[L_2, L_{-1}\right] L_{-1}\left|h\right\rangle \\ &= \left(4\, h + \frac{c}{2}\right) \left|h\right\rangle + 6ah \left|h\right\rangle \\ &= \left(4h + \frac{c}{2} + 6ah\right) \left|h\right\rangle \qquad\qquad \Rightarrow \qquad c = \frac{2h}{2h+1}\left(5 - 8h\right)\,. \end{aligned}$$

Therefore, we have shown that for a theory with central charge $c = \frac{2h}{2h+1}(5 - 8h)$ the null state at level $N = 2$ satisfies

$$\left(L_{-2} - \frac{3}{2(2h+1)}\, L_{-1}^2\right) \left|h\right\rangle = 0\,. \tag{2.122}$$

Let us quickly check Eq. (2.122) for the unitary series from the last section. We found that in theories with central charges $c = 1 - \frac{6}{m(m+1)}$, $m \geq 3$, there is a highest weight at level two characterised by

$$h_{2,1}(m) = \frac{\left(\,2(m+1) - m\,\right)^2 - 1}{4\, m\, (m+1)} = \frac{(m+2)^2 - 1}{4\, m\, (m+1)}\,.$$

Solving this equation for m leads to $m = \frac{3}{4h-1}$, which gives a central charge $c = \frac{2h(5-8h)}{2h+1}$ in agreement with our result above.

Descendant Fields and Correlation Functions

We already mentioned that the fields in the Verma module obtained by acting with L_m are called *descendant fields* and we will now formalise the concept of descendants to some extend. Given a primary field $\phi(w)$, the descendant fields $\hat{L}_{-n}\phi$ for $n > 0$ are defined to be the fields appearing in the OPE

$$T(z)\,\phi(w) = \sum_{n\geq 0} (z-w)^{n-2}\, \widehat{L}_{-n}\,\phi(w)\,.$$

Performing a contour integration, we find for the descendant fields

$$\boxed{\widehat{L}_{-n}\,\phi(w) = \oint \frac{dz}{2\pi i}\, \frac{1}{(z-w)^{n-1}}\, T(z)\,\phi(w)}\,. \tag{2.123}$$

If the conformal dimension of ϕ is integer, we can determine the descendant fields for the first values of n from Eq. (2.40) as

$$\widehat{L}_0\,\phi(w) = h\,\phi(w)\,, \quad \widehat{L}_{-1}\,\phi(w) = \partial\phi(w)\,, \quad \widehat{L}_{-2}\,\phi(w) = N\big(\phi\, T\big)(w)\,. \tag{2.124}$$

Let us now derive an expression for the correlator of a descendant field with a number of other primaries. For convenience, we work with chiral primaries, however, the result for the anti-chiral part is obtained along similar lines. We use Eq. (2.123) for the descendant field and the deformation of contours illustrated in Fig. 2.7 to find

$$\begin{aligned}
&\big\langle\, \widehat{L}_{-n}\,\phi(w)\,\phi_1(w_1)\ldots\phi_N(w_N)\,\big\rangle \\
&= \oint_{C(w)} \frac{dz}{2\pi i}\,(z-w)^{1-n}\Big\langle \Big(T(z)\,\phi(w)\Big)\,\phi_1(w_1)\ldots\phi_N(w_N)\Big\rangle \\
&= -\sum_{i=1}^{N}\oint_{C(w_i)} \frac{dz}{2\pi i}\,(z-w)^{1-n}\Big\langle\,\phi(w)\,\phi_1(w_1)\ldots\Big(T(z)\,\phi_i(w_i)\Big)\ldots\phi_N(w_N)\Big\rangle \\
&= -\sum_{i=1}^{N}\oint_{C(w_i)} \frac{dz}{2\pi i}\,(z-w)^{1-n}\,\times \\
&\quad\times\left(\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i}\partial_{w_i}\right)\big\langle\,\phi(w)\,\phi_1(w_1)\ldots\phi_N(w_N)\,\big\rangle \\
&= -\sum_{i=1}^{N}\Big((1-n)\,\big(w_i-w\big)^{-n}\,h_i\, + \big(w_i-w\big)^{1-n}\,\partial_{w_i}\Big)\big\langle\,\phi(w)\,\phi_1(w_1)\ldots\phi_N(w_N)\,\big\rangle\,,
\end{aligned}$$

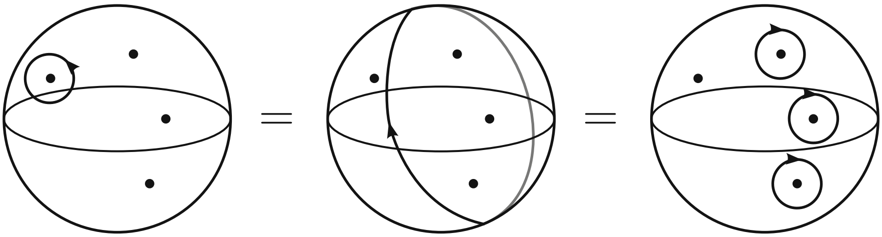


Fig. 2.7 Transformation of contour integrals on the sphere. Note that the orientation on the very right is clock-wise so the residue picks up a minus sign

where from the third to the fourth line we used the OPE of a primary field and in the last step we employed the residue theorem. We have therefore shown that

> The correlator involving a descendant field $\widehat{L}_{-n}\phi$ can be computed from the correlator involving the corresponding primary field ϕ by applying the differential operator $\mathcal{L}_{-n}$ in the following way:
>
> $$\left\langle \widehat{L}_{-n}\,\phi(w)\,\phi_1(w_1)\ldots\phi_N(w_N)\right\rangle = \mathcal{L}_{-n}\left\langle \phi(w)\,\phi_1(w_1)\ldots\phi_N(w_N)\right\rangle$$
>
> where the operator $\mathcal{L}_{-n}$ has the form
>
> $$\mathcal{L}_{-n} = \sum_{i=1}^{N}\left(\frac{(n-1)\,h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\,\partial_{w_i}\right). \qquad (2.125)$$

Two Particular Examples

Let us consider again Eq. (2.122) for a null state at level two. We see that the corresponding descendant field

$$\widehat{L}_{-2}\,\phi(z) - \frac{3}{2\,(2h+1)}\widehat{L}^2_{-1}\,\phi(z)$$

is a null field where $\widehat{L}^2_{-1}\,\phi(z)$ is understood as $\big(\widehat{L}_{-1}(\widehat{L}_{-1}\phi)\big)(z) = \partial^2\phi(z)$. Furthermore, this relation implies an expression for the differential operators $\mathcal{L}_{-n}$ acting on correlation functions involving $\phi(z)$, i.e.

$$0 = \left(\mathcal{L}_{-2} - \frac{3}{2\,(2h+1)}\,\mathcal{L}^2_{-1}\right)\left\langle \phi(w)\,\phi_1(w_1)\ldots\phi_N(w_N)\right\rangle.$$

From Eq. (2.124), we recall that $\widehat{L}_{-1}\,\phi(w) = \partial_w\,\phi(w)$ and therefore $\mathcal{L}_{-1}$ acts as ∂_w. Employing then the definition (2.125) for $\mathcal{L}_{-2}$, we find

$$
\begin{aligned}
0 = &\Bigg(\sum_{i=1}^{N}\left(\frac{h_i}{(w_i-w)^2} - \frac{1}{w_i - w}\,\partial_{w_i}\right) \\
&-\frac{3}{2\,(2h+1)}\,\partial_w^2\Bigg)\big\langle \phi(w)\,\phi_1(w_1)\dots\;\phi_N(w_N)\big\rangle\,.
\end{aligned}
\tag{2.126}
$$

Working out this differential equation for the example of the two-point function yields

$$
\begin{aligned}
0 &= \left(\frac{h}{(w_1-w)^2} - \frac{1}{w_1-w}\,\partial_{w_1} - \frac{3}{2\,(2h+1)}\,\partial_{w_1}^2\right)\frac{d}{(w-w_1)^{2h}} \\
0 &= \left(h + 2h - \frac{3}{2\,(2h+1)}\,2h\,(2h+1)\right)\frac{d}{(w-w_1)^{2h+2}}\,,
\end{aligned}
$$

and we realise that it is trivially satisfied. However, for the three-point function we will find a non-trivial condition. Recalling the precise form of this correlator

$$
\big\langle \phi(w)\,\phi_1(w_1)\,\phi_2(w_2)\big\rangle = \frac{C_{\phi\,\phi_1\,\phi_2}}{(w-w_1)^{h+h_1-h_2}\,(w_1-w_2)^{h_1+h_2-h}\,(w-w_2)^{h+h_2-h_1}}\,,
$$

and inserting it into the differential equation (2.126), after a tedious calculation one obtains the following constraint on the conformal weights $\{h, h_1, h_2\}$:

$$
2\,(2h+1)\,(h+2h_2-h_1) = 3\,(h-h_1+h_2)\,(h-h_1+h_2+1)\,.
$$

This expression can be solved for h_2 leading to

$$
h_2 = \frac{1}{6} + \frac{h}{3} + h_1 \pm \frac{2}{3}\sqrt{h^2 + 3hh_1 - \frac{1}{2}h + \frac{3}{2}h_1 + \frac{1}{16}}\,.
\tag{2.127}
$$

Fusion Rules for Unitary Minimal Models

Next, let us apply Eq. (2.127) to the primary fields $\phi_{(p,q)}$ of the rational models with central charges $c(m)$ studied in Sect. 2.10. In particular, if we choose $h = h_{2,1}(m)$ and $h_1 = h_{p,q}(m)$ then the two solutions for h_2 are precisely $\{h_{p-1,q}(m), h_{p+1,q}(m)\}$. Therefore, at most two of the coefficients $C_{\phi\phi_1\phi_2}$ of a three-point function will be non-zero. The OPE of $\phi_2 = \phi_{(2,1)}$ with any other primary field $\phi_{(p,q)}$ in a unitary minimal model is then restricted to be of the form

$$
\big[\phi_{(2,1)}\big]\times\big[\phi_{(p,q)}\big] = \big[\phi_{(p+1,q)}\big] + \big[\phi_{(p-1,q)}\big]\,,
\tag{2.128}
$$

where $[\phi_{(p,q)}]$ denotes the conformal family descending from $\phi_{(p,q)}$[2]. Still, the coefficients $C_{\phi\phi_1\phi_2}$ could be zero, but at most two other conformal families appear on the right-hand side of Eq. (2.128).

This strategy can be generalised to higher level null states. Without detailed derivation, we note the final result that the conformal families in a unitary minimal model form a closed algebra

$$\boxed{\left[\phi_{(p_1,q_1)}\right] \times \left[\phi_{(p_2,q_2)}\right] = \sum_{\substack{k=1+|p_1-p_2| \\ k+p_1+p_2 \text{ odd}}}^{p_1+p_2-1} \sum_{\substack{l=1+|q_1-q_2| \\ l+q_1+q_2 \text{ odd}}}^{q_1+q_2-1} \left[\phi_{(k,l)}\right]} \; . \tag{2.129}$$

These are the so-called *fusion rules* for the conformal families in the unitary minimal models of the Virasoro algebra.

Let us briefly illustrate these rules for the Ising model, i.e. for $m = 3$. We label the relevant fields $\phi_{(p,q)}$ as

$$\begin{aligned} \phi_{(1,1)} &= 1 \; , & \phi_{(1,2)} &= \sigma \; , & \phi_{(1,3)} &= \epsilon \; , \\ \phi_{(2,3)} &= 1 \; , & \phi_{(2,2)} &= \sigma \; , & \phi_{(2,1)} &= \epsilon \; . \end{aligned}$$

Using the formula above, we then find

$$\begin{aligned} &[1] \times [\sigma] = [\sigma] \; , && [\epsilon] \times [\epsilon] = [1] \; , && [\sigma] \times [\sigma] = [1] + [\epsilon] \; , \\ &[1] \times [\epsilon] = [\epsilon] \; , && [\epsilon] \times [\sigma] = [\sigma] \; . \end{aligned}$$

Fusion Algebra

The fusion rules (2.129) for the unitary minimal models of the Virasoro algebra can be generalised to arbitrary RCFTs. In particular, the OPE between conformal families $[\phi_i]$ and $[\phi_j]$ gives rise to the concept of a *Fusion algebra*

$$[\, \phi_i \,] \times [\, \phi_j \,] = \sum_k N_{ij}^{k} \, [\, \phi_k \,] \tag{2.130}$$

where $N_{ij}^{k} \in \mathbb{Z}_0^+$. Furthermore, one finds $N_{ij}^{k} = 0$ if and only if $C_{ijk} = 0$, and for unitary minimal models of $\mathrm{Vir}_{(h,c)}$, we get $N_{ij}^{k} \in \{0, 1\}$.

Let us note that the algebra (2.130) is commutative as well as associative and that the vacuum representation [1] containing just the energy–momentum tensor as

---

[2] Let us make clear how to interpret Eq. (2.128). This equation means that the OPE between a field in the conformal family of $\phi_{(2,1)}$ and a field in the conformal family of $\phi_{(p,q)}$ involves only fields belonging to the conformal families of $\phi_{(p+1,q)}$ and $\phi_{(p-1,q)}$. However, more work is needed to determine the precise form of the OPE.

well as its descendants is the unit element because $N_{i1}^{k} = \delta_{ik}$. Commutativity of Eq. (2.130) implies $N_{ij}^{k} = N_{ji}^{k}$, and for the consequences of associativity consider

$$\begin{aligned}
[\,\phi_i\,] \times \big([\,\phi_j\,] \times [\,\phi_k\,]\big) &= [\,\phi_i\,] \times \sum_l N_{jk}^{l}\, [\,\phi_l\,] = \sum_{l,m} N_{jk}^{l}\, N_{il}^{m}\, [\,\phi_m\,] \\
\big([\,\phi_i\,] \times [\,\phi_j\,]\big) \times [\,\phi_k\,] &= \sum_{l,m} N_{ij}^{l}\, N_{lk}^{m}\, [\,\phi_m\,]\,,
\end{aligned}$$

from which we conclude that

$$\boxed{\sum_l N_{kj}^{l}\, N_{il}^{m} = \sum_l N_{ij}^{l}\, N_{lk}^{m}}\;.$$

Defining finally the matrices $(\overline{N}_i)_{jk} := N_{ij}^{k}$, we can write this formula as

$$\boxed{\overline{N}_i\, \overline{N}_k = \overline{N}_k\, \overline{N}_i}\;. \tag{2.131}$$

We will come back to these fusion rules in the discussion of one-loop partition functions, where we will find an intriguing relation between the fusion coefficients N_{ij}^{k} and the so-called modular S-matrix.

2.12 Non-Holomorphic OPE and Crossing Symmetry

In this chapter, so far we have mainly focussed on the chiral sector of two-dimensional conformal field theories. However, for non-chiral fields the structure is very similar and so we can briefly summarise it here.

Two- and Three-Point Functions

In particular, the results for the two- and three-point functions of chiral quasi-primary fields $\phi_i(z)$ carry over directly to non-chiral fields $\phi_i(z,\bar{z})$. The two-point function is determined by the $SL(2,\mathbb{C})/\mathbb{Z}_2 \times \overline{SL(2,\mathbb{C})/\mathbb{Z}_2}$ conformal symmetry up to a normalisation factor as

$$\big\langle \phi_1(z_1,\bar{z}_1)\, \phi_2(z_2,\bar{z}_2) \big\rangle = \frac{d_{12}}{z_{12}^{h_1+h_2}\, \bar{z}_{12}^{\bar{h}_1+\bar{h}_2}}\, \delta_{h_1,h_2}\, \delta_{\bar{h}_1,\bar{h}_2}\;,$$

where again z_{12} is defined as $z_{12} = z_1 - z_2$ and similarly for $\bar{z}_{12}$. The three-point function is fixed up to the structure constant C_{123} in the following way:

$$\big\langle \phi_1(z_1,\bar{z}_1)\,\phi_2(z_2,\bar{z}_2)\,\phi_3(z_3,\bar{z}_3)\big\rangle = \\ = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\,z_{23}^{h_2+h_3-h_1}\,z_{13}^{h_1+h_3-h_2}\quad \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\,\bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\,\bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2}}\,.$$

OPE of Primary Fields

The general form of the OPE of two non-chiral quasi-primary fields can be determined following the same steps as for the case of chiral fields studied in Sect. 2.6.3. The OPE takes the form

$$\phi_i(z,\bar{z})\,\phi_j(w,\overline{w}) = \sum_p \sum_{\{k,\bar{k}\}} C_{ij}^p\, \frac{\beta_{ij}^{p,\{k\}}\ \overline{\beta}_{ij}^{p,\{\bar{k}\}}\ \ \phi_p^{\{k,\bar{k}\}}(w,\overline{w})}{(z-w)^{h_i+h_j-h_p-K}\ (\bar{z}-\overline{w})^{\bar{h}_i+\bar{h}_j-\bar{h}_p-\overline{K}}}\,, \tag{2.132}$$

where the multi-index $\{k,\bar{k}\}$ labels all the descendant fields

$$\big\{\hat{L}_{-k_1}\ldots\hat{L}_{-k_n}\,\hat{\overline{L}}_{-k_1}\ldots\hat{\overline{L}}_{-k_n}\,\phi_p(z,\bar{z})\big\}$$

in the conformal family of the primary field $\phi_p(z,\bar{z})$. Moreover, we have introduced $K=\sum_i k_i$ and $\overline{K}=\sum_i \bar{k}_i$ as well as the coefficients $\beta_{ij}^{p,\{k\}}$ and $\overline{\beta}_{ij}^{p,\{\bar{k}\}}$. The later govern the coupling of the descendants and depend only on the central charge of the theory as well as on the conformal dimensions of the fields involved[3]. Therefore, in principle, the only unknown parameters are the structure constants C_{ij}^p among the primary fields.

Four-Point Functions and Crossing Symmetry

However, from the operator algebra point of view there are additional constraints for the structure constants C_{ij}^p coming from Jacobi identities (see also Sect. 3.7). At the level of the OPE, these constraints arise from what is called crossing symmetries. These appear first at the level of four-point functions to which we turn now. Due to the $SL(2,\mathbb{C})/\mathbb{Z}_2 \times \overline{SL(2,\mathbb{C})/\mathbb{Z}_2}$ symmetry, a general four-point function

$$G\big(\mathbf{z},\bar{\mathbf{z}}\big) = \big\langle \phi_i(z_1,\bar{z}_1)\,\phi_j(z_2,\bar{z}_2)\,\phi_l(z_3,\bar{z}_3)\,\phi_m(z_4,\bar{z}_4)\big\rangle \tag{2.133}$$

can only depend on the so-called crossing ratios

$$x=\frac{z_{12}\,z_{34}}{z_{13}\,z_{24}}, \qquad \bar{x}=\frac{\bar{z}_{12}\,\bar{z}_{34}}{\bar{z}_{13}\,\bar{z}_{24}}\,.$$

[3] We will compute some of these coefficients in Sect. 3.7, when we discuss the construction of $\mathcal{W}$ algebras.

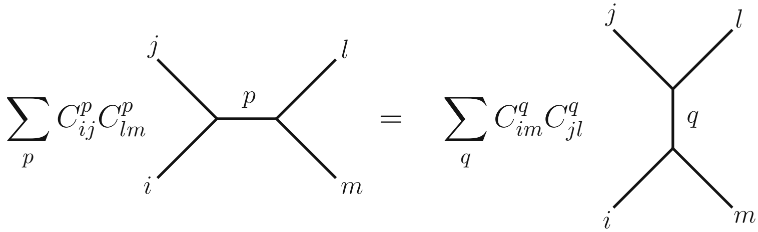


Fig. 2.8 Illustration of the crossing symmetry for four-point functions

This can be made plausible for instance by using the conformal symmetry to map the four points z_i to $z_1 = 0$, $z_2 = x$, $z_3 = 1$ and $z_4 = \infty$ and similarly for $\overline{z}_i$. One can then evaluate the amplitude (2.133) in several different ways.

- First, one can employ the OPE for $\phi_i(z_1, \overline{z}_1)\,\phi_j(z_2, \overline{z}_2)$ and then the one for $\phi_l(z_3, \overline{z}_3)\,\phi_m(z_4, \overline{z}_4)$. As a result, the amplitude (2.133) can be expressed as

$$G(\mathbf{z}, \overline{\mathbf{z}}) = \sum_p C_{ij}^p\, C_{lm}^p\, \mathcal{F}_{ij}^{lm}(p\,|\,x)\, \overline{\mathcal{F}}_{ij}^{lm}(p\,|\,\overline{x})\ , \tag{2.134}$$

  where the contributions of the descendants of the primary field ϕ_p factorise into a holomorphic and an anti-holomorphic piece. These in general quite complicated expressions $\mathcal{F}_{ij}^{lm}(p\,|\,x)$ are called conformal blocks and depend only on the conformal dimensions of the primary fields involved and on the central charge of the CFT.
- One can evaluate the four-point function (2.133) also by first using the OPE for $\phi_j(z_2, \overline{z}_2)\,\phi_l(z_3, \overline{z}_3)$. Effectively this means exchanging $\phi_j(z_2, \overline{z}_2)$ and $\phi_m(z_4, \overline{z}_4)$ which on the level of crossing ratios is achieved by $x \mapsto 1 - x$. The resulting four-point amplitude can now be expressed as

$$G(\mathbf{z}, \overline{\mathbf{z}}) = \sum_p C_{im}^p\, C_{jl}^p\, \mathcal{F}_{im}^{jl}(p\,|\,1 - x)\, \overline{\mathcal{F}}_{im}^{jl}(p\,|\,1 - \overline{x})\ . \tag{2.135}$$

- Similarly, one can first evaluate the OPE $\phi_j(z_2, \overline{z}_2)\,\phi_m(z_4, \overline{z}_4)$ which leads to the following form of the four-point amplitude:

$$G(\mathbf{z}, \overline{\mathbf{z}}) = x^{-2h_j}\, \overline{x}^{-2\overline{h}_j} \sum_p C_{il}^p\, C_{jm}^p\, \mathcal{F}_{il}^{jm}\left(p\,\Big|\,\frac{1}{x}\right) \overline{\mathcal{F}}_{il}^{jm}\left(p\,\Big|\,\frac{1}{\overline{x}}\right) . \tag{2.136}$$

Equating the three expressions (2.134), (2.135) and (2.136) for the four-point function gives a number of consistency conditions for the structure constants C_{ij}^p among the primary fields. These so-called crossing symmetry conditions are depicted in Fig. 2.8.

In the boot-strap approach to quantum field theories the hope is that these conditions eventually determine all such structure constants, so that the whole theory is solved. In the case of chiral fields with necessarily (half-)integer conformal dimensions, these crossing symmetry conditions are equivalent to the Jacobi-identity for the corresponding operator algebra.

2.13 Fusing and Braiding Matrices

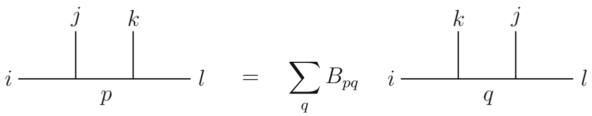
In the previous section, we have discussed the crossing symmetry of the four-point function of primary fields. For RCFTs a simplification occurs, as there are only a finite number of conformal families which can propagate as intermediate states. This means that the conformal blocks for the three different channels form a finite-dimensional vector space. The crossing symmetry then says that the different classes of conformal blocks are nothing else than three different choices of basis which must be related by linear transformations. We can therefore write

$$\mathcal{F}^{kl}_{ij}(p\,|\,x) = \sum_q B\begin{bmatrix} j & k \\ i & l \end{bmatrix}_{p,q} \mathcal{F}^{jl}_{ik}\left(q\,|\,\frac{1}{x}\right) . \tag{2.137}$$

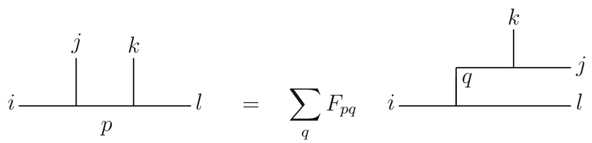
The matrices B are called *braiding matrices* and in the example above $\{i, j, k, l\}$ are indices of B while (p, q) denote a particular matrix element. For the second crossing symmetry, one similarly defines the so-called *fusing matrices*

$$\mathcal{F}^{kl}_{ij}(p\,|\,x) = \sum_q F\begin{bmatrix} j & k \\ i & l \end{bmatrix}_{p,q} \mathcal{F}^{jk}_{il}(q\,|\,1-x) . \tag{2.138}$$

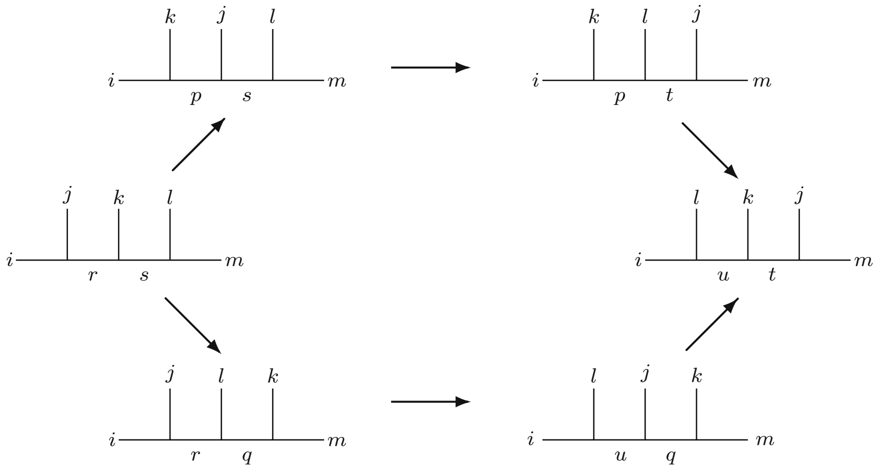
It is very useful to introduce a graphical notation for these two transformations, which also clarifies the choice of name for them. The braiding matrices defined in Eq. (2.137) are depicted as



and the fusing matrices defined in Eq. (2.138) as



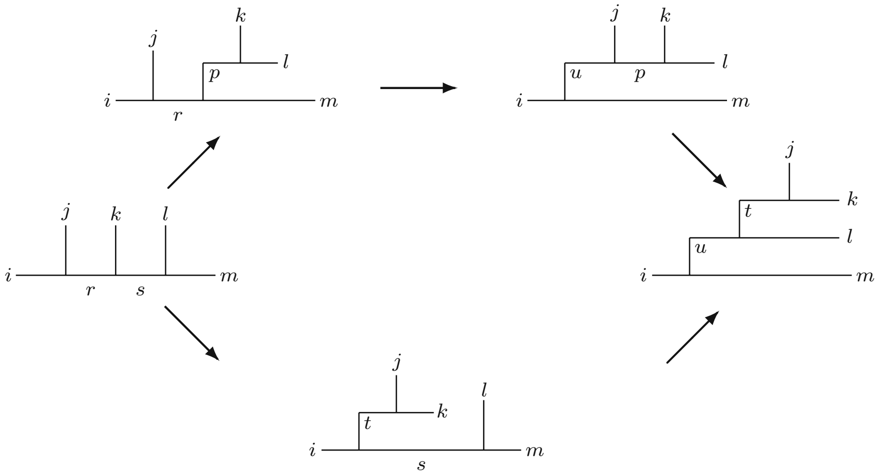
These matrices satisfy two important identities which can be derived by considering a five-point function and successively applying the braiding, and fusing operations, respectively. Again the origin of these relation is more transparent using the graphical notation. First, the commutativity of the diagram



leads to the so-called hexagon identify for the braiding matrices

$$\sum_p B\begin{bmatrix} j & k \\ i & s \end{bmatrix}_{rp} B\begin{bmatrix} j & l \\ p & m \end{bmatrix}_{st} B\begin{bmatrix} k & l \\ i & t \end{bmatrix}_{pu} = \sum_q B\begin{bmatrix} k & l \\ r & m \end{bmatrix}_{sq} B\begin{bmatrix} j & l \\ i & q \end{bmatrix}_{ru} B\begin{bmatrix} j & k \\ u & m \end{bmatrix}_{qt} ,$$

which is very similar to the Yang–Baxter equation arising for integrable models. Similarly, the commutativity of the diagram



implies a pentagon identity for the fusing matrices

$$F\begin{bmatrix} j & k \\ i & s \end{bmatrix}_{rt} F\begin{bmatrix} t & l \\ i & m \end{bmatrix}_{su} = \sum_p F\begin{bmatrix} k & l \\ r & m \end{bmatrix}_{sp} F\begin{bmatrix} j & p \\ i & m \end{bmatrix}_{ru} F\begin{bmatrix} j & k \\ u & l \end{bmatrix}_{pt} .$$

We will see in Sect. 4.3 that the pentagon identity also plays a very important role in the proof of the so-called Verlinde formula.

Further Reading

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