Chapter 1
Dimensional Analysis

1.1 Introduction

Before beginning the material on dimensional analysis, it is worth considering a simple example that demonstrates what we are doing. One that qualifies as simple is the situation of when a object is thrown upwards. The resulting mathematical model for this is an equation for the height $x(t)$ of the projectile from the surface of the Earth at time t. This equation is determined using Newton's second law, $F = ma$, and the law of gravitation. The result is

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(R+x)^2}, \quad \text{for} \;\; 0 < t, \tag{1.1}$$

where g is the gravitational acceleration constant and R is the radius of the Earth. Finding the solution x of this equation requires two integrations. Each will produce an integration constant, and we need more information to find these constants. This is done by specifying the initial conditions. Assuming the projectile starts at the surface with velocity v_0 then the initial conditions are as follows

$$x(0) = 0, \tag{1.2}$$

$$\frac{dx}{dt}(0) = v_0. \tag{1.3}$$

The resulting initial value problem for x consists in finding the solution of (1.1) that satisfies (1.2) and (1.3). Mathematically, the problem is challenging because it involves solving a second-order nonlinear differential equation. One option for finding the solution is simply to use a computer. However, the limitation with this is that it does not provide much insight into how the solution depends on the terms in the equation. One of the primary objectives of this text is to use mathematics to derive a fundamental understanding of how and why things work the way they do, and so, we are very interested in

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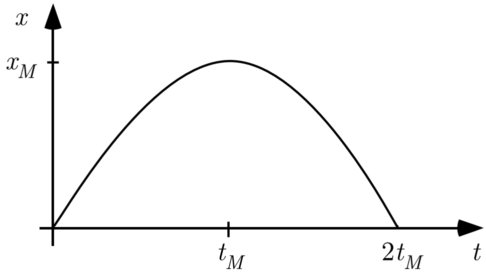


Figure 1.1 The solution (1.5) of the projectile problem in a uniform gravitational field.

obtaining at least an approximate solution of this problem. This is the same point-of-view taken in most physics books and it is worth looking at how they might address this issue. Adopting for the moment the typical Physics I approach, in looking at the equation in (1.1) it is not unreasonable to assume R is significantly larger than even the largest value of x. If true then we should be able to replace the $x+R$ term with just R. In this case the problem reduces to solving

$$\frac{d^2x}{dt^2} = -g, \quad \text{for} \;\; 0 < t. \tag{1.4}$$

Integrating and then using the two initial conditions yields

$$x(t) = -\frac{1}{2}gt^2 + v_0 t. \tag{1.5}$$

This solution is shown schematically in Figure 1.1. We have what we wanted, a relatively simple expression that serves as an approximation to the original nonlinear problem. To complete the derivation we should check that the assumption made in the derivation is satisfied, namely x is much smaller than R. Now, the maximum height for (1.5) occurs when

$$\frac{dx}{dt} = 0. \tag{1.6}$$

Solving this equation yields $t = v_0/g$ and from this it follows that the maximum height is

$$x_M = \frac{v_0^2}{2g}\,. \tag{1.7}$$

Therefore, we must require that $v^2/(2g)$ is much less than R, which we write as $v_0^2/(2g) << R$.

It is now time to critique the above derivation. The first criticism is that the approach is heuristic. The reason is that even though the argument for replacing $x + R$ with R seems plausible, we simply ignored a particular term in the equation. The projectile problem is not particularly complicated so dropping a term as we did is straightforward. However, in the real world where problems can be quite complicated, dropping a term in one part of the problem can lead to inconsistencies in another part. A second criticism can be made by asking a question. Specifically, what exactly is the effect of the nonlinearity on the projectile? Our reduction replaced the nonlinear gravitational force, which is the right-hand side of (1.1), with a uniform gravitational field given by $-g$. Presumably if gravity decreases with height then the projectile will be going higher than we would expect based on our approximation in (1.5). It is of interest to understand quantitatively what this nonlinear effect is and whether it might interfere with our reduction.

Based on the comments of the previous paragraph we need to make the reduction process more systematic. The procedure that is used to simplify the problem should enable us to know exactly what is large or small in the problem, and it should also enable us to construct increasingly more accurate approximations to the problem. Explaining what is involved in a systematic reduction occurs in two steps. The first, which is the objective of this chapter, involves the study of dimensions and how these can be used to simplify the mathematical formulation of the problem. After this, in Chapter 2, we develop techniques to construct accurate approximations of the resulting equations.

1.2 Examples of Dimensional Reduction

The first idea that we explore will, on the surface, seem to be rather simple, but it is actually quite profound. It has to do with the dimensions of the physical variables, or parameters, in a problem. To illustrate, suppose we know that the speed s of a ball is determined by its radius r and the length of time t it has been moving. Implicit in this statement is the assumption that the speed does not depend on any other physical variable. In mathematical terms we have that $s = f(r, t)$. The function f is not specified and all we know is that there is some expression that connects the speed with r and t. The only possible way to combine these two quantities to produce the dimension of speed is through their ratio r/t. For example, it is impossible to have $s = \alpha r + \beta t$ without α and β having dimensions. This would mean α and β are physical parameters, and we have assumed there are no others in the problem. This observation enables us to conclude that based on the original assumptions that the only function we can have is $s = \alpha r/t$, where α is a number.

What we are seeing in this example is that the dimensions of the variables in the problem end up dictating the form of the function. This is very useful information and we will spend some time exploring how to exploit this idea. To set the stage we need to introduce some of the terminology. The first is the concept of a fundamental dimension. As is well known, physical variables such as force, density, and velocity can be broken down into length L, time

Quantity	**Dimensions**	**Quantity**	**Dimensions**
Acceleration	LT^{-2}	Enthalpy	ML^2T^{-2}
Angle	1	Entropy	$ML^2T^{-2}\theta^{-1}$
Angular Acceleration	T^{-2}	Gas Constant	$L^2T^{-2}\theta^{-1}$
Angular Momentum	ML^2T^{-1}	Internal Energy	ML^2T^{-2}
Angular Velocity	T^{-1}	Specific Heat	$L^2T^{-2}\theta^{-1}$
Area	L^2	Temperature	θ
Energy, Work	ML^2T^{-2}	Thermal Conductivity	$MLT^{-3}\theta^{-1}$
Force	MLT^{-2}	Thermal Diffusivity	L^2T^{-1}
Frequency	T^{-1}	Heat Transfer Coefficient	$MT^{-3}\theta^{-1}$
Concentration	L^{-3}		
Length	L	Capacitance	$M^{-1}L^{-2}T^4I^2$
Mass	M	Charge	TI
Mass Density	ML^{-3}	Charge Density	$L^{-3}TI$
Momentum	MLT^{-1}	Conductivity	$M^{-1}L^{-3}T^3I^2$
Power	ML^2T^{-3}	Electric Current Density	$L^{-2}I$
Pressure, Stress, Elastic Modulus	$ML^{-1}T^{-2}$	Electric Current	I
Surface Tension	MT^{-2}	Electric Displacement	$L^{-2}TI$
Time	T	Electric Potential	$ML^2T^{-3}I^{-1}$
Torque	ML^2T^{-2}	Electric Field Intensity	$MLT^{-3}I^{-1}$
Velocity	LT^{-1}	Inductance	$ML^2T^{-2}I^{-2}$
Viscosity (Dynamic)	$ML^{-1}T^{-1}$	Magnetic Field Intensity	$L^{-1}I$
Viscosity (Kinematic)	L^2T^{-1}	Magnetic flux	$L^2MT^{-2}I^{-1}$
Volume	L^3	Permeability	$MLT^{-2}I^{-2}$
Wave Length	L	Permittivity	$M^{-1}L^{-3}T^4I^2$
Strain	1	Electric Resistance	$ML^2T^{-3}I^{-2}$

Table 1.1 Fundamental dimensions for commonly occurring quantities. A quantity with a one in the dimensions column is dimensionless.

T, and mass M (see Table 1.1). Moreover, length, time, and mass are independent in the sense that one of them cannot be written in terms of the other two. For these two reasons we will consider L, T, and M as *fundamental dimensions*. For problems involving thermodynamics we will expand this list to include temperature (θ) and for electrical problems we add current (I). In conjunction with this, given a physical variable x we will designate the fundamental dimensions of x using the notation $[x]$. For example, $[velocity] = L/T$, $[force] = ML/T^2$, $[g] = L/T^2$, and $[density] = M/L^3$.

It is important to understand that nothing is being assumed about which specific system of units is used to determine the values of the variables or parameters. Dimensional analysis requires that the equations be independent of the system of units. For example, both Newton's law $F = ma$ and the differential equation (1.1) do not depend on the specific system one selects. For this reason these equations are said to be *dimensionally homogeneous*. If one were to specialize (1.1) to SI units and set $R = 6378\,\text{km}$ and $g = 9.8\,\text{m/sec}^2$ they would end up with an equation that is not dimensionally homogeneous.

1.2.1 Maximum Height of a Projectile

The process of dimensional reduction will be explained by applying it to the projectile problem To set the stage, suppose we are interested in the maximum height x_M of the projectile. Based on Newton's second law, and the initial conditions in (1.2) and (1.3), it is assumed that the only physical parameters that x_M depends on are g, v_0, and the mass m of the projectile. Mathematically this assumption is written as $x_M = f(g, m, v_0)$. The function f is unknown but we are going to see if the dimensions can be used to simplify the expression. The only way to combine g, m, v_0 to produce the correct dimensions is through a product or ratio. So, our start-off hypothesis is that there are numbers a, b, c so that

$$[x_M] = [m^a v_0^b g^c]. \tag{1.8}$$

Using the fundamental dimensions for these variables the above equation is equivalent to

$$\begin{aligned} L &= M^a (L/T)^b (L/T^2)^c \\ &= M^a L^{b+c} T^{-b-2c}. \end{aligned} \tag{1.9}$$

Equating the exponents of the respective terms in this equation we conclude

$$\begin{aligned} L: &\quad b + c = 1, \\ T: &\quad -b - 2c = 0, \\ M: &\quad a = 0. \end{aligned}$$

Solving these equations we obtain $a = 0$, $b = 2$, and $c = -1$. This means the only way to produce the dimensions of length using m, v_0, g is through the ratio v_0^2/g. Given our start-off assumption (1.8), we conclude that x_M is proportional to v_0^2/g. In other words, the original assumption that $x_M = f(g, m, v_0)$ dimensionally reduces to the expression

$$x_M = \alpha \frac{v_0^2}{g}\,, \tag{1.10}$$

where α is an arbitrary number. With (1.10) we have come close to obtaining our earlier result (1.7) and have done so without solving a differential equation or using calculus to find the maximum value. Based on this rather minimal effort we can make the following observations:

- If the initial velocity is increased by a factor of 2 then the maximum height will increase by a factor of 4. This observation offers an easy method for experimentally checking on whether the original modeling assumptions are correct.
- The constant α can be determined by running one experiment. Namely, for a given initial velocity $v_0 = \bar{v}_0$ we measure the maximum height $x_M = \bar{x}_M$. With these known values, $\alpha = g\bar{x}_M/\bar{v}_0^2$. Once this is done, the formula in (1.10) can be used to determine x_M for any v_0.

The steps we have used are the basis for the method of dimensional reduction, where an expression is simplified based on the fundamental dimensions of the quantities involved. Given how easy it was to obtain (1.10) the method is very attractive as an analysis tool. It does have limitations and one is that we do not know the value of the number α. It also requires us to be able to identify at the beginning what parameters are needed. The importance of this and how this relies on understanding the physical laws underlying the problem will be discussed later.

The purpose of the above example is to introduce the idea of dimensional reduction. What it does not show is how to handle problems with several parameters and this is the purpose of the next two examples.

1.2.2 Drag on a Sphere

In the design of automobiles, racing bicycles, and aircraft there is an overall objective to keep the drag on the object as small as possible. It is interesting to see what insight dimensional analysis might provide in such a situation, but since we are beginners it will be assumed the object is very simple and is a sphere (see Figure 1.2). The modeling assumption that is made is that the drag force D_F on the sphere depends on the radius R of the sphere, the velocity v of the sphere, the density ρ of the air, and the dynamic viscosity μ of the air. The latter is a measure of the resistance force of the air to motion and we will investigate this in Chapter 8. For the moment all we need is its fundamental dimensions and these are given in Table 1.1. In mathematical terms the modeling assumption is

$$D_F = f(R, v, \rho, \mu), \tag{1.11}$$

(a)

(b)

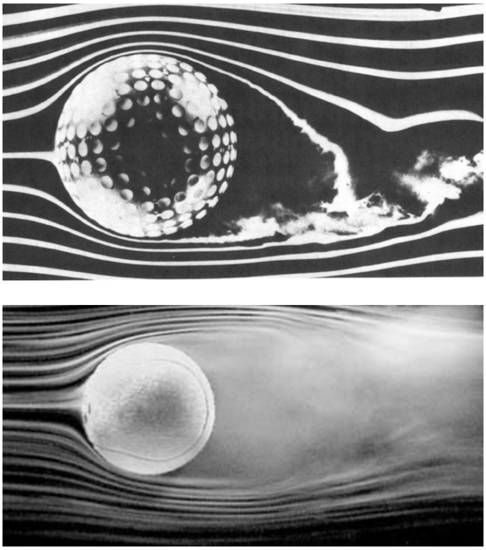


Figure 1.2 Air flow around an object can be visualized using smoke. The flow around a golf ball is shown in (a) (Brown [1971]), and around a tennis ball in (b) (Bluck [2000]). In both cases the air is moving from left to right.

and we want to use dimensional reduction to find a simplified version of this expression. Similar to the last example, the first question is whether we can find numbers a, b, c, d so that

$$[D_F] = [R^a v^b \rho^c \mu^d]. \tag{1.12}$$

Expressing these using fundamental dimensions yields

$$\begin{aligned} MLT^{-2} &= L^a (L/T)^b (M/L^3)^c (M/LT)^d \\ &= L^{a+b-3c-d} T^{-b-d} M^{c+d}. \end{aligned}$$

As before we equate the respective terms and conclude

$$\begin{aligned} L:&\quad a+b-3c-d = 1, \\ T:&\quad -b-d = -2, \\ M:&\quad c+d = 1. \end{aligned} \tag{1.13}$$

We have four unknowns and three equations, so it is anticipated that in solving the above system of equations one of the unknowns will be undetermined. From the T equation we have $b = 2-d$, and from the M equation $c = 1-d$. The L equation then gives us $a = 2-d$. With these solutions, and based on our assumption in (1.12), we have that

$$D_F = \alpha\rho R^{2-d} v^{2-d} \rho^{1-d} \mu^d$$
$$= \alpha\rho R^2 v^2 \left(\frac{\mu}{Rv\rho}\right)^d ,$$

where α is an arbitrary number. This can be written as

$$D_F = \alpha\rho R^2 v^2 \Pi^d , \tag{1.14}$$

where

$$\Pi = \frac{\mu}{Rv\rho} , \tag{1.15}$$

and d, α are arbitrary numbers. This is the *general product solution* for how D_F depends on the given variables. The quantity Π is dimensionless, and it is an example of what is known as a dimensionless product. Physically, it can be thought of as the ratio of the viscous force (μ) to the inertial force ($Rv\rho$) in the air. Calling it a product is a bit misleading as Π involves both multiplications and divisions. Some avoid this by calling it a dimensionless group. We will use both expressions in this book.

The formula for D_F in (1.14) is not the final answer. What remains is to determine the consequence of the arbitrary exponent d. The key observation is that given any two sets of values for (α, d), say (α_1, d_1) and (α_2, d_2), then

$$D_F = \alpha_1 \rho R^2 v^2 \Pi^{d_1} + \alpha_2 \rho R^2 v^2 \Pi^{d_2}$$
$$= \rho R^2 v^2 (\alpha_1 \rho \Pi^{d_1} + \alpha_2 \Pi^{d_2})$$

is also a solution. Extending this observation we conclude that another solution is

$$D_F = \rho R^2 v^2 (\alpha_1 \Pi^{d_1} + \alpha_2 \Pi^{d_2} + \alpha_3 \Pi^{d_3} + \cdots), \tag{1.16}$$

where $d_1, d_2, d_3, \ldots$ are arbitrary numbers as are the coefficients $\alpha_1, \alpha_2, \alpha_3, \ldots$. To express this in a more compact form, note that the expression within the parentheses in (1.16) is simply a function of Π. From this observation we obtain the *general solution*, which is

$$D_F = \rho R^2 v^2 F(\Pi), \tag{1.17}$$

where F is an arbitrary function of the dimensionless product Π. We have, therefore, been able to use dimensional analysis to reduce (1.11), which involves an unknown of four variables, down to an unknown function of one variable. Although this is a significant improvement, the result is perhaps not as satisfying as the one obtained for the projectile example, given in (1.10), because we have not been able to determine F. However, there are various ways to address this issue, and some of them will be considered below.

Representation of Solution

Now that the derivation is complete a few comments are in order. First, it is possible for two people to go through the above steps and come to what looks to be very different conclusions. For example, the general solution can also be written as

$$D_F = \frac{\mu^2}{\rho} H(\Pi), \tag{1.18}$$

where H is an arbitrary function of Π. The proof that this is equivalent to (1.17) comes from the requirement that the two expressions must produce the same result. In other words, it is required that

$$\frac{\mu^2}{\rho} H(\Pi) = \rho R^2 v^2 F(\Pi).$$

Solving this for H yields

$$H(\Pi) = \frac{1}{\Pi^2} F(\Pi).$$

The fact that the right-hand side of the above equation only depends on Π shows that (1.18) is equivalent to (1.17). As an example, if $F(\Pi) = \Pi$ in (1.17), then $H(\Pi) = 1/\Pi$ in (1.18).

Another representation for the general solution is

$$D_F = \rho R^2 v^2 G(Re), \tag{1.19}$$

where

$$Re = \frac{Rv\rho}{\mu}, \tag{1.20}$$

and G is an arbitrary function of Re. This form is the one usually used in fluid dynamics, where the dimensionless product Re is known as the Reynolds number. The function G is related to the drag coefficient C_D, through the equation $G = \frac{\pi}{2} C_D$. Because of its importance in fluids, G has been measured for a wide range of Reynolds numbers, producing the curve shown in Figure 1.3. To transform between the representation in (1.19), and the one in (1.17), note $Re = 1/\Pi$. From the requirement

$$\rho R^2 v^2 G(Re) = \rho R^2 v^2 F(\Pi),$$

we obtain

$$G(Re) = F(1/Re).$$

The reason for the different representations is that there are four unknowns in (1.12) yet only three equations. This means one of the unknowns is used in the general solution and, as expressed in (1.14), we used d. If you were to use one of the others then a different looking, but mathematically equivalent,

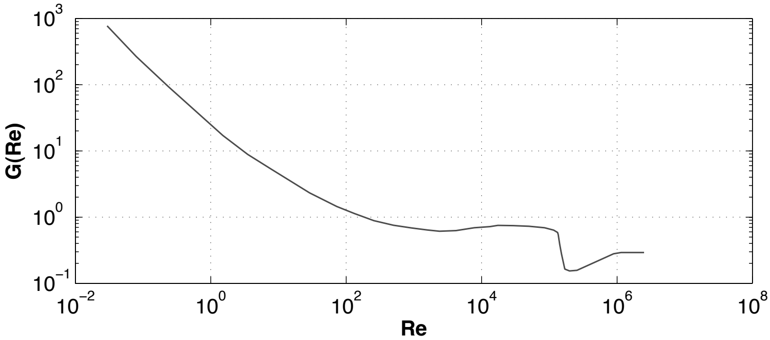


Figure 1.3 The measured values of the function $G(Re)$ that arises in the formula for the drag on a sphere, as given in (1.19).

expression would be obtained. The fact that there are multiple ways to express the solution can be used to advantage. For example, if one is interested in the value of D_F for small values of the velocity then (1.19) would be a bit easier to use. The reason is that to investigate the case of small v it is somewhat easier to determine what happens to G for Re near zero than to expand F for large values of Π. For the same reason, (1.17) is easier to work with for studying large velocities. One last comment to make is that even though there are choices on the form of the general solution, they all have exactly the same number of dimensionless products.

Determining F

A more challenging question concerns how to determine the function F in (1.17). The mathematical approach would be to solve the equations for fluid flow around a sphere and from this find F. This is an intriguing idea and one that will be used from time to time in this book. There is, however, another more applied approach that makes direct use of (1.17). Specifically, a sequence of experiments is run to measure $F(r)$ for $0 < r < \infty$. To do this a sphere with a given radius R_0, and a fluid with known density ρ_0 and viscosity μ_0, are selected. In this case (1.17) can be written as

$$F(r) = \frac{\gamma D_F}{v^2} \tag{1.21}$$

where $\gamma = 1/(\rho_0 R_0^2)$ is known and fixed. The experiment consists of taking various values of v and then measuring the resulting drag force D_F on the sphere. To illustrate, suppose our choice for the sphere and fluid give $R = 1$, $\rho_0 = 2$, and $\mu_0 = 3$. Also, suppose that running the experiment using $v = 4$ produces a measured drag of $D_F = 5$. In this case $r = \mu_0/(R_0 v \rho_0) = 3/8$ and $\gamma D_F/v^2 = 5/32$. Our conclusion is therefore that $F(3/8) = 5/32$. In this way, picking a wide range of v values we will be able to determine the values for the function $F(r)$. This approach is used extensively in the real world and the example we are considering has been a particular favorite for study. The data determined from such experiments are shown in Figure 1.3.

A number of conclusions can be drawn from Figure 1.3. For example, there is a range of Re values where G is approximately constant. Specifically, if $10^3 < Re < 10^5$ then $G \approx 0.7$. This is the reason why in the fluid dynamics literature you will occasionally see the statement that the drag coefficient $C_D = \frac{2}{\pi}G$ for a sphere has a constant value of approximately 0.44. For other Re values, however, G is not constant. Of particular interest, is the dependence of G for small values of Re. This corresponds to velocities v that are very small, what is known as Stokes flow. The data in Figure 1.3 show that G decreases linearly with Re in this region. Given that this is a log-log plot, then this means that $\log(G) = a - b\log(Re)$, or equivalently, $G = \alpha/Re^b$ where $\alpha = 10^a$. Curve fitting this function to the data in Figure 1.3 it is found that $\alpha \approx 17.6$ and $\beta \approx 1.07$. These are close to the exact values of $\alpha = 6\pi$ and $\beta = 1$, which are obtained by solving the equations of motion for Stokes flow. Inserting these values into (1.19), the conclusion is that the drag on the sphere for small values of the Reynolds number is

$$D_F \approx 6\pi\mu R v. \tag{1.22}$$

This is known as Stokes formula for the drag on a sphere, and we will have use for it in Chapter 4 when studying diffusion.

Scale Models

Why all the work to find F? Well, knowing this function allows for the use of scale model testing. To explain, suppose it is required to determine the drag on a sphere with radius R_f for a given velocity v_f when the fluid has density ρ_f and viscosity μ_f. Based on (1.17) we have $D_F = \rho_f R_f^2 v_f^2 F(\Pi_f)$, where

$$\Pi_f = \frac{\mu_f}{R_f v_f \rho_f}. \tag{1.23}$$

Consequently, we can determine D_F if we know the value of F at Π_f. Also, suppose that this cannot be measured directly as R_f is large and our experimental equipment can only handle small spheres. We can still measure $F(\Pi_f)$ using a small value of R if we change one or more of the parameters in such a way that the value of Π_f does not change. If R_m, μ_m, ρ_m and v_m are the values used in the experiment then we want to select them so that

$$\frac{\mu_m}{R_m v_m \rho_m} = \frac{\mu_f}{R_f v_f \rho_f}, \tag{1.24}$$

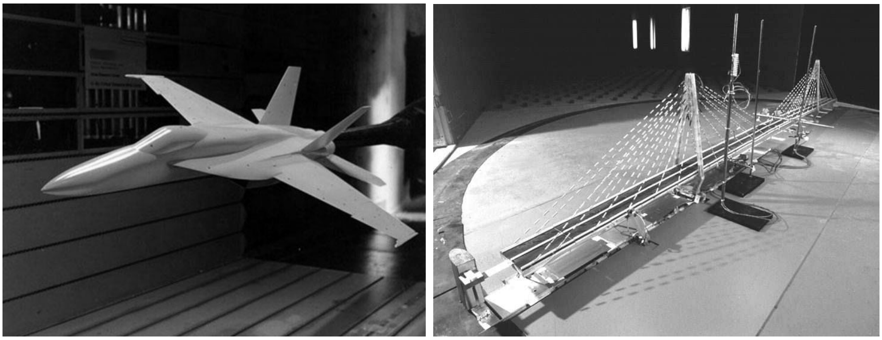


Figure 1.4 Scale model testing. Dimensional analysis is used in the development of scale models used in wind tunnels. On the left there is a flight test of an F-18 model in NASA's 11 ft transonic wind tunnel (NASA [2008]), and on the right a wind tunnel test of a 1:160 scale model of the Owensboro Bridge (Hsu [2009]).

or equivalently

$$v_m = \frac{\mu_m R_f \rho_f}{\mu_f R_m \rho_m} v_f. \tag{1.25}$$

This equation relates the values for the full-scale ball (subscript f) to those for the model used in the experiment (subscript m). As an example, suppose we are interested in the drag on a very large sphere, say $R_f = 100\,m$, but our equipment can only handle smaller values, say $R_m = 2\,m$. If the fluid for the two cases is the same, so $\rho_m = \rho_f$ and $\mu_m = \mu_f$, then according to (1.25), in our experiment we should take $v_m = 50v_f$. If the experimental apparatus is unable to generate velocities 50 times the value of v_f then it would be necessary to use a different fluid to reduce this multiplicative factor.

The result in the above example is the basis of scale model testing used in wind tunnels (see Figure 1.4). Usually these tests involve more than just keeping one dimensionless product constant as we did in (1.24). Moreover, it is evident in Figure 1.4 that the models look like the originals, they are just smaller. This is the basis of geometric similarity, where the lengths of the model are all a fraction of the original. For example, the bridge in Figure 1.4 is a $\frac{1}{20}$th scale model of the Owensboro Bridge. Other scalings are sometimes used and the most common are kinematic similarity, where velocities are scaled, and dynamic similarity, where forces are scaled.

Endnotes

One question that has not been considered so far is, how do you know to assume that the drag force depends on the radius, velocity, density, and dynamic viscosity? The assumption comes from knowing the laws of fluid dynamics, and identifying the principal terms that contribute to the drag.

For the most part, in this chapter the assumptions will be stated explicitly, as they were in this example. Later in the text, after the basic physical laws are developed, it will be possible to construct the assumptions directly. However, one important observation can be made, and that is the parameters used in the assumption should be independent. For example, even though the drag on a sphere likely depends on the surface area and volume of the sphere it is not necessary to include them in the list. The reason is that it is already assumed that D_F depends on the radius R and both the surface area and volume are determined using R.

The problem of determining the drag on a sphere is one of the oldest in fluid dynamics. Given that the subject is well over 150 years old, you would think that whatever useful information can be derived from this particular problem was figured out long ago. Well, apparently not, as research papers still appear regularly on this topic. A number of them come from the sports industry, where there is interest in the drag on soccer balls (Asai et al. [2007]), golf balls (Smits and Ogg [2004]), tennis balls (Goodwill et al. [2004]), as well as nonspherical-shaped balls (Mehta [1985]). Others have worked on how to improve the data in Figure 1.3, and an example is the use of a magnetic suspension system to hold the sphere (Sawada and Kunimasu [2004]). A more novel idea is to drop different types of spheres down a deep mine shaft, and then use the splash time as a means to determine the drag coefficient (Maroto et al. [2005]). The point here is that even the most studied problems in science and engineering still have interesting questions that remain unanswered.

1.2.3 Toppling Dominoes

Domino toppling refers to the art of setting up dominoes, and then knocking them down. The current world record for this is 4,000,000 plus dominoes for a team, and 300,000 plus for an individual. One of the more interesting aspects of this activity is that as the dominoes fall it appears as if a wave is propagating along the line of dominoes. The objective of this example is to examine what dimensional analysis might be able to tell us about the velocity of this wave. A schematic of the situation is shown in Figure 1.5. The assumption is that the velocity v depends on the spacing d, height h, thickness t, and the gravitational acceleration constant g. Therefore, the modeling assumption is $v = f(d, h, t, g)$ and we want to use dimensional reduction to find a simplified version of this expression. As usual, the first step is to find numbers a, b, c, e so that

$$[v] = [d^a h^b t^c g^e].$$

Expressing these using fundamental dimensions yields

$$LT^{-1} = L^a L^b L^c (L/T^2)^e$$
$$= L^{a+b+c+e} T^{-2e}.$$

Equating the respective terms we obtain

$$\begin{aligned} L:&\ \ a+b+c+e = 1, \\ T:&\ \ \qquad\qquad -2e = -1. \end{aligned}$$

Solving these two equations gives us that $e = \frac{1}{2}$ and $b = \frac{1}{2} - a - c$. With this we have that

$$\begin{aligned} v &= \alpha d^a h^{1/2-a-c} t^c g^{1/2} \\ &= \alpha\sqrt{hg}\left(\frac{d}{h}\right)^a \left(\frac{t}{h}\right)^c \\ &= \alpha\sqrt{hg}\Pi_1^a \Pi_2^c, \end{aligned} \tag{1.26}$$

where α is an arbitrary number, and the two dimensionless products are

$$\Pi_1 = \frac{d}{h},$$
$$\Pi_2 = \frac{t}{h}.$$

The expression in (1.26) is the general product solution. Therefore, the general solution for how the velocity depends on the given parameters is

$$v = \sqrt{hg}\, F(\Pi_1, \Pi_2), \tag{1.27}$$

where F is an arbitrary function of the two dimensionless products. The proof of how (1.27) follows from (1.26) is very similar to the method used to derive (1.17) from (1.15).

Dimensional analysis has been able to reduce the original assumption involving a function of four-dimensional parameters down to one involving two dimensionless products. This example is also informative as it demonstrates how to obtain the general solution when more than one dimensionless product is involved. The question remains, however, if this really applies to toppling

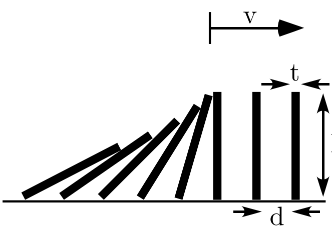


Figure 1.5 Schematic of toppling dominoes, creating a wave that propagates with velocity v.

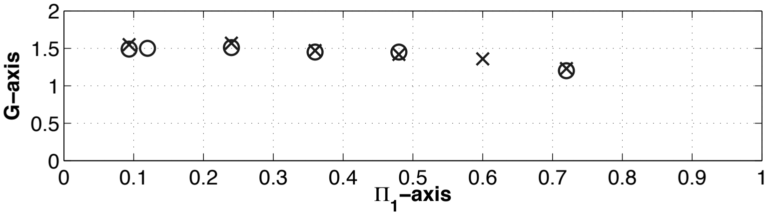


Figure 1.6 Data for two different types of toppling dominoes (Stronge and Shu [1988]). In these experiments, $t = 0.12h$, so the thin domino approximation is appropriate.

dominoes. It does, but in using this formula it is usually assumed the dominoes are very thin, or more specifically that $t << h$. This means that it is possible to assume $\Pi_2 = 0$, and (1.27) simplifies to

$$v = \sqrt{hg}\, G(\Pi_1), \tag{1.28}$$

where G is an arbitrary function. Some effort has been made to measure G, and the measurements for two different types of dominoes are given in Figure 1.6. Although the data show that G decreases with Π_1, it is approximately constant over the range of Π_1 values used in the experiments. Therefore, as an approximation we conclude that the speed at which dominoes topple is $v \approx 1.5\sqrt{hg}$. A typical domino has $h = 5$ cm, which results in a velocity of $v \approx 1$ m/s. To obtain a more explicit formula for G, however, requires the solution of a challenging mathematical problem, and an expanded discussion of this can be found in Efthimiou and Johnson [2007].

1.2.4 Endnotes

Based on the previous examples, the benefits of using dimensional reduction are apparent. However, a word of caution is needed here as the method gives the impression that it is possible to derive useful information without getting involved with the laws of physics or potentially difficult mathematical problems. One consequence of this is that the method is used to comment on situations and phenomena that are simply inappropriate (e.g., to study psychoacoustic behavior). The method relies heavily on knowing the fundamental laws for the problem under study, and without this whatever conclusions made using dimensional reduction are limited. For example, we earlier considered the drag on a sphere and in the formulation of the problem we assumed that the drag depends on the dynamic viscosity. Without knowing the equations of motion for fluids it would not have been possible to know that this term needed to be included or what units it might have. By not including it we would have concluded that $d = 0$ in (1.14) and instead of (1.17) we would have $D_F = \alpha\rho R^2 v^2$ where α is a constant. In Figure 1.3 it does appear that D_F is approximately independent of Re when $10^3 < Re < 10^5$. However, outside of this interval, D_F is strongly dependent on Re, and this means ignoring the viscosity would be a mistake. Another example illustrating the need to know the underlying physical laws arises in the projectile problem when we included the gravitational constant. Again, this term is essential and without some understanding of Newtonian mechanics it would be missed completely. The point here is that dimensional reduction can be a very effective method for simplifying complex relationships, but it is based heavily on knowing what the underlying laws are that govern the systems being studied.

1.3 Theoretical Foundation

The theoretical foundation for dimensional reduction is contained in the Buckingham Pi Theorem. To derive this result assume we have a physical quantity q that depends on physical parameters or variables $p_1, p_2, \ldots, p_n$. In this context, the word physical means that the quantity is measurable. Each can be expressed in fundamental dimensions and we will assume that the L, T, M system is sufficient for this task. In this case we can write

$$[q] = L^{\ell_0} T^{t_0} M^{m_0}, \tag{1.29}$$

and

$$[p_i] = L^{\ell_i} T^{t_i} M^{m_i}. \tag{1.30}$$

Our modeling assumption is that $q = f(p_1, p_2, \ldots, p_n)$. To dimensionally reduce this expression we will determine if there are numbers $a_1, a_2, \ldots, a_n$ so that

$$[q] = [p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}]. \tag{1.31}$$

Introducing (1.29) and (1.30) into the above expression, and then equating exponents, we obtain the equations

$$\begin{aligned}
L:&\quad \ell_1 a_1 + \ell_2 a_2 + \cdots + \ell_n a_n = \ell_0,\\
T:&\quad t_1 a_1 + t_2 a_2 + \cdots + t_n a_n = t_0,\\
M:&\quad m_1 a_1 + m_2 a_2 + \cdots + m_n a_n = m_0.
\end{aligned}$$

This can be expressed in matrix form as

$$\mathbf{A}\mathbf{a} = \mathbf{b}, \tag{1.32}$$

where

$$\mathbf{A} = \begin{pmatrix} \ell_1 & \ell_1 & \cdots & \ell_n \\ t_1 & t_2 & \cdots & t_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}, \tag{1.33}$$

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \ell_0 \\ t_0 \\ m_0 \end{pmatrix}. \tag{1.34}$$

The matrix $\mathbf{A}$ is known as the dimension matrix. As expressed in (1.33) it is $3 \times n$ but if we were to have used L, T, M, θ as the fundamental system then it would be $4 \times n$. In other words, the number of rows in the dimension matrix equals the number of fundamental units needed, and the number of columns equals the number of parameters that q is assumed to depend on.

With (1.32) we have transformed the dimensional reduction question into a linear algebra problem. To determine the consequences of this we first consider the situation that (1.32) has no solution. In this case the assumption that q depends on $p_1, p_2, \ldots, p_n$ is incomplete and additional parameters are needed. This situation motivates the following definition.

Definition 1.1. The set $p_1, p_2, \ldots, p_n$ is dimensionally incomplete for q if it is not possible to combine the p_i's to produce a quantity with the same dimension as q. If it is possible, the set is dimensionally complete for q.

From this point on we will assume the p_i's are complete and there is at least one solution of (1.32). To write down the general solution we consider the associated homogeneous equation, namely $\mathbf{Aa} = \mathbf{0}$. The set of solutions of this equation form a subspace $K(\mathbf{A})$, known as the kernel of $\mathbf{A}$. Letting k be the dimension of this subspace then the general solution of $\mathbf{Aa} = \mathbf{0}$ can be written as $\mathbf{a} = \gamma_1\mathbf{a}_1 + \gamma_2\mathbf{a}_2 + \cdots + \gamma_k\mathbf{a}_k$, where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ is a basis for $K(\mathbf{A})$ and $\gamma_1, \gamma_2, \ldots, \gamma_k$ are arbitrary. It is understood here that if $k = 0$ then $\mathbf{a} = \mathbf{0}$. With this, the general solution of (1.32) can be written as

$$\mathbf{a} = \mathbf{a}_p + \gamma_1\mathbf{a}_1 + \gamma_2\mathbf{a}_2 + \cdots + \gamma_k\mathbf{a}_k, \tag{1.35}$$

where $\mathbf{a}_p$ is any vector that satisfies (1.32) and $\gamma_1, \gamma_2, \ldots, \gamma_k$ are arbitrary numbers.

Example: Drag on a Sphere

To connect the above discussion with what we did earlier consider the drag on a sphere example. Writing (1.13) in matrix form we obtain

$$\begin{pmatrix} 1 & 1 & -3 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

This is the matrix equation (1.32) for this particular example. Putting this in augmented form, and row reducing, yields the following

$$\left(\begin{array}{rrrr|r} 1 & 1 & -3 & -1 & 1 \\ 0 & -1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right).$$

From this we conclude that $a = 2 - d$, $b = 2 - d$, and $c = 1 - d$. To be consistent with the notation in (1.35), set $d = \gamma$, so the solution is

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$

where γ is arbitrary. Comparing this with (1.35) we have that $k = 1$,

$$\mathbf{a}_p = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{a}_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}. \quad \blacksquare$$

It is now time to take our linear algebra conclusions and apply them to the dimensional reduction problem. Just as the appearance of d in (1.14) translated into the appearance of a dimensionless product in the general solution given in (1.17), each of the γ_i's in (1.35) gives rise to a dimensionless product in the general solution for the problem we are currently studying. To be specific, writing the ith basis vector $\mathbf{a}_i$ in component form as

$$\mathbf{a}_i = \begin{pmatrix} \alpha \\ \beta \\ \vdots \\ \gamma \end{pmatrix}, \tag{1.36}$$

then the corresponding dimensionless product is

$$\Pi_i = p_1^{\alpha} p_2^{\beta} \cdots p_n^{\gamma}. \tag{1.37}$$

Moreover, because the $\mathbf{a}_i$'s are independent vectors, the dimensionless products $\Pi_1, \Pi_2, \ldots, \Pi_k$ are independent.

As for the particular solution $\mathbf{a}_p$ in (1.35), assuming it has components

$$\mathbf{a}_p = \begin{pmatrix} a \\ b \\ \vdots \\ c \end{pmatrix}, \tag{1.38}$$

then the quantity

$$Q = p_1^a p_2^b \cdots p_n^c \tag{1.39}$$

has the same dimensions as q.

Based on the conclusions of the previous two paragraphs, the general product solution is $q = \alpha Q \Pi_1^{\kappa_1} \Pi_2^{\kappa_2} \cdots \Pi_k^{\kappa_k}$, where $\alpha, \kappa_1, \kappa_2, \ldots, \kappa_k$ are arbitrary constants. From this we obtain the following theorem.

Theorem 1.1. *Assuming the formula $q = f(p_1, p_2, \ldots, p_n)$ is dimensionally homogeneous and dimensionally complete, then it is possible to reduce it to one of the form $q = Q\,F(\Pi_1, \Pi_2, \ldots, \Pi_k)$, where $\Pi_1, \Pi_2, \ldots, \Pi_k$ are independent dimensionless products of $p_1, p_2, \ldots, p_n$. The quantity Q is a dimensional product of $p_1, p_2, \ldots, p_n$ with the same dimensions as q.*

According to this theorem, the original formula for q can be reduced from a function of n variables down to one with k. The value of k, which equals the nullity of the dimension matrix, ranges from 0 to $n-1$ depending on the given quantities $p_1, p_2, \ldots, p_n$. In the case that $k = 0$ the function F reduces to a constant and the conclusion is that $q = \alpha Q$, where α is an arbitrary number.

The importance of this theorem is that it establishes that the process used to reduce the drag on a sphere and toppling dominoes examples can be applied to much more complex problems. It also provides insight into how the number of dimensionless products is determined. There are still, however, fundamental questions left unanswered. For example, those with a more mathematical bent might still be wondering if this result can really be true no matter how discontinuous the original function f might be. Others might be wondering if the fundamental units used here, particularly length and time, are really independent. This depth of inquiry, although quite interesting, is beyond the scope of this text. Those wishing to pursue further study of these and related topics should consult Penrose [2007] and Bluman and Anco [2002].

1.3.1 Pattern Formation

The mechanism responsible for the colorful patterns on seashells, butterfly wings, zebras, and the like has intrigued scientists for decades. An experiment that has been developed to study pattern formation involves pouring chemicals into one end of a long tube, and then watching what happens as they interact while moving along the tube. This apparatus is called a plug-flow

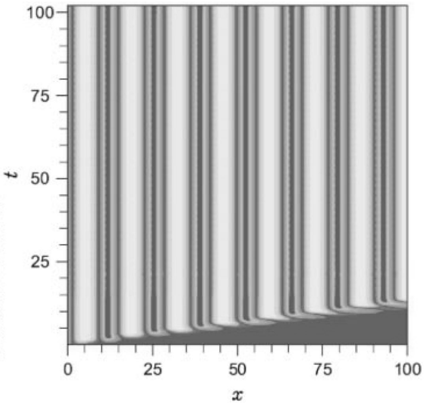


Figure 1.7 Spatial pattern created in a plug-flow reactor (Bamforth et al. [2000]). The tube occupies the interval $0 \leq x \leq 100$, and starting at $t = 0$ the chemicals are poured into the left end. As they flow along the tube a striped pattern develops.

reactor and the outcome of one such experiment is shown in Figure 1.7. It was found in these experiments that patterns appear only for certain pouring velocities v. According to what is known as the Lengyel-Epstein model, this velocity depends on the concentration U of the chemical used in the experiment, the rate k_2 at which the chemicals interact, the diffusion coefficient D of the chemicals, and a parameter k_3 that has the dimensions of concentration squared. The model is therefore assuming

$$v = f(U, k_2, D, k_3). \tag{1.40}$$

From Table 1.1 we have that $[v] = L/T$, $[U] = 1/L^3$, $[D] = L^2/T$, and $[k_3] = 1/L^6$. Also, from the Lengyel-Epstein model one finds that $[k_2] = L^3/T$. Using dimensional reduction we require

$$[v] = [U^a k_2^b D^c k_3^d]. \tag{1.41}$$

Expressing these using fundamental dimensions yields

$$\begin{aligned} LT^{-1} &= (L^{-3})^a (L^3 T^{-1})^b (L^2 T^{-1})^c (L^{-6})^d \\ &= L^{-3a+3b+2c-6d} T^{-b-c}. \end{aligned}$$

As before we equate the respective terms and conclude

$$\begin{aligned} L &: -3a + 3b + 2c - 6d = 1 \\ T &: \qquad\qquad -b - c = -1. \end{aligned}$$

These equations will enable us to express two of the unknowns in terms of the other two. There is no unique way to do this, and one choice yields

$b = -1 + 3a + 6d$ and $c = 2 - 3a - 6d$. From this it follows that the general product solution is

$$\begin{aligned} v &= \alpha U^a k_2^{3a+6d-1} D^{2-3a-6d} k_3^d \\ &= \alpha k_2^{-1} D^2 (U k_2^3 D^{-3})^a (k_2^6 D^{-6} k_3)^d. \end{aligned}$$

This can be rewritten as

$$v = \alpha k_2^{-1} D^2 \Pi_1^a \Pi_2^d, \tag{1.42}$$

where

$$\Pi_1 = \frac{U k_2^3}{D^3}, \tag{1.43}$$

and

$$\Pi_2 = \frac{k_2^6 k_3}{D^6}. \tag{1.44}$$

The dimensionless products Π_1 and Π_2 are independent, and this follows from the method used to derive these expressions. Independence is also evident from the observation that Π_1 and Π_2 do not involve exactly the same parameters. From this result it follows that the general form of the reduced equation is

$$v = k_2^{-1} D^2 F(\Pi_1, \Pi_2). \tag{1.45}$$

It is of interest to compare (1.45) with the exact formula obtained from solving the differential equations coming from the Lengyel-Epstein model. It is found that

$$v = \sqrt{k_2 D U}\, G(\beta), \tag{1.46}$$

where $\beta = k_3/U^2$ and G is a rather complicated square root function (Bamforth et al. [2000]). This result appears to differ from (1.45). To investigate this, note that $\beta = \Pi_2/\Pi_1^2$. Equating (1.45) and (1.46) it follows that

$$\begin{aligned} F(\Pi_1, \Pi_2) &= \frac{k_2^{3/2} U^{1/2}}{D^{3/2}} G(\beta) \\ &= \sqrt{\Pi_1}\, G(\Pi_2/\Pi_1^2). \end{aligned}$$

Because the right-hand side is a function of only Π_1 and Π_2 then (1.45) does indeed reduce to the exact result (1.46). Dimensional reduction has therefore successfully reduced the original unknown function of four variables in (1.40) down to one with only two variables. However, the procedure is not able to reduce the function down to one dimensionless variable, as given in (1.46). In this problem that level of reduction requires information only available from the differential equations, something that dimensional arguments are not able to discern.

1.4 Similarity Variables

Dimensions can be used not just to reduce formulas, they can be also used to simplify complex mathematical problems. The degree of simplification depends on the parameters, and variables, in the problem. One of the more well-known examples is the problem of finding the density $u(x,t)$ of a chemical over the interval $0 < x < \infty$. In this case the density satisfies the diffusion equation

$$D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \tag{1.47}$$

where the boundary conditions are

$$u|_{x=0} = u_0, \quad u|_{x\to\infty} = 0, \tag{1.48}$$

and the initial condition is

$$u|_{t=0} = 0. \tag{1.49}$$

The constant D is called the diffusion coefficient, and its dimensions can be determined from the terms in the differential equation. In particular, the dimensions of the left and right sides of (1.47) must be the same, and this means $[Du_{xx}] = [u_t]$. Because $[u] = M/L^3$ then $[u_{xx}] = [u]/L^2 = M/L^5$ and $[u_t] = [u]/T = M/(TL^3)$. From this we have $[D]M/L^5 = M/(TL^3)$, and therefore $[D] = L^2/T$. In a similar manner, in boundary condition (1.48), $[u_0] = [u] = M/L^3$. As a final comment, the physical assumptions underlying the derivation of (1.47) are the subject of Chapter 4. In fact, the solution we are about to derive is needed in Section 4.5.2 to solve the diffusion equation.

Dimensional Reduction

The conventional method for solving the diffusion equation on a semi-infinite spatial interval is to use an integral transform, and this will be considered in Chapter 4. It is also possible to find u using dimensional reduction. The approach is based on the observation that the only dimensional variables, and parameters, appearing in the problem are u, u_0, D, x, and t. In other words, it must be true that $u = f(x, t, D, u_0)$. With this we have the framework for dimensional reduction, and the question is whether we can find numbers a, b, c, d so that

$$[u] = [x^a t^b D^c (u_0)^d]. \tag{1.50}$$

Using fundamental dimensions yields

$$\begin{aligned} ML^{-3} &= L^a T^b (L^2/T)^c (M/L^3)^d \\ &= L^{a+2c-3d} T^{b-c} M^d, \end{aligned}$$

and then equating the respective terms gives us

$$\begin{aligned} L &: a + 2c - 3d = -3, \\ T &: \qquad\quad b - c = 0, \\ M &: \qquad\qquad\;\; d = 1. \end{aligned} \tag{1.51}$$

The solution of the above system can be written as $d = 1$ and $b = c = -a/2$. Given the assumption in (1.50), we conclude that the general product solution is

$$u = \alpha u_0 \left(\frac{x}{\sqrt{Dt}} \right)^a .$$

The general solution therefore has the form

$$u = u_0 F(\eta), \tag{1.52}$$

where

$$\eta = \frac{x}{\sqrt{Dt}} . \tag{1.53}$$

In this case, η is called a *similarity variable* as it is a dimensionless product that involves the independent variables in the problem.

When working out the drag on a sphere example, we discussed how it is possible to derive different representations of the solution. For the current example, when solving (1.51), instead of writing $b = c = -a/2$, we could just as well state that $a = -2b$ and $c = b$. In this case (1.52) is replaced with $u = u_0 G(\xi)$ where $\xi = Dt/x^2$. Although the two representations are equivalent, in the sense that one can be transformed into the other, it does make a difference which one is used when deriving a similarity solution. The reason is that we will be differentiating the solution, and (1.52) leads to much simpler formulas than the other representation. The rule of thumb here is that you want x in the numerator of the similarity variable. If you would like a hands on example of why this is true, try working out the steps below using the representation $u = u_0 G(\xi)$ instead of (1.52).

Similarity Solution

Up to this point we have been using a routine dimensional reduction argument. Our result, given in (1.52), is interesting as it states that the solution has a very specific dependence on the independent variables x and t. Namely, u can be written as a function of a single intermediate variable η. To determine F we substitute (1.52) back into the problem and find what equation F satisfies. With this in mind note, using the chain rule,

$$\begin{aligned}\frac{\partial u}{\partial t} &= u_0 F'(\eta)\frac{\partial \eta}{\partial t} \\ &= u_0 F'(\eta)\left(-\frac{x}{2D^{1/2}t^{3/2}}\right) \\ &= -u_0 F'(\eta)\frac{\eta}{2t}\,.\end{aligned}$$

In a similar manner one can show that

$$\frac{\partial^2 u}{\partial x^2} = u_0 F''(\eta)\frac{1}{Dt}\,.$$

Substituting these into (1.47) yields

$$F'' = -\frac{1}{2}\eta F', \quad \text{for } \ 0 < \eta < \infty. \tag{1.54}$$

We must also transform the boundary and initial conditions. The boundary condition at $x = 0$ takes the form

$$F(0) = 1, \tag{1.55}$$

while the condition as $x \to \infty$ and the one at $t = 0$ both translate into

$$F(\infty) = 0. \tag{1.56}$$

With this we have transformed a problem involving a partial differential equation (PDE) into one with an ordinary differential equation (ODE). As required, the resulting problem for F is only in terms of η. All of the original dimensional quantities, including the independent variables x and t, do not appear anywhere in the problem. This applies not just to the differential equation, but also to the boundary and initial conditions.

The problem for F is simpler than the original diffusion problem and, by itself, makes the use of dimensional analysis worthwhile. In this particular problem it is so simple that it is possible to solve for F. This can be done by letting $G = F'$, so the equation takes the form $G' = -\frac{1}{2}\eta G$. The general solution of this is $G = \alpha \exp(-\eta^2/4)$. Because $F' = G$, we conclude that the general solution is

$$F(\eta) = \beta + \alpha \int_0^{\eta} e^{-s^2/4} ds. \tag{1.57}$$

From (1.55) we have that $\beta = 1$ and from (1.56) we get

$$1 + \alpha \int_0^{\infty} e^{-s^2/4} ds = 0. \tag{1.58}$$

Given that $\int_0^\infty e^{-s^2/4} ds = \sqrt{\pi}$, then

$$F(\eta) = 1 - \frac{1}{\sqrt{\pi}} \int_0^{\eta} e^{-s^2/4} ds. \tag{1.59}$$

Expressions like this arise so often that they have given rise to a special function known as the complementary error function $\text{erfc}(\eta)$. This is defined as

$$\text{erfc}(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-r^2} dr. \tag{1.60}$$

Therefore, we have found that the solution of the diffusion problem is

$$u(x,t) = u_0 \, \text{erfc}\left(\frac{x}{2\sqrt{Dt}} \right). \tag{1.61}$$

As the above example demonstrates, using similarity variables and dimensional analysis provides a powerful tool for solving PDEs. It is, for example, one of the very few methods known that can be used to solve nonlinear problems. Its limitation is that the problem must have a specific form to work. We were able to solve the above diffusion problem because dimensional analysis reduced the form of the solution down to a function of one variable. This does not always happen and in such cases the method provides no insight into how the problem can be solved. As an example, if the spatial interval in the above diffusion problem is changed to one that is finite, so $0 < x < \ell$, then dimensional analysis will show that there are two independent similarity variables. This represents no improvement as we already know it is a function of two independent variables, so a reduction is not possible. Even with these limitations, however, similarity variables and their use in solving differential equations is a thriving area and a good introduction of the material can be found in Bluman and Cole [1974].

1.5 Nondimensionalization and Scaling

Another use we will have for dimensional analysis is to transform a problem into dimensionless form. The reason for this is that the approximation methods that are used to reduce difficult problems are based on comparisons. For example, in the projectile problem we simplified the differential equation by assuming that x was small compared to R. In contrast there are problems where the variable of interest is large, or it is slow or that it is fast compared to some other term in the problem. Whatever the comparison, it is important to know how all of the terms in the problem compare and for this we need the concept of scaling.

1.5.1 Projectile Problem

The reduction of the projectile equation (1.1) was based on the assumption that x is not very large, and so $x + R$ could be replaced with just R. We will routinely use arguments like this to find an approximate solution and it is therefore essential we take more care in making such reductions. The way this is done is by first scaling the variables in the problem using characteristic values. The best way to explain what this means is to work out an example and the projectile problem is an excellent place to start.

Change Variables

The first step in nondimensionalizing a problem is to introduce a change of variables, which for the projectile problem will have the form

$$t = t_c s,$$
$$x = x_c u.$$

In the above formula, x_c is a constant and it is a characteristic value of the variable x. It is going to be determined using the physical parameters in the problem, which for the projectile problem are g, R, and v_0. In a similar manner, t_c is a constant that has the dimensions of time and it represents a characteristic value of the variable t. In some problems it will be clear at the beginning how to select x_c and t_c. However, it is assumed here that we have no clue at the start what to choose and will not select them until the problem is studied a bit more. All we know at the moment is that whatever the choice, the new variables u, s are dimensionless. To make the change of variables note that from the chain rule

$$\begin{aligned} \frac{d}{dt} &= \frac{ds}{dt}\frac{d}{ds} \\ &= \frac{1}{t_c}\frac{d}{ds}\,, \end{aligned} \tag{1.62}$$

and

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{1}{t_c^2}\frac{d^2}{ds^2}\,. \tag{1.63}$$

With this the projectile equation (1.1) takes the form

$$\frac{1}{t_c^2}\frac{d^2}{ds^2}(x_c u) = -\frac{gR^2}{(R + x_c u)^2}\,. \tag{1.64}$$

The method requires us to collect the parameters into dimensionless groups. There is no unique way to do this, and this can cause confusion when first learning the procedure as there is no fixed method or answer. For example, to nondimensionalize the denominator in (1.64) one can factor it as either $R(1+x_c u/R)$ or as $x_c(R/x_c+u)$. The first has the benefit of enabling us to cancel the R in the numerator. Making this choice yields

$$\frac{x_c}{gt_c^2}\frac{d^2u}{ds^2} = -\frac{1}{(1+x_c u/R)^2}, \tag{1.65}$$

where the initial conditions (1.2), (1.3) are

$$u(0) = 0, \tag{1.66}$$

$$\frac{du}{ds}(0) = \frac{t_c}{x_c}v_0. \tag{1.67}$$

Find the Dimensionless Groups

Our change of variables has resulted in three dimensionless groups appearing in the transformed problem. They are

$$\Pi_1 = \frac{x_c}{gt_c^2}, \tag{1.68}$$

$$\Pi_2 = \frac{x_c}{R}, \tag{1.69}$$

$$\Pi_3 = \frac{t_c v_0}{x_c}. \tag{1.70}$$

There are a few important points that need to made here. First, the Π's do not involve the variables u, s and only depend on the parameters in the problem. Second, they are dimensionless and to accomplish this it was necessary to manipulate the projectile problem so the parameters end up grouped together to form dimensionless ratios. The third, and last, point is that the above three dimensionless groups are independent in the sense that it is not possible to write any one of them in terms of the other two. For example, Π_1 is the only one that contains the parameter g while Π_2 is the only one containing R. It is understood that in making the statement that the three groups are independent that x_c and t_c can be selected, if desired, independently of any of the parameters in the problem.

Before deciding on how to select x_c and t_c, it is informative to look a little closer at the above dimensionless groups. We begin with Π_2. In physical terms it is a measure of a typical, or characteristic, height of the projectile compared to the radius of the Earth. In comparison, Π_3 is a measure of a typical, or characteristic, velocity x_c/t_c compared to the velocity the projectile starts with. Finally, the parameter group Π_1 measures a typical, or characteristic, acceleration x_c/t_c^2 in comparison to the acceleration due to gravity in a uniform field. These observations can be helpful when deciding on how to nondimensionalize a problem as will be shown next.

Use Dimensionless Groups to Determine Scaling

It is now time to actually decide on what to take for x_c and t_c. There are whole papers written on what to consider as you select these parameters, but we will take a somewhat simpler path. For our problem we have two parameters to determine, and we will do this by setting two of the above dimensionless groups equal to one. What we need to do is decide on which two to pick, and we will utilize what might be called rules of thumb.

Rule of Thumb 1: Pick the Π's that appear in the initial and/or boundary conditions.

We only have initial conditions in our problem, and the only dimensionless group involved with them is Π_3. So we set $\Pi_3 = 1$ and conclude

$$x_c = v_0 t_c. \tag{1.71}$$

Rule of Thumb 2: Pick the Π's that appear in the reduced problem.

To use this rule it is first necessary to explain what the reduced problem is. This comes from the earlier assumption that the object does not get very high in comparison to the radius of the Earth, in other words, Π_2 is small. The reduced problem is the one obtained in the extreme limit of $\Pi_2 \to 0$. Taking this limit in (1.65)-(1.67), and using (1.71), the reduced problem is

$$\Pi_1 \frac{d^2u}{ds^2} = -1\,,$$

where

$$u(0) = 0, \quad \text{and} \quad \frac{du}{ds}(0) = 1.$$

According to the stated rule of thumb, we set $\Pi_1 = 1$, and so

$$x_c = v_0^2/g. \tag{1.72}$$

This choice for x_c seems reasonable based on our earlier conclusion that the maximum height for the uniform field case is $v_0^2/(2g)$.

Combining (1.71) and (1.72), we have that $x_c = v_0^2/g$ and $t_c = v_0/g$. With this scaling then (1.65) - (1.67) take the form

$$\frac{d^2u}{ds^2} = -\frac{1}{(1+\epsilon u)^2}\,, \tag{1.73}$$

where

$$u(0) = 0, \tag{1.74}$$

$$\frac{du}{ds}(0) = 1. \tag{1.75}$$

The dimensionless parameter appearing in the above equation is

$$\epsilon = \frac{v_0^2}{gR} \, . \tag{1.76}$$

This parameter will play a critical role in our constructing an accurate approximation of the solution of the projectile problem. This will be done in the next chapter but for the moment recall that since $R \approx 6.4 \times 10^6$ m and $g \approx 9.8\,\mathrm{m/s}^2$ then $\epsilon \approx 1.6 \times 10^{-8} v_0^2$. Consequently for baseball bats, sling shots, BB-guns, and other everyday projectile-producing situations, where v_0 is not particularly large, the parameter ϵ is very small. This observation is central to the subject of the next chapter.

Changing Your Mind

Before leaving this example it is worth commenting on the nondimensionalization procedure by asking a question. Namely, how bad is it if different choices would have been made for x_c and t_c? For example, suppose for some reason one decides to take $\Pi_2 = 1$ and $\Pi_3 = 1$. The resulting projectile problem is

$$\epsilon \frac{d^2u}{ds^2} = -\frac{1}{(1+u)^2} \, , \tag{1.77}$$

where $u(0) = 0$, $\frac{du}{ds}(0) = 1$, and ϵ is given in (1.76). No approximation has been made here and therefore this problem is mathematically equivalent to the one given in (1.73)-(1.75). Based on this, the answer to the question would be that using this other scaling is not so bad. However, the issue is amenability and what properties of the solution one is interested in. To explain, earlier we considered how the solution behaves if v_0 is not very large. With the scaling that produced (1.77), small v_0 translates into looking at what happens when ϵ is near zero. Unfortunately, the limit of $\epsilon \to 0$ results in the loss of the highest derivative in the differential equation and (1.77) reduces to $0 = -1$. How to handle such singular limits will be addressed in the next chapter but it requires more work than is necessary for this problem. In comparison, letting ϵ approach zero in (1.73) causes no such complications and for this reason it is more amenable to the study of the small v_0 limit. The point here is that if there are particular limits, or conditions, on the parameters that it is worth taking them into account when constructing the scaling.

1.5.2 Weakly Nonlinear Diffusion

To explore possible extensions of the nondimensionalization procedure we consider a well-studied problem involving nonlinear diffusion. The problem consists of finding the concentration $c(x,t)$ of a chemical over an interval $0 < x < \ell$. The concentration satisfies

$$D\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t} - \lambda(\gamma - c)c, \tag{1.78}$$

where the boundary conditions are

$$c|_{x=0} = c|_{x=\ell} = 0, \tag{1.79}$$

and the initial condition is

$$c|_{t=0} = c_0 \sin(5\pi x/\ell). \tag{1.80}$$

The nonlinear diffusion equation (1.78) is known as Fisher's equation, and it arises in the study of the movement of genetic traits in a population. A common simplifying assumption made when studying this equation is that the nonlinearity is weak, which means that the term λc^2 is small in comparison to the others in the differential equation. This assumption will be accounted for in the nondimensionalization.

Before starting the nondimensionalization process we should look at the fundamental dimensions of the variables and parameters in the problem. First, c is a concentration, which corresponds to the number of molecules per unit volume, and so $[c] = L^{-3}$. The units for the diffusion coefficient D were determined earlier, and it was found that $[D] = L^2/T$. As for γ, the $\gamma - c$ term in the differential equation requires these two quantities to have the same dimensions, and so $[\gamma] = [c]$. Similarly, from the differential equation we have $[\lambda(\gamma - c)c] = [\frac{\partial c}{\partial t}]$, and from this it follows that $[\lambda] = L^3T^{-1}$. Finally, from the initial condition we have that $[c_0] = [c]$. It is important to make an observation related to dimensions, and this will be done by asking a question: is it possible to replace the initial condition (1.80) with $c|_{t=0} = c_0 \sin(5\pi x)$ or with $c|_{t=0} = c_0 \sin(x)$? The answer in both cases is no, and the reason is that the argument of the sine function must be dimensionless. For exactly the same reason it is not possible to use $c|_{t=0} = c_0 e^x$. It is possible, however, to use $c|_{t=0} = c_0 x$ or $c|_{t=0} = c_0 x^2$, although the dimensions of c_0 differ from what we found earlier.

Now, to nondimensionalize the problem we introduce the change of variables

$$x = x_c y, \tag{1.81}$$

$$t = t_c s, \tag{1.82}$$

$$c = c_c u. \tag{1.83}$$

In this context, x_c has the dimensions of length and is a characteristic value of the variable x. Similar statements apply to t_c and c_c. Using the chain rule as in (1.62) the above differential equation takes the form

$$\frac{Dc_c}{x_c^2}\frac{\partial^2 u}{\partial y^2} = \frac{c_c}{t_c}\frac{\partial u}{\partial s} - \lambda c_c(\gamma - c_c u)u.$$

It is necessary to collect the parameters into dimensionless groups, and so in the above equation we rearrange things a bit to obtain

$$\frac{Dt_c}{x_c^2}\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial s} - \lambda t_c c_c(\gamma/c_c - u)u. \tag{1.84}$$

In conjunction with this we have the boundary conditions

$$u|_{y=0} = u|_{y=\ell/x_c} = 0, \tag{1.85}$$

and the initial condition is

$$u|_{s=0} = (c_0/c_c)\sin(5\pi x_c y/\ell). \tag{1.86}$$

The resulting dimensionless groups are

$$\Pi_1 = \frac{Dt_c}{x_c^2}, \tag{1.87}$$

$$\Pi_2 = \lambda t_c c_c, \tag{1.88}$$

$$\Pi_3 = \gamma/c_c, \tag{1.89}$$

$$\Pi_4 = \ell/x_c, \tag{1.90}$$

$$\Pi_5 = c_0/c_c. \tag{1.91}$$

It is important to note that the five dimensionless groups given above are independent in the sense that it is not possible to write one of them in terms of the other four. As before this statement is based on our ability to select, if desired, the scaling parameters x_c, t_c, c_c independently of each other and the other parameters in the problem. Also, in counting the dimensionless groups one might consider adding a sixth. Namely, in the initial condition (1.86) there is $\Pi_6 = 5\pi x_c/\ell$. The reason it is not listed above is that it is not independent of the others because $\Pi_6 = 5\pi/\Pi_4$. The 5π is a number and does not play a role in determining dimensional independence.

We have three scaling parameters to specify, namely x_c, t_c, c_c. Using Rule of Thumb 1, the Π's that appear in the boundary and initial conditions are set equal to one. In other words, we set $\Pi_4 = 1$ and $\Pi_5 = 1$, from which it follows that $x_c = \ell$ and $c_c = c_0$.

To use Rule of Thumb 2, we need to investigate what it means to say that the nonlinearity is weak. The equation (1.84) is nonlinear due to the term $\lambda t_c c_c u^2$, and the coefficient $\lambda t_c c_c$ is the associated strength of the nonlinearity. For a weakly nonlinear problem one is interested in the solution for small values of $\lambda t_c c_c$. Taking the extreme limit we set $\lambda t_c c_c = 0$ in (1.84) to produce the reduced equation. The only group that remains in this limit is Π_1, and for this reason this is the group we select. So, setting $\Pi_1 = 1$ then we conclude $t_c = \ell^2/D$.

The resulting nondimensional diffusion equation is

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial s} - \epsilon(b-u)u, \tag{1.92}$$

with boundary conditions

$$u(0,s) = u(1,s) = 0, \tag{1.93}$$

and the initial condition

$$u(y,0) = \sin(5\pi y). \tag{1.94}$$

The dimensionless parameters appearing in the above equation are $\epsilon = \lambda c_0 \ell^2/D$ and $b = \gamma/c_0$. With this, weak nonlinearity corresponds to assuming that ϵ is small.

1.5.3 Endnotes

As you might have noticed, the assumption of a weak nonlinearity was used in the projectile problem, although it was stated in more physical terms. In both examples the reduced problem, obtained setting $\epsilon = 0$, is linear. It is certainly possible that a physical problem is not weakly nonlinear but involves some other extreme behavior. As an example, in nonlinear diffusion problems you come across situations involving weak diffusion. What this means for (1.84) is that Dt_c/x_c^2 has a small value. In the extreme limit that this term is zero then the only group that remains in the reduced problem is Π_2. Setting $\Pi_2 = 1$ then $t_c = c_0/\lambda$. With this, (1.84) becomes

$$\epsilon\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + (b-u)u, \tag{1.95}$$

where $\epsilon = Dc_0/(\lambda\ell^2)$ and $b = \gamma/c_0$. With this, weak diffusion corresponds to assuming that ϵ is small.

For those keeping track of the rules of thumb used to nondimensionalize a problem we have two. The first we ran across is the rule that the dimensionless groups in the initial and boundary conditions are set to one. The second rule arose when setting the dimensionless groups in the reduced problem to one. Although these can be effective rules, it is certainly possible to find problems where another scaling should be considered, and examples are given in Exercises 1.17 and 3.8. The overall objective in all cases is that the nondimensionalization is based on characteristic values of the variables.

Exercises

1.1. The amount of noise permitted from the large rollers used in road construction was recently limited by changes in the environmental laws. Rather than build multiple full-sized rollers in an attempt to find one that satisfied the new law a manufacturer decided that dimensional analysis could be used. The assumption they made was that the frequency f of the sound coming off the roller depends on the elastic modulus E and the density ρ of the steel used to construct the roller as well as on the length ℓ of the roller.

(a) Find a dimensionally reduced form for f.

(b) In building a scale model for testing the manufacturer selected the parameters so that

$$\frac{f_m}{f_f} = \frac{\ell_f}{\ell_m}\sqrt{\frac{\rho_f E_m}{\rho_m E_f}},$$

where the subscript f designates full-sized and the subscript m designates scale model. Explain why this was done.

1.2. For a pendulum that starts from rest, the period p depends on the length ℓ of the rod, on gravity g, on the mass m of the ball, and on the initial angle θ_0 at which the pendulum is started.

(a) Use dimensional analysis to determine the functional dependence of p on these four quantities.

(b) For the largest pendulum ever built, the rod is 70 ft and the ball weighs 900 lbs. Assuming that $\theta_0 = \pi/6$ explain how to use a pendulum that fits on your desk to determine the period of this largest pendulum.

(c) Suppose it is found that p depends linearly on θ_0, with $p = 0$ if $\theta_0 = 0$. What does your result in part (a) reduce to in this case?

1.3. The velocity v at which flow in a pipe will switch from laminar to turbulent depends on the diameter d of the pipe as well as on the density ρ and dynamic viscosity μ of the fluid.

(a) Find a dimensionally reduced form for v.

(b) Suppose the pipe has diameter $d = 100$ and for water (where $\rho = 1$ and $\mu = 10^{-2}$) it is found that $v = 0.25$. What is v for olive oil (where $\rho = 1$ and $\mu = 1$)? The units here are in cgs.

1.4. The luminosity of certain giant and supergiant stars varies in a periodic manner. It is hypothesized that the period p depends upon the star's average radius r, its mass m, and the gravitational constant G.

(a) Newton's law of gravitation asserts that the attractive force between two bodies is proportional to the product of their masses divided by the square of the distance between them, that is,

$$F = \frac{Gm_1m_2}{d^2},$$

where G is the gravitational constant. From this determine the (fundamental) dimensions of G.

(b) Use dimensional analysis to determine the functional dependence of p on m, r, and G.

(c) Arthur Eddington used the theory for thermodynamic heat engines to show that

$$p = \sqrt{\frac{3\pi}{2\gamma G\rho}},$$

where ρ is the average density of the star and γ is the ratio of specific heats for stellar material. How does this differ from your result?

(d) In Figure 1.8 the data for a pulsating star are given. Explain how you could use data like this to complete the formula you derived in part (b).

1.5. When a drop of liquid hits a wetted surface a crown formation appears, as shown in Figure 1.9(a). It has been found that the number of points N on the crown depends on the speed U at which the drop hits the surface, the radius r and density ρ of the drop, and the surface tension σ of the liquid making up the drop. How N depends on these quantities has been studied extensively and some of the reasons why are given in Rioboo et al. [2003].

(a) Use dimensional reduction to determine the functional dependence of N on U, r, ρ, and σ. Express your answer in terms of the Weber number $W_e = \rho U^2 r/\sigma$.

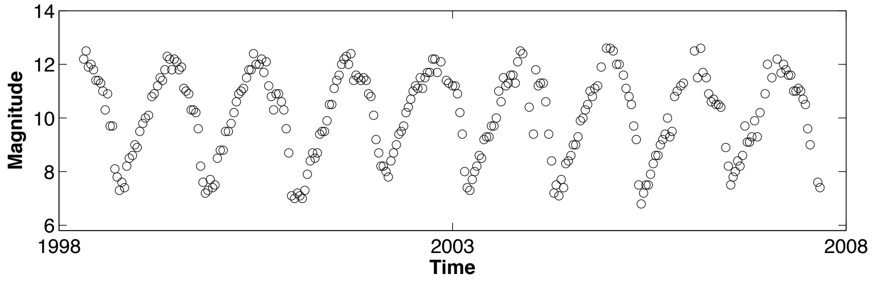


Figure 1.8 Luminosity of a Mira type variable star, 1621+19 U Herculis (AAVSO [2009]).

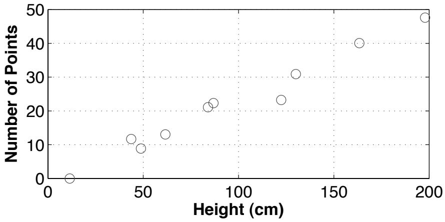
(a)  (b) 

Figure 1.9 (a) Formation of a crown when a liquid drop hits a wetted surface. (b) The measured values of the number of points N (Hobbs and Kezweent [1967]).

(b) The value of N has been measured as a function of the initial height h of the drop and the results are shown in Figure 1.9(b). Express your answer in part (a) in terms of h by writing U in terms of h and g. Assume the drop starts with zero velocity.
(c) The data in Figure 1.9(b) show a piecewise linear dependence on h, specifically, N can be described as a continuous function made up of two linear segments. Use this, and your result from part (b), to find the unknown function in part (a). In the experiments, $r = 3.6$ mm, $\rho = 1.1014$ gm/cm^3, and $\sigma = 50.5$ dyn/cm.
(d) According to your result from part (c), what must the initial height of the drop be to produce at least 80 points?
(e) According to your result from part (c), how many points are generated for a drop of mercury when $h = 200$ cm? Assume $r = 3.6$ mm, $\rho = 13.5$ gm/cm^2, and $\sigma = 435$ dyn/cm.

1.6. The frequency ω of waves on a deep ocean is found to depend on the wavelength λ of the wave, the surface tension σ of the water, the density ρ of the water, and gravity.
(a) Use dimensional reduction to determine the functional dependence of ω on λ, σ, ρ, and g.
(b) In fluid dynamics it is shown that

$$\omega = \sqrt{gk + \frac{\sigma k^3}{\rho}},$$

where $k = 2\pi/\lambda$ is the wavenumber. How does this differ from your result in (a)?

1.7. A ball is dropped from a height h_0 and it rebounds to a height h_r. The rebound height depends on the elastic modulus E, radius R, and the mass density ρ of the ball. It also depends on the initial height h_0 and the gravitational constant g.
(a) Find a dimensionally reduced form for h_r.
(b) Suppose it is found that h_r depends linearly on h_0, with $h_r = 0$ if $h_0 = 0$. What does your formula from part (a) reduce to in this case?

(c) Suppose the density of the ball is doubled. Use the result in (a) to explain how to change E so the rebound height stays the same.

1.8. A ball, when released underwater, will rise towards the surface with velocity v. This velocity depends on the density ρ_b and radius R of the ball, on gravity g, and on the density ρ_f and kinematic viscosity ν of the water.
(a) Find a dimensionally reduced form for v.
(b) In fluid mechanics, using Stokes' Law, it is found that

$$v = \frac{2gR^2(\rho_b - \rho_f)}{9\nu\rho_f} .$$

How does this differ from your result from part (a)? It is interesting to note that this formula is used by experimentalists to determine the viscosity of fluids. They do this by measuring the velocity in an apparatus called a falling ball viscometer, and then solving for ν in the above formula.

1.9. In electric image tomography the objective is to determine the properties inside an object and this is done by applying a potential U to the surface. What is measured is the resulting electric current j on the surface. Suppose that it is found that the electric potential u within a spherical body depends on the object's radius R and conductivity σ as well as depends on U and j.
(a) Find a dimensionally reduced form for u.
(b) Suppose that given a particular object that doubling the applied potential U causes the internal potential u to increase by a factor of four. How does this help simply your result in (a)?
(c) Suppose it is necessary to know the internal potential u when using a large applied potential, say $U = 2500V$. However, for legal reasons it is required that only applied potentials less than $250V$ can be used. Explain, using your result from (a), how to legally determine the large applied potential value.

1.10. The velocity v of water through a circular pipe depends on the pressure difference p between the two ends of the pipe, the length ℓ and radius r of the pipe, as well as on the dynamic viscosity μ and density ρ of the water.
(a) Use dimensional analysis to determine the functional dependence of v on the above quantities.
(b) Suppose it is found that v depends linearly on p, with $v = 0$ if $p = 0$. What does your formula from part (a) reduce to in this case?
(c) Your formula from part (b) should contain a general function of one, or more, dimensionless products. Explain how to experimentally determine this function. Be specific about which parameters are fixed, and which are varied, in the experiment. Also, your experiment should vary as few of the parameters as possible in determining this function.

1.11. In a high energy explosion there is a very rapid release of energy E that produces an approximately spherical shock wave that expands in time (Figure 1.10).

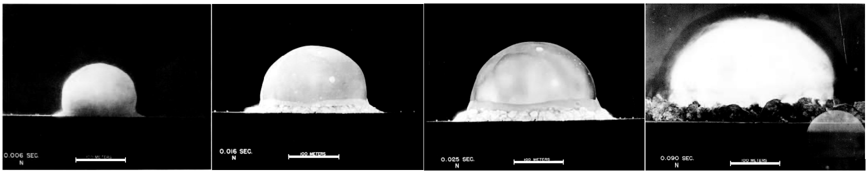


Figure 1.10 Shock wave produced by a nuclear explosion, at 6 msec, 16 msec, 25 msec, and 90 msec. The width of the white bar in each figure is $100m$ (Brixner [2009]).

(a) Assuming the radius R of the shock wave depends on E, the length of time t since the explosion, and the density ρ of the air, use dimensional reduction to determine how the radius depends on these quantities. This expression is known as the Taylor-Sedov formula.
(b) It was shown by G. I. Taylor that if $E = 1J$ and $\rho = 1\,kg/m^3$ then $R = t^{2/5}\,m/s^{2/5}$. Use this information and the result from (a) to find the exact formula for R.
(c) Use the photographs in Figure 1.10, and your result from (b) to estimate the energy released. The air density is $\rho = 1\,kg/m^3$.
(d) The blast wave from a supernova can be modeled using the Taylor-Sedov formula. Explain how this can be used to estimate the date the supernova took place, using your result from part (b). As an example, use Tycho, which currently has a radius of about 33.2 light years, an estimated energy of $10^{44}J$, and density $\rho = 2 \times 10^{-21}$ kg/m^3.

1.12. The vertical displacement $u(x)$ of an elastic string of length ℓ satisfies the boundary value problem

$$\tau\frac{d^2u}{dx^2} + \mu u = p, \quad \text{for } \; 0 < x < \ell,$$

where $u(0) = 0$, $u(\ell) = U$. Also, p is a constant and has the dimensions of force per length.
(a) What are the dimensions for the constants τ and μ?
(b) Show how it is possible to nondimensionalize this problem so it takes the form

$$\frac{d^2v}{ds^2} + \alpha v = \beta, \quad \text{for } \; 0 < s < 1,$$

where $v(0) = 0$, $v(1) = 1$. Make sure to state what α, β are.

1.13. From Newton's second law, the displacement $y(t)$ of the mass in a mass, spring, dashpot system satisfies

$$m\frac{d^2y}{dt^2} = F_s + F_d, \quad \text{for } \; 0 < t,$$

where m is the mass, F_s is the restoring force in the spring, and F_d is the damping force. To have a compete IVP we need to state the initial conditions, and for this problem assume

$$y(0) = 0, \quad \frac{dy}{dt}(0) = v_0.$$

(a) Suppose there is no damping, so $F_d = 0$, and the spring is linear, so $F_s = -ky$. What are the dimensions for the spring constant k? Nondimensionalize the resulting IVP. Your choice for y_c and t_c should result in no dimensionless products being left in the IVP.

(b) Now, in addition to a linear spring, suppose linear damping is included, so,

$$F_d = -c\frac{dy}{dt}.$$

What are the dimensions for the damping constant c? Using the same scaling as in part (a), nondimensionalize the IVP. Your answer should contain a dimensionless parameter ϵ that measures the strength of the damping. In particular, if c is small then ϵ is small. The system in this case is said to have weak damping.

1.14. The velocity $v(t)$ of the waves on a deep ocean satisfies the equation

$$\frac{dv}{dt} + kv^2 = \ell v, \quad \text{for } \; 0 < t,$$

where $v(0) = V$.

(a) What are the dimensions of the constants k, ℓ, and V?

(b) Assuming a weak nonlinearity, use the Rules of Thumb given in Section 1.5 to nondimensionalize this problem.

1.15. The equation for an elastic beam is

$$EI\frac{\partial^4 u}{\partial x^4} + \rho\frac{\partial^2 u}{\partial t^2} = 0,$$

where the boundary conditions are $u = u_0 \sin(\omega t)$ and $\frac{\partial u}{\partial x} = 0$ at $x = 0$, while $u = \frac{\partial u}{\partial x} = 0$ at $x = \ell$. Assume the initial conditions are $u = 0$ and $\frac{\partial u}{\partial t} = 0$ at $t = 0$. Here E is the elastic modulus, I is the moment of inertia, and ρ is the mass per unit length of the beam. Nondimensionalize the problem in such a way that the resulting boundary conditions contain no nondimensional groups.

1.16. When an end of a slender strip of paper is put into a cup of water, because of absorption, the water rises up the paper. The density ρ of the water along the strip satisfies the differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0,$$

where J is known as the flux.

(a) What are the dimensions of J?
(b) The flux J depends on the gravitational constant g, the strip width d, the density gradient $\frac{\partial \rho}{\partial x}$, and the surface tension σ of the water. Find a dimensionally reduced form for J.
(c) What does your result in (b) reduce to if it is found that J depends linearly on the density gradient, with $J = 0$ if $\rho_x = 0$? What is the resulting differential equation?
(d) If the strip has length h the boundary conditions are $\rho = \rho_0$ at $x = 0$ and $J = 0$ at $x = h$. The initial condition is $\rho = 0$ at $t = 0$. With this, and your differential equation from (c), nondimensionalize the problem for ρ in such a way that no nondimensional groups appear in the final answer.

1.17. A thermokinetic model for the concentration u and temperature q of a mixture consists of the following equations (Gray and Scott [1994])

$$\frac{du}{dt} = k_1 - k_2 u e^{k_3 q},$$
$$\frac{dq}{dt} = k_4 u e^{k_3 q} - k_5 q.$$

The initial conditions are $u = 0$ and $q = 0$ at $t = 0$.
(a) What are the dimensions of the k_i's?
(b) Explain why the rule of thumb for scaling used in the projectile problem does not help here.
(c) Find the steady-state solution, that is, the solution of the differential equations with $u' = 0$ and $q' = 0$.
(d) Nondimensionalize the problem using the steady-state solution from (c) to scale u and q. Make sure to explain how you selected the scaling for t.

1.18. The equations that account for the relativistic motion of a planet around the sun are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{Gm}{r^2} + \frac{b}{r^3},$$
$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0,$$

where b is a constant. Assume the initial conditions are $r = r_0$, $r' = 0$, and $\theta = 0$ at $t = 0$.
(a) What are the dimensions of r_0, b?
(b) Nondimensionalize the problem. The scaling should be chosen so the only nondimensional group appearing in the problem involves b.

1.19. Suppose you are given a dimensionless function $f(\Pi)$ where Π is a dimensionless group. Also, suppose $\Pi = A^a B^b C^c$ where A, B, C are dimensional parameters and the exponents a, b, c are nonzero numbers.

(a) Show that if $f(\Pi)$ is found to be linear in A then it must be that $f(\Pi) = \alpha\Pi^{1/a} + \beta$ where α, β are arbitrary numbers.
(b) What can you conclude if it is found that $\sqrt{AB}f(\Pi)$ is linear in A?
(c) Suppose it is found that if A is doubled that the value of F increases by a factor of four. Can this be used to determine F?

1.20. This problem explores some consequences of dimensional quantities.
(a) If g is the gravitational acceleration constant, explain why $\sin(g)$ and e^g make no sense.
(b) Explain why density, volume, and velocity can be used in place of length, mass, and time as fundamental units.
(c) Explain why volume, velocity, and acceleration cannot be used in place of length, mass, and time as fundamental units.

1.21. In quantum chromodynamics three parameters that play a central role are the speed of light c, Planck's constant $\hbar$, and the gravitational constant G.
(a) Explain why it is possible to use $[c], [\hbar], [G]$ as fundamental units.
(b) The distance ℓ_p at which the strong, electromagnetic and weak forces become equal depends on $c, \hbar, G$. Find a dimensionally reduced form for how ℓ_p depends on these three parameters. Based on this result, if the speed of light were to double what happens to ℓ_p?
(c) The Bohr radius a of an electron depends on $\hbar$, the electron's charge e, and the mass m_e of the electron. Find a dimensionally reduced form for a.

1.22. The speed c_m at which magnetonsonic waves travel through a plasma depends on the intensity B of the magnetic field, the permeability μ_0 of free space, and the density ρ and pressure p of the plasma.
(a) Use dimensional reduction to determine the functional dependence of c_m on B, μ_0, ρ, and p.
(b) From the basic laws for plasmas it is shown that

$$c_m = \sqrt{V_A^2 + c_s^2}\,,$$

where $V_A = B/\sqrt{\mu_0\rho}$ is the Alfven speed and $c_s = \sqrt{\gamma p/\rho}$ is the sound speed in the gas. In the latter expression, γ is a number. How does this differ from your result in (a)?

1.23. In the study of the motion of particles moving along the x-axis one comes across the problem of finding the velocity u that satisfies the nonlinear partial differential equation

$$u_t + uu_x = 0, \tag{1.96}$$

where

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ u_0 & \text{if } 0 < x. \end{cases} \tag{1.97}$$

Assume that u_0 is a positive constant. The equation (1.96) is derived in Chapter 5, and it is known as the inviscid Burgers' equation. It, along with the jump condition in (1.97), form what is known as a Riemann problem.

(a) What three physical quantities does u depend on?
(b) Use dimensional reduction, and a similarity variable, to reduce this problem to a nonlinear ordinary differential equation with two boundary conditions.
(c) Use the result from part (b) to solve the Riemann problem. The solution, which is known as an expansion fan, must be continuous for $t > 0$.
(d) What is the solution if the initial condition (1.97) is replaced with $u(x, 0) = u_0$?
(e) Suppose that, rather than velocity, the variable u is displacement. Explain why it is not possible for u to satisfy (1.96).

1.24. Consider the partial differential equation

$$u_t + Du_{xxxx} = 0,$$

where $u = u_0$ at $x = 0$, $u \to 0$ as $x \to \infty$, and $u = 0$ at $t = 0$. Use dimensional reduction, and a similarity variable, to reduce this problem to an ordinary differential equation.

1.25. The equation of the concentration c, on an interval of length ℓ, is

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + \mu c,$$

where the boundary conditions are $c(x, 0) = 0$, $c(0, t) = c_0$, and $c(\ell, t) = 0$.

(a) What are the dimensions of D, c_0, and μ?
(b) Nondimensionalize the problem so it has the form

$$\frac{\partial u}{\partial s} = \frac{\partial^2 u}{\partial y^2} + \alpha u,$$

where the boundary conditions are $u(y, 0) = 0$, $u(0, s) = 1$, and $u(1, s) = 0$.

1.26. One of the standard experimental tests used in the study of fluid motion through porous materials consists of determining the displacement u when the material is given a constant load. The governing differential equation in this case is

$$H\left[1 + \left(\frac{\partial u}{\partial x}\right)^3\right]\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

The boundary conditions are

$$\frac{\partial u}{\partial x} = -1, \quad \text{at} \ \ x = 0,$$

and

$$u = 0, \quad \text{as} \ \ x \to \infty.$$

The initial condition is

$$u = 0, \quad \text{at} \ \ t = 0.$$

(a) What are the dimensions of the constant H?
(b) Find a dimensionally reduced form for the solution and then use this to transform the above diffusion problem into one involving a nonlinear ordinary differential equation. Make sure to state what happens to the boundary and initial conditions. You do not need to solve this problem.
(c) In the experiment the surface displacement $u(0,t)$ is measured. Without solving the problem use your results from (b) to sketch $u(0,t)$ as a function of t.
(d) Suppose the experimental data show that $u(0,t) = 16t$ cm/sec. Using your result from part (c), explain why the mathematical model is incorrect. Also, explain why changing the differential equation to either $Hu_{xx} = u_t$ or to $H[1 + (u_x)^5]u_{xx} = u_t$ will also produce an incorrect model.

1.27. Consider the problem of solving the diffusion equation

$$D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

where the boundary conditions are

$$u = 0, \quad \text{as} \ \ x \to \pm\infty.$$

Instead of an initial condition, assume the solution satisfies

$$\int_{-\infty}^{\infty} u dx = \gamma, \ \ \forall t > 0.$$

(a) What are the dimensions of γ?
(b) Find a dimensionally reduced form for the solution and then use this to transform the above diffusion equation into an ordinary differential equation. How do the boundary conditions transform? The integral condition should be considered in the dimensional reduction but its conversion using the similarity variable will wait until part (d).
(c) Find the solution of the problem from part (b). You can assume $F' \to 0$ and $\eta F \to 0$ as $\eta \to \pm\infty$. As a hint, you might want to look for the expression $(\eta F)'$ in your equation.
(d) The solution from part (c) should contain an arbitrary constant. Find its value using the given integral condition and with this show that

$$u = \frac{\gamma}{\sqrt{\pi D t}} e^{-x^2/(4Dt)}.$$

This is known as the fundamental, or point source, solution of the diffusion equation.

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