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Abstract Parabolic Evolution Equations and their Applications

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Chapter 12
Chemotaxis Models

Budrene–Berg discovered [BB91, BB95] a remarkable aggregation pattern generated by chemotactic bacteria. In 1995, Woodward–Tyson–Myerscough–Murray–Budrene–Berg presented the mathematical model

$$\begin{cases} \frac{\partial u}{\partial t} = a\Delta u - \mu\nabla\cdot\left[\frac{u}{(1+\alpha\rho)^2}\nabla\rho\right] + cu\left(1-\frac{u}{s}\right) & \text{in } \Omega\times(0,\infty), \\ \frac{\partial\rho}{\partial t} = b\Delta\rho + \frac{su^p}{1+\beta u^q} - \frac{u^r\rho}{1+\gamma\rho} & \text{in } \Omega\times(0,\infty), \end{cases} \tag{12.1}$$

for describing the process of pattern formation. Here, Ω is a two-dimensional domain where bacteria are bred. The function $u(x,t)$ denotes the population density of bacteria in Ω at time t. The function $\rho(x,t)$ denotes the concentration of chemical attractant in Ω at time t. The movement of biological individuals consists of two effects: one is a random walk, and the other is a directed movement in a sense that they have a tendency to move toward a higher concentration direction of the chemical substance. The latter directed movement is called chemotaxis.

In 1970, Keller–Segel introduced a mathematical formulation for chemotaxis by the diffusion–advection equation

$$\frac{\partial u}{\partial t} = a\Delta u - \mu\nabla\cdot[u\nabla\chi(\rho)],$$

where $\chi(\rho)$ is a sensitivity function of bacteria toward the chemoattractant, and $\mu > 0$ is an attraction rate. Several forms are proposed as sensitivity functions. The typical ones are such that $\chi(\rho) = \rho,\ \rho^2,\ \log(\rho+1),\ \log\rho,\ \frac{\rho}{\rho+1}$, etc. In (12.1), the authors took the function $\chi(\rho) = \frac{\rho}{1+\alpha\rho}$.

In (12.1), the term $cu(1-\frac{u}{s})$ denotes the proliferation of bacteria by the logistic law, where s denotes the concentration of the substrate. In this model, s is assumed to be spatially homogeneous and constant. The term $\frac{su^p}{1+\beta u^q}$ denotes the production of chemoattractant by bacteria, where p, q are some nonnegative exponents and $\beta \geq 0$ is a constant. The term $-\frac{u^r\rho}{1+\gamma\rho}$ denotes the consumption of chemoattractant by bacteria, where r is some nonnegative exponent, and $\gamma \geq 0$ is a constant. The bacteria and the chemical substance diffuse with rates $a > 0$ and $b > 0$, respectively.

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We will consider in this chapter the following simplified model:

$$\begin{cases} \frac{\partial u}{\partial t} = a\Delta u - \mu\nabla\cdot[u\nabla\chi(\rho)] + cu - \gamma u^2 & \text{in } \Omega\times(0,\infty), \\ \frac{\partial \rho}{\partial t} = b\Delta\rho - d\rho + \nu u & \text{in } \Omega\times(0,\infty). \end{cases} \tag{12.2}$$

The sensitivity function will not be specified but will be assumed to satisfy condition (12.4). Actually, $\chi(\rho)=\rho$, $\log(\rho+1)$, and $\frac{\rho}{\rho+1}$ are allowed. For the production term, we take $p=1$ and $\beta=0$. We ignore the consumption of chemical substance by bacteria, i.e., $r=0$; instead we consider the unconstrained resolution which is denoted by $-d\rho$ with rate $d>0$.

First, we construct a dynamical system for the initial-boundary-value problem of (12.2) imposing the homogeneous Neumann boundary conditions for u and ρ and show that the dynamical system possesses exponential attractors. Secondly, we investigate the stability and instability of the homogeneous stationary solution $(\overline{u},\overline{\rho})=\left(\frac{c}{\gamma},\frac{c\nu}{d\gamma}\right)$. In fact, we show that, when the product $\mu\nu$ is small, $(\overline{u},\overline{\rho})$ is stable and, on the other hand, that, when $\mu\nu$ is large enough, $(\overline{u},\overline{\rho})$ is no longer stable. Moreover, we show that the degree of instability tends to infinity as $\mu\nu\to\infty$ and consequently the fractal dimensions of exponential attractors also tend to infinity.

1 Chemotaxis Model Without Proliferation

Before studying (12.2), let us consider a chemotaxis model without proliferation. We are concerned with the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} = a\Delta u - \mu\nabla\cdot[u\nabla\chi(\rho)] & \text{in } \Omega\times(0,\infty), \\ \frac{\partial \rho}{\partial t} = b\Delta\rho - d\rho + \nu u & \text{in } \Omega\times(0,\infty), \\ \frac{\partial u}{\partial n} = \frac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x), \quad \rho(x,0)=\rho_0(x) & \text{in } \Omega. \end{cases} \tag{12.3}$$

As mentioned above, such a chemotaxis model was founded by Keller–Segel [KS70]. The form of the equations of (12.3) is of some particular case of the general ones presented in the paper.

Here, Ω is a two-dimensional $\mathcal{C}^2$ or convex, bounded domain. We assume that the sensitivity function $\chi(\rho)$ is a real smooth function for $0\le\rho<\infty$ satisfying the condition

$$\sup_{0\le\rho<\infty}\left|\frac{d^i\chi}{d\rho^i}(\rho)\right|<\infty \quad \text{for } i=1,2. \tag{12.4}$$

We assume also that a, b, d, μ, and ν are positive constants (>0).

First, we construct local solutions for problem (12.3). Secondly, in the case where $\chi(\rho)$ is linear ($=\rho$), we establish a priori estimates of local solutions for the initial functions $u_0(x)\ge 0$ and $\rho_0(x)\ge 0$ with sufficiently small norm $\|u_0\|_{L_1}$. Consequently, for such initial functions, we can show the global existence of solutions.

However, we must notice the fact due to Herrero–Velázquez [HV96] that the local solutions to (12.3) blow up in finite time, provided that the norms $\|u_0\|_{L_1}$ of their initial functions are large.

1.1 Abstract Formulation

Let us formulate problem (12.3) as the Cauchy problem for an abstract evolution equation. It will be spontaneous to try to handle the first equation of (12.3) on u in the space $L_2(\Omega)$. Then, we are naturally led to handle the second one in the Sobolev space $H^2(\Omega)$. As will be observed in establishing a priori estimates for (12.3) (see Propositions 12.2 and 12.3), we have to couple the norms $\|u(t)\|_{L_2}^2$ and $\|\rho(t)\|_{H^2}^2$. This means that it is natural to handle (12.3) in the product space of $u \in L_2(\Omega)$ and $\rho \in H^2(\Omega)$. By this reason, our underlying space is taken as

$$X = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in L_2(\Omega) \text{ and } \rho \in H_N^2(\Omega) \right\}. \tag{12.5}$$

Since ρ satisfies the homogeneous Neumann boundary conditions, the space for ρ is taken as $H_N^2(\Omega)$.

In the underlying space X, the chemotactic term $\nabla \cdot [u \nabla \chi(\rho)]$ can be treated as a lower term. In other words, we are able to formulate problem (12.3) as a semilinear problem of the form (4.1), although the first diffusion equation of (12.3) on u is quasilinear in form. This is the merit of taking an underlying space of the form (12.5). In this way, we rewrite (12.3) by

$$\begin{cases} \frac{dU}{dt} + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0, \end{cases} \tag{12.6}$$

in X, where $U = {}^t(u, \rho)$. Here, the linear operator A is given by

$$A = \begin{pmatrix} A_1 & 0 \\ -\nu & A_2 \end{pmatrix}. \tag{12.7}$$

The two operators $A_1 = -a\Delta + 1$ and $A_2 = -b\Delta + d$ are positive definite self-adjoint operators of $L_2(\Omega)$ with domains $\mathcal{D}(A_1) = \mathcal{D}(A_2) = H_N^2(\Omega)$. However, A_2 is regarded as an operator from $\mathcal{H}_{N^2}^4(\Omega) = \mathcal{D}(A_2^2) = \{\rho \in H_N^2(\Omega);\ \Delta\rho \in H_N^2(\Omega)\}$ into $\mathcal{D}(A_2) = H_N^2(\Omega)$. (Note that, as Ω is not assumed to be of $\mathcal{C}^4$ class, we cannot expect that $\mathcal{H}_{N^2}^4(\Omega) \subset H^4(\Omega)$.) So, the domain of A is given by

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in H_N^2(\Omega) \text{ and } \rho \in \mathcal{H}_{N^2}^4(\Omega) \right\}.$$

The nonlinear operator F is given by

$$F(U) = \begin{pmatrix} -\mu\nabla \cdot [u \nabla \chi(\operatorname{Re}\rho)] + u \\ 0 \end{pmatrix}. \tag{12.8}$$

Since $\nabla \cdot [u \nabla \chi] = \nabla u \cdot \nabla \chi + u \Delta \chi$, it is possible to see that

$$\|\nabla \cdot [u \nabla \chi]\|_{L_2} \leq C \|u\|_{H^{1+\varepsilon}} \|\chi\|_{H^2}, \quad u \in H^{1+\varepsilon}(\Omega),\ \chi \in H^2_N(\Omega) \tag{12.9}$$

with an arbitrary fixed exponent $0 < \varepsilon < \frac{1}{2}$. Indeed, note that $H^{1+\varepsilon}(\Omega) \subset L_\infty(\Omega)$ due to (1.76) and $H^{1+\varepsilon}(\Omega) \subset H^1_p(\Omega)$ for p such that $2 < p < 2/(1-\varepsilon)$ due to (1.74). We then observe that $F(U)$ is an operator from $\mathcal{D}(A^\eta)$ into X, where the exponent η is given by $\eta = \frac{1+\varepsilon}{2}$. Note also by (1.93) that $\rho \mapsto \chi(\mathrm{Re}\,\rho)$ is a bounded operator from $H^2_N(\Omega)$ into itself.

1.2 Construction of Local Solutions

In order to show the existence of local solutions to (12.6), let us verify that A and F fulfill the structural assumptions made in Chap. 4, Sect. 1.

Denote by B the multiplicative operator by the number $-\nu$. Then, A is obviously written by

$$A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}. \tag{12.10}$$

We regard B as a linear operator from $\mathcal{D}(B) \subset L_2(\Omega)$ to $H^2_N(\Omega)$; clearly, $\mathcal{D}(B) = H^2_N(\Omega)$, and B is a bounded operator from $\mathcal{D}(A_1)$ onto $H^2_N(\Omega)$. Hence, it is an immediate consequence of Theorem 2.16 that the operator A is a sectorial operator of the product space X. Let us next investigate the domains of fractional powers A^θ for $0 \leq \theta \leq 1$. According to Theorems 16.7 and 16.7 in Chap. 16, the domains $\mathcal{D}(A_1^\theta)$ and $\mathcal{D}(A_2^\theta)$ are characterized by

$$\mathcal{D}(A_1^\theta) = \mathcal{D}(A_2^\theta) = H^{2\theta}(\Omega), \quad 0 \leq \theta < \frac{3}{4},$$
$$\mathcal{D}(A_1^\theta) = \mathcal{D}(A_2^\theta) = H^{2\theta}_N(\Omega), \quad \frac{3}{4} < \theta \leq 1.$$

Consequently, we have

$$\mathcal{D}(A_2^\theta) = \mathcal{H}^{2\theta}_N(\Omega) = \{\rho \in H^2_N(\Omega);\ \Delta\rho \in H^{2(\theta-1)}(\Omega)\}, \quad 1 \leq \theta < \frac{7}{4},$$
$$\mathcal{D}(A_2^\theta) = \mathcal{H}^{2\theta}_{N^2}(\Omega) = \{\rho \in H^2_N(\Omega);\ \Delta\rho \in H^{2(\theta-1)}_N(\Omega)\} \quad \frac{7}{4} < \theta \leq 2.$$

Consider a diagonal operator $A_D = \mathrm{diag}\{A_1, A_2\}$ acting in X. Then, A_D is a self-adjoint operator of X; therefore, $A_D^\theta = \mathrm{diag}\{A_1^\theta, A_2^\theta\}$. More precisely,

$$\mathcal{D}(A_D^\theta) = \left\{\begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in H^{2\theta}(\Omega) \text{ and } \rho \in \mathcal{H}^{2(\theta+1)}_N(\Omega)\right\}, \quad 0 < \theta < \frac{3}{4},$$
$$\mathcal{D}(A_D^\theta) = \left\{\begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in H^{2\theta}_N(\Omega) \text{ and } \rho \in \mathcal{H}^{2(\theta+1)}_{N^2}(\Omega)\right\}, \quad \frac{3}{4} < \theta \leq 1.$$

We then verify the following property.

Proposition 12.1 *The domains of fractional powers of A are given by*

$$\mathcal{D}(A^\theta) = \mathcal{D}\big(A_D^\theta\big), \quad 0 \le \theta \le 1, \tag{12.11}$$

with the norm equivalence

$$C^{-1}\big\|A_D^\theta\cdot\big\|_X \le \|A^\theta\cdot\|_X \le C\big\|A_D^\theta\cdot\big\|_X, \quad 0 \le \theta \le 1.$$

Proof We will apply Theorem 16.4 to A. It is clear that the analogous statement to Theorem 16.4 is true for matrix operators of the form (12.10).

The multiplicative operator B is clearly an operator from $L_2(\Omega)$ to $H_N^2(\Omega) = \mathcal{D}(A_2)$ with the domain $\mathcal{D}(B) = H_N^2(\Omega)$. Therefore, B is a bounded operator from $\mathcal{D}(A_1)$ into $H_N^2(\Omega)$. In the meantime, $u \in H_N^2(\Omega)$ belongs to $\mathcal{D}(B^*)$ if and only if the functional $(u, B\cdot)_{H_N^2}$ is continuous with respect to the L_2 topology. Since $(u, Bv)_{H_N^2} = (A_2u, -A_2\nu v)_{L_2} = -\nu(A_2u, A_2v)_{L_2}$, $u \in \mathcal{D}(A_2^2)$ implies $u \in \mathcal{D}(B^*)$. This indeed means that $\mathcal{D}(A_2^2) \subset \mathcal{D}(B^*)$ and B^* is a bounded operator from $\mathcal{D}(A_2^2)$ into $L_2(\Omega)$. Hence, we have verified that Theorem 16.4 is available to A.

We now know that $\mathcal{D}(A^\theta) = [\mathcal{D}(A), X]_\theta$. From this we deduce that $\mathcal{D}(A^\theta) = [\mathcal{D}(A), X]_\theta = [\mathcal{D}(A_D), X]_\theta = \mathcal{D}(A_D^\theta)$ because of the coincidence $\mathcal{D}(A) = \mathcal{D}(A_D)$. The norm equivalence is also verified by Theorem 16.4. □

Note that $\rho \mapsto \chi(\mathrm{Re}\,\rho)$ is a continuous and bounded operator from $H_N^2(\Omega)$ into itself due to (1.93). Furthermore, by (12.9) and (12.11) ($\theta = \eta$), we observe that F satisfies the Lipschitz condition (4.21). We therefore know that Theorem 4.4 is available to problem (12.6).

As (4.21) is valid in the present case, we are accordingly led to set the space of initial values by

$$\mathcal{K} = \left\{\begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix};\ 0 \le u_0 \in L_2(\Omega) \text{ and } 0 \le \rho_0 \in H_N^2(\Omega)\right\}. \tag{12.12}$$

We have thus proved that, for each $U_0 \in \mathcal{K}$, (12.6) (and hence (12.3)) possesses a unique local solution $U = {}^t(u, \rho)$ in the function space:

$$\begin{cases} u \in \mathcal{C}((0, T_{U_0}]; H_N^2(\Omega)) \cap \mathcal{C}([0, T_{U_0}]; L_2(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; L_2(\Omega)), \\ \rho \in \mathcal{C}((0, T_{U_0}]; \mathcal{H}_{N^2}^4(\Omega)) \cap \mathcal{C}([0, T_{U_0}]; H_N^2(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; H_N^2(\Omega)). \end{cases} \tag{12.13}$$

In addition, it holds for U that

$$t\left\|\frac{dU}{dt}(t)\right\|_X + t\|AU(t)\|_X + \|U(t)\|_X \le C_{U_0}, \quad 0 < t \le T_{U_0}. \tag{12.14}$$

Here, the positive time $T_{U_0} > 0$ and the constant $C_{U_0} > 0$ are determined by the norm $\|U_0\|_X$ only.

1.3 Nonnegativity of Local Solutions

For $U_0 \in \mathcal{K}$, let $U = {}^t(u, \rho)$ be the local solution of (12.6) constructed above. Our goal is to prove that $u(t) \geq 0$ and $\rho(t) \geq 0$ for every $0 < t \leq T_{U_0}$ by the truncation method.

Let us first verify that $U(t)$ is real valued. Indeed, the complex conjugate $\overline{U(t)}$ of $U(t)$ is also a local solution of (12.6) with the same initial value. Therefore, the uniqueness of solution implies that $\overline{U(t)} = U(t)$; hence, $U(t)$ is real valued.

Let $H(u)$ be a $\mathcal{C}^{1,1}$ cutoff function such that $H(u) = \frac{1}{2}u^2$ for $-\infty < u < 0$ and $H(u) \equiv 0$ for $0 \leq u < \infty$. According to (1.100), the function $\psi(t) = \int_\Omega H(u(t))\,dx$ is continuously differentiable with the derivative

$$\psi'(t) = (H'(u), a\Delta u - \mu\nabla \cdot [u\nabla\chi(\rho)])_{L_2}.$$

Applying (1.96) to $H'(u)$, we observe that

$$(H'(u), a\Delta u)_{L_2} = -a\int_\Omega \nabla H'(u) \cdot \nabla u\,dx = -a\int_\Omega |\nabla H'(u)|^2\,dx.$$

In the meantime, by (1.96) again,

$$\begin{aligned}
-(H'(u), \mu\nabla \cdot [u\nabla\chi(\rho)])_{L_2} &= \mu\int_\Omega u\nabla H'(u) \cdot \nabla\chi(\rho)\,dx \\
&= \mu\int_\Omega H'(u)\nabla H'(u) \cdot \nabla\chi(\rho)\,dx \\
&= -\frac{\mu}{2}\int_\Omega H'(u)^2\Delta\chi(\rho)\,dx.
\end{aligned}$$

Since

$$\Delta\chi(\rho) = \chi'(\rho)\Delta\rho + \chi''(\rho)|\nabla\rho|^2, \tag{12.15}$$

(12.13) shows that $\|\Delta\chi(\rho)\|_{L_2} \leq C_U$ for $0 \leq t \leq T_{U_0}$. Therefore,

$$\begin{aligned}
-(H'(u), \mu\nabla \cdot [u\nabla\chi(\rho)])_{L_2} &\leq \frac{\mu}{2}\|H'(u)\|_{L_4}^2\|\Delta\chi(\rho)\|_{L_2} \\
&\leq C_U\|H'(u)\|_{H^1}\|H'(u)\|_{L_2} \\
&\leq \frac{a}{2}\|H'(u)\|_{H^1}^2 + C_U\|H'(u)\|_{L_2}^2.
\end{aligned}$$

Hence, $\psi'(t) \leq C_U\psi(t)$; consequently, $\psi(t) \leq \psi(0)e^{C_U t}$. Then, $\psi(0) = 0$ implies $\psi(t) \equiv 0$, i.e., $u(t) \geq 0$ for $0 < t \leq T_{U_0}$. It is similar for the proof of $\rho(t) \geq 0$ for $0 < t \leq T_{U_0}$.

2 Case where $\chi(\rho) = \rho$

In this section, let us consider problem (12.6) in a particular case where the sensitivity function is given by

$$\chi(\rho) = \rho, \quad 0 \le \rho < \infty. \tag{12.16}$$

We can establish a priori estimates for sufficiently small initial functions u_0, and consequently we obtain the global existence of solutions.

2.1 Global Solutions for Small Initial Functions

Let $U_0 \in \mathcal{K}$, and let U be any local solution of (12.6) on $[0, T_U]$ in the function space:

$$\begin{cases} 0 \le u \in \mathcal{C}((0, T_U]; H_N^2(\Omega)) \cap \mathcal{C}([0, T_U]; L_2(\Omega)) \cap \mathcal{C}^1((0, T_U]; L_2(\Omega)), \\ 0 \le \rho \in \mathcal{C}((0, T_U]; \mathcal{H}_{N^2}^4(\Omega)) \cap \mathcal{C}([0, T_U]; H_N^2(\Omega)) \cap \mathcal{C}^1((0, T_U]; H_N^2(\Omega)). \end{cases} \tag{12.17}$$

Proposition 12.2 *There exists a positive number $r > 0$ and a continuous increasing function $p(\cdot)$ such that the estimate*

$$\|u(t)\|_{L_2} + \|\rho(t)\|_{H^2} \le p(\|u_0\|_{L_2} + \|\rho_0\|_{H^2}), \quad 0 \le t \le T_U, \tag{12.18}$$

holds for any local solution $U = {}^t(u, \rho)$ of (12.6) in (12.17) with initial value $U_0 \in \mathcal{K}$ satisfying the condition $\|u_0\|_{L_1} \le r$.

Proof In the proof, the notation $p(\cdot)$ stands for some continuous increasing functions which are determined only by Ω and by the initial constants in the equations of (12.3). Similarly, the notation C stands for some constants, which are determined in the same way as $p(\cdot)$.

Step 1. Integrate the first equation of (12.6) in Ω. Then, obviously,

$$\frac{d}{dt} \int_\Omega u \, dx = 0.$$

Since $u(t) \ge 0$, it follows that

$$\|u(t)\|_{L_1} = \|u_0\|_{L_1}, \quad 0 \le t \le T_U. \tag{12.19}$$

Step 2. We here introduce the quantity

$$N_{1,\log}(u) = \int_\Omega u \log(u + 1) \, dx, \quad 0 \le u \in L_2(\Omega).$$

The purpose of this step is to estimate $N_{1,\log}(u(t))$ for the local solution U.

Multiply the first equation of (12.6) by $\log(u+1)$ and integrate the product in Ω. Then, since

$$\int_\Omega \log(u+1)\Delta u\,dx = -\int_\Omega \frac{1}{u+1}|\nabla(u+1)|^2\,dx = -4\int_\Omega |\nabla\sqrt{u+1}|^2 dx$$

and

$$\begin{aligned}\int_\Omega \log(u+1)\nabla\cdot[u\nabla\rho]\,dx &= \int_\Omega \nabla[(u+1)\log(u+1)-u]\cdot\nabla\rho\,dx \\ +\int_\Omega u\log(u+1)\Delta\rho\,dx &= \int_\Omega [u-\log(u+1)]\Delta\rho\,dx,\end{aligned}$$

it follows that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega [(u+1)\log(u+1)-u]\,dx + 4a\int_\Omega |\nabla\sqrt{u+1}|^2\,dx \\ &= \mu\int_\Omega [\log(u+1)-u]\Delta\rho\,dx \le \zeta\|\Delta\rho\|_{L_2}^2 + C_\zeta\|u\|_{L_2}^2 \qquad (12.20)\end{aligned}$$

with any $\zeta > 0$.

In the meantime, from the second equation of (12.6) we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2\,dx + b\int_\Omega |\nabla\rho|^2\,dx + d\int_\Omega \rho^2\,dx = \nu\int_\Omega u\rho\,dx \le \frac{d}{2}\|\rho\|_{L_2}^2 + C\|u\|_{L_2}^2$$

and

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\rho|^2\,dx + b\int_\Omega |\Delta\rho|^2\,dx + d\int_\Omega |\nabla\rho|^2\,dx \\ &= \nu\int_\Omega u\Delta\rho\,dx \le \frac{b}{4}\|\Delta\rho\|_{L_2}^2 + C\|u\|_{L_2}^2.\end{aligned}$$

Fixing the parameter ζ as $\zeta = \frac{b}{4}$, we sum up these three differential inequalities to obtain that

$$\frac{d}{dt}\psi(t) + d\psi(t) + 4a\int_\Omega |\nabla\sqrt{u+1}|^2\,dx + b\int_\Omega |\Delta\rho|^2\,dx \le C\big(\|u\|_{L_2}^2 + 1\big),$$

where $\psi(t) = \|(u+1)\log(u+1)-u\|_{L_1} + \|\rho\|_{H^1}^2$. We notice the fact that

$$\begin{aligned}\|u\|_{L_2} &\le \|u\|_{L_1}^{1/3}\|u\|_{L_4}^{2/3} \le \|u\|_{L_1}^{1/3}\|\sqrt{u+1}\|_{L_8}^{4/3} \\ &\le C\|u\|_{L_1}^{1/3}\|\sqrt{u+1}\|_{L^2}^{1/3}(\|\nabla\sqrt{u+1}\|_{L_2} + \|\sqrt{u+1}\|_{L_2}), \quad 0 \le u \in H^1(\Omega).\end{aligned}$$

Hence, on account of (12.19),

$$\frac{d}{dt}\psi(t)+d\psi(t)+4a\int_\Omega|\nabla\sqrt{u+1}|^2\,dx+b\int_\Omega|\Delta\rho|^2\,dx$$
$$\le \|u_0\|_{L_1}^{\frac{2}{3}}p(\|u_0\|_{L_1})\big(\|\nabla\sqrt{u+1}\|_{L^2}^2+1\big).$$

We now fix a radius $r>0$ small enough to satisfy that $r^{\frac{2}{3}}p(r)\le 3a$. Then, if $\|u_0\|_{L_1}\le r$, it holds that

$$\frac{d}{dt}\psi(t)+d\psi(t)+\int_\Omega\big(a|\nabla\sqrt{u+1}|^2+b|\Delta\rho|^2\big)\,dx\le C.$$

As a consequence (cf. (1.58)),

$$\psi(t)\le e^{-dt}\psi(0)+C,\quad 0\le t\le T_U.$$

In particular,

$$N_{1,\log}(u(t))+\|\rho(t)\|_{H^1}^2\le C\big\{e^{-dt}\big[N_{1,\log}(u_0)+\|\rho_0\|_{H^1}^2\big]+1\big\},\quad 0\le t\le T_U. \tag{12.21}$$

Proposition 1.5 is also available to conclude that

$$\int_s^t\big[\|\sqrt{u(\tau)+1}\|_{H^1}^2+\|\Delta\rho(\tau)\|_{L_2}^2\big]\,d\tau$$
$$\le C\big[(t-s)+N_{1,\log}(u_0)+\|\rho_0\|_{H^1}^2+1\big],\quad 0\le s<t\le T_U.$$

Step 3. The present step is devoted to showing the estimate

$$\|u(t)\|_{L_2}^2+\|\rho(t)\|_{H^2}^2\le p(N_{1,\log}(u_0)+\|\rho_0\|_{H^1})$$
$$\times\big[e^{-\delta t}\big(\|u_0\|_{L_2}^2+\|\rho_0\|_{H^2}^2\big)+1\big],\quad 0\le t\le T_U,$$

with some exponent $\delta>0$. Now, when estimate (12.21) is established, the techniques of its proof are quite analogous to that of the proof (Step 4) of Proposition 12.3 below, which establishes a similar a priori estimate for a general sensitivity function including the case of (12.16). So, the continued proof will be given there. □

It is now possible to apply Corollary 4.3 to conclude the global existence of solutions. In fact, there is a number $r>0$ such that, for any $U_0\in\mathcal{K}$ satisfying $\|u_0\|_{L_1}\le r$, (12.3) possesses a unique global solution in the function space:

$$\begin{cases}0\le u\in\mathcal{C}((0,\infty);H_N^2(\Omega))\cap\mathcal{C}([0,\infty);L_2(\Omega))\cap\mathcal{C}^1((0,\infty);L_2(\Omega)),\\0\le\rho\in\mathcal{C}((0,\infty);\mathcal{H}_{N^2}^4(\Omega))\cap\mathcal{C}([0,\infty);H_N^2(\Omega))\cap\mathcal{C}^1((0,\infty);H_N^2(\Omega)).\end{cases} \tag{12.22}$$

2.2 Lyapunov Function

When (12.16) holds, we can construct a Lyapunov function. Let $U = {}^t(u, \rho)$ denote a local or global solution of (12.6) in the function space (12.17). Let $\varepsilon > 0$ be a positive parameter and put $u_\varepsilon = u + \varepsilon$. It is clear that $\frac{\partial u}{\partial t} = \frac{\partial u_\varepsilon}{\partial t}$ and that

$$a\Delta u - \mu\nabla \cdot [u\nabla\rho] = a\Delta u_\varepsilon - \mu\nabla \cdot [u_\varepsilon\nabla\rho] + \varepsilon\mu\Delta\rho.$$

Therefore, the first equation of (12.6) is written as

$$\frac{\partial u_\varepsilon}{\partial t} = \nabla \cdot [u_\varepsilon\nabla(a\log u_\varepsilon - \mu\rho)] + \varepsilon\mu\Delta\rho.$$

Multiply this equation by $(a\log u_\varepsilon - \mu\rho)$ and integrate the product in Ω. Then,

$$a\int_\Omega \frac{\partial u_\varepsilon}{\partial t}\log u_\varepsilon\,dx - \mu\int_\Omega \frac{\partial u}{\partial t}\rho\,dx = -\int_\Omega u_\varepsilon|\nabla(a\log u_\varepsilon - \mu\rho)|^2\,dx + R_\varepsilon,$$

where $R_\varepsilon = \varepsilon\mu\int_\Omega(a\log u_\varepsilon - \mu\rho)\Delta\rho\,dx$. In the meantime, multiplying the equation of ρ by $\frac{\partial\rho}{\partial t}$ and integrating the product in Ω, we have

$$\int_\Omega\left(\frac{\partial\rho}{\partial t}\right)^2 dx = -\frac{b}{2}\frac{d}{dt}\int_\Omega|\nabla\rho|^2\,dx - \frac{d}{2}\frac{d}{dt}\int_\Omega\rho^2\,dx + \nu\int_\Omega u\frac{\partial\rho}{\partial t}\,dx.$$

Therefore,

$$\frac{d}{dt}\int_\Omega\left[a\nu(u_\varepsilon\log u_\varepsilon - u_\varepsilon) + \frac{b\mu}{2}|\nabla\rho|^2 + \frac{d\mu}{2}\rho^2 - \mu\nu u\rho\right]dx$$
$$= -\int_\Omega\left[\nu u_\varepsilon|\nabla(a\log u_\varepsilon - \mu\rho)|^2 + \mu\left(\frac{\partial\rho}{\partial t}\right)^2\right]dx + \nu R_\varepsilon \le \nu R_\varepsilon.$$

Let $0 < s < t \le T_U$. Integrating this inequality in $[s, t]$, we obtain that

$$\left[\int_\Omega\left[a\nu(u_\varepsilon(\tau)\log u_\varepsilon(\tau) - u_\varepsilon(\tau)) + \frac{b\mu}{2}|\nabla\rho(\tau)|^2 + \frac{d\mu}{2}\rho(\tau)^2\right.\right.$$
$$\left.\left. - \mu\nu u(\tau)\rho(\tau)\right]dx\right]_{\tau=s}^{\tau=t} \le \int_s^t \nu R_\varepsilon(\tau)\,d\tau.$$

It is not difficult to verify that $\lim_{\varepsilon\to 0}\int_s^t R_\varepsilon(\tau)\,d\tau = 0$. Hence, if we set

$$\Phi(U) = \int_\Omega\left[a\nu(u\log u - u) + \frac{b\mu}{2}|\nabla\rho|^2 + \frac{d\mu}{2}\rho^2 - \mu\nu u\rho\right]dx, \quad U \in X, \quad (12.23)$$

then $\Phi(U(t)) \le \Phi(U(s))$. This means that $\Phi(U)$ plays a role of the Lyapunov function for (12.3). Note that the function $u\log u$ can be extended for all $0 \le u < \infty$ as a continuous function.

2.3 Blowup of Solutions

Herrero–Velázquez [HV96, HV97] proved that, for suitable initial functions (at least, when $\|u_0\|_{L_1}$ are sufficiently large), the local solutions to (12.6) blow up in finite time.

Theorem 12.1 *There are initial values $U_0 \in \mathcal{K}$ for which the local solutions to (12.6) satisfy $\limsup_{t\to T} \|U(t)\|_X = \infty$ with finite time $T < \infty$. Hence, (12.6) with the initial values U_0 have no global solutions.*

3 Chemotaxis Model with Proliferation

Let us consider the initial-boundary-value problem for the chemotaxis model with proliferation

$$\begin{cases} \frac{\partial u}{\partial t} = a\Delta u - \mu\nabla\cdot[u\nabla\chi(\rho)] + cu - \gamma u^2 & \text{in } \Omega\times(0,\infty),\\ \frac{\partial \rho}{\partial t} = b\Delta\rho - d\rho + \nu u & \text{in } \Omega\times(0,\infty),\\ \frac{\partial u}{\partial n} = \frac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x) & \text{in } \Omega, \end{cases} \tag{12.24}$$

in a two-dimensional $\mathcal{C}^2$ or convex, bounded domain Ω.

We still assume that $\chi(\rho)$ satisfies (12.4) and assume that $\gamma > 0$ is a positive constant. As before, we handle problem (12.24) in the product space X given by (12.5), and, as a space of initial functions, we take the space $\mathcal{K}$ given by (12.12).

3.1 Local Solutions

In order to construct local solutions to (12.24), we can argue in a quite analogous way as for problem (12.3). In fact, we rewrite (12.24) as an abstract problem of the form (12.6) in X. The linear operator A is given in a similar way by (12.7). However, now, the nonlinear operator F is given (instead of (12.8)) by

$$F(U) = \begin{pmatrix} -\mu\nabla\cdot[u\nabla\chi(\operatorname{Re}\rho)] + (c+1)u - \gamma u^2 \\ 0 \end{pmatrix}. \tag{12.25}$$

Let $0 < \varepsilon < \frac{1}{2}$ be arbitrarily fixed and put $\eta = \frac{1+\varepsilon}{2}$. Then, as before (by (12.9) and (12.11)), F is an operator from $\mathcal{D}(A^\eta)$ into X. It is clear that

$$\|u^2\|_{L_2} \le C\|u\|_{L_2}\|u\|_{H^{1+\varepsilon}}, \quad u \in H^{1+\varepsilon}(\Omega).$$

Then, this, together with (12.9), implies that F fulfills the Lipschitz condition (4.21).

For each $U_0 \in \mathcal{K}$, we can therefore construct a unique local solution to (12.24). Furthermore, by the truncation method, we can verify the nonnegativity of the local solution. In this way, for any initial value $U_0 \in \mathcal{K}$, problem (12.24) possesses a unique local solution in the function space:

$$\begin{cases} 0 \le u \in \mathcal{C}((0, T_{U_0}]; H_N^2(\Omega)) \cap \mathcal{C}([0, T_{U_0}]; L_2(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; L_2(\Omega)), \\ 0 \le \rho \in \mathcal{C}((0, T_{U_0}]; \mathcal{H}_{N^2}^4(\Omega)) \cap \mathcal{C}([0, T_{U_0}]; H_N^2(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; H_N^2(\Omega)), \end{cases}$$

where the positive time $T_{U_0} > 0$ is determined by the norm $\|U_0\|_X$ only. In addition, estimate (12.14) holds for the present local solution, too.

3.2 A Priori Estimates of Local Solutions

For $U_0 \in \mathcal{K}$, let $U = {}^t(u, \rho)$ denote a local solution of (12.24) on $[0, T_U]$ in the function space:

$$\begin{cases} 0 \le u \in \mathcal{C}((0, T_U]; H_N^2(\Omega)) \cap \mathcal{C}([0, T_U]; L_2(\Omega)) \cap \mathcal{C}^1((0, T_U]; L_2(\Omega)), \\ 0 \le \rho \in \mathcal{C}((0, T_U]; \mathcal{H}_{N^2}^4(\Omega)) \cap \mathcal{C}([0, T_U]; H_N^2(\Omega)) \cap \mathcal{C}^1((0, T_U]; H_N^2(\Omega)). \end{cases} \tag{12.26}$$

We can establish the following a priori estimates for local solutions.

Proposition 12.3 *There exists a continuous increasing function $p(\cdot)$ such that the estimate*

$$\|U(t)\|_X \le p(\|U_0\|_X), \quad 0 \le t \le T_U, \tag{12.27}$$

holds for any local solution U of (12.24) *in the function space* (12.26).

Proof We put

$$f(u) = cu - \gamma u^2, \quad 0 \le u < \infty.$$

The notation $p(\cdot)$ stands for some continuous increasing functions which are determined only by $\chi(\cdot)$ and Ω and by the initial constants in the equations of (12.24). Similarly, the notation C stands for some constants which are determined in the same way as $p(\cdot)$.

Step 1. Integration of the first equation of (12.24) in Ω gives

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega (cu - \gamma u^2)\,dx \le \int_\Omega \left(-u + \frac{(c+1)^2}{4\gamma}\right) dx.$$

Solving this differential inequality (cf. (1.58)), we see that

$$\int_\Omega u\,dx \le e^{-t}\|u_0\|_{L_1} + \frac{(c+1)^2}{4\gamma}|\Omega|.$$

Since $u(t) \geq 0$, we conclude that

$$\|u(t)\|_{L_1} \leq e^{-t}\|u_0\|_{L_1} + \frac{(c+1)^2}{4\gamma}|\Omega|, \quad 0 \leq t < T_U. \tag{12.28}$$

Furthermore, for any $0 \leq s < t \leq T_U$, we have

$$\|u(t)\|_{L_1} - \|u(s)\|_{L_1} = \int_s^t \int_\Omega (cu - \gamma u^2)\, dx\, d\tau.$$

So,

$$\gamma \int_s^t \|u(\tau)\|_{L_2}^2\, d\tau \leq c \int_s^t \|u(\tau)\|_{L_1} d\tau + \|u(s)\|_{L_1};$$

hence, on account of (12.28),

$$\int_s^t \|u(\tau)\|_{L_2}^2\, d\tau \leq \gamma^{-1}\left(\|u_0\|_{L^1} + \frac{(c+1)^2}{4\gamma}|\Omega|\right)[c(t-s)+1], \quad 0 \leq s < t \leq T_U. \tag{12.29}$$

Step 2. Multiply the second equation of (12.24) by $\Delta\rho$ and integrate the product in Ω. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\rho|^2\, dx + b\int_\Omega |\Delta\rho|^2\, dx + d\int_\Omega |\nabla\rho|^2\, dx \\ = -\nu\int_\Omega u\Delta\rho\, dx \leq \frac{b}{2}\|\Delta\rho\|_{L^2}^2 + \frac{\nu^2}{2b}\|u\|_{L^2}^2.\end{aligned}$$

Similarly, multiplying the second equation of (12.24) by ρ, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2\, dx + b\int_\Omega |\nabla\rho|^2\, dx + d\int_\Omega \rho^2\, dx = -\nu\int_\Omega u\rho\, dx \\ \leq \frac{d}{2}\|\rho\|_{L^2}^2 + \frac{\nu^2}{2d}\|u\|_{L^2}^2.\end{aligned}$$

Summing up these differential inequalities, we obtain that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \left(|\nabla\rho|^2 + \rho^2\right) dx + \int_\Omega \left[b|\Delta\rho|^2 + 2(d+b)|\nabla\rho|^2 + d\rho^2\right] dx \\ \leq \nu^2(b+d)(bd)^{-1}\|u(t)\|_{L^2}^2.\end{aligned} \tag{12.30}$$

As (12.29) holds for $u(t)$, it is possible to solve this by applying Proposition 1.4. Consequently, we deduce that

$$\|\rho(t)\|_{H^1}^2 \leq e^{-dt}\|\rho_0\|_{H^1}^2 + C(\|u_0\|_{L_1} + 1), \quad 0 \leq t \leq T_U. \tag{12.31}$$

At the same time,

$$\int_s^t \|\Delta\rho(\tau)\|_{L_2}^2\,d\tau \le C(\|u_0\|_{L_1}+1)[(t-s)+\|\rho_0\|_{H^1}+1], \quad 0\le s<t\le T_U.$$

Furthermore, since it holds that

$$\|\rho\|_{H^2} \le C(\|\Delta\rho\|_{L_2}+\|\rho\|_{L_2}), \quad \rho\in H_N^2(\Omega), \tag{12.32}$$

due to (2.33), it is also verified that

$$\int_s^t \|\rho(\tau)\|_{H^2}^2\,d\tau \le C(\|u_0\|_{L_1}+1)[(t-s)+\|\rho_0\|_{H^1}+1], \quad 0\le s<t\le T_U. \tag{12.33}$$

Step 3. In this step, we use the notation

$$p_1(U_0)=p(\|u_0\|_{L_1}+\|\rho_0\|_{H^1}),$$

where $p(\cdot)$ is some continuous increasing function.

Multiply the first equation of (12.24) by $\log(u+1)$ and integrate the product in Ω. Then, by the same calculation as for (12.20), we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega [(u+1)\log(u+1)-u]\,dx+4a\int_\Omega |\nabla\sqrt{u+1}|^2\,dx\\ &\quad=\mu\int_\Omega [\log(u+1)-u]\Delta\chi(\rho)\,dx+\int_\Omega f(u)\log(u+1)\,dx.\end{aligned}$$

In addition, we observe that

$$\Delta\chi(\rho)=\chi'(\rho)\Delta\rho+\chi''(\rho)|\nabla\rho|^2.$$

Then, the first integral in the right-hand side can be estimated as follows. By (12.4),

$$\int_\Omega [\log(u+1)-u]\chi'(\rho)\Delta\rho\,dx \le C\|u\|_{L_2}\|\Delta\rho\|_{L_2} \le C\big(\|u\|_{L_2}^2+\|\Delta\rho\|_{L^2}^2\big),$$

while, by (12.4), (12.31), and (12.32),

$$\begin{aligned}\int_\Omega [\log(u+1)-u]\chi''(\rho)|\nabla\rho|^2\,dx &\le C\|u\|_{L^2}\|\nabla\rho\|_{L^4}^2 \le C\|u\|_{L^2}\|\rho\|_{H^2}\|\rho\|_{H^1}\\ &\le C\|\rho\|_{H^1}\big(\|u\|_{L^2}^2+\|\rho\|_{H^2}^2\big)\\ &\le p_1(U_0)\big(\|u\|_{L^2}^2+\|\rho\|_{H^2}^2\big).\end{aligned}$$

As for the second integral, it is easy to see that

$$f(u)\log(u+1)\le cu^2 \quad \text{for all } 0\le u<\infty.$$

Similarly,

$$(u+1)\log(u+1) \le \frac{1}{2}u^2 + u \quad \text{for all } 0 \le u < \infty.$$

These estimates thus yield that

$$\int_\Omega [f(u)+u+1]\log(u+1)\,dx \le \left(c+\frac{1}{2}\right)\|u\|_{L_2}^2 + \|u\|_{L_1}.$$

Our inequality is then reduced to

$$\frac{d}{dt}\int_\Omega [(u+1)\log(u+1)-u]\,dx + 4a\int_\Omega |\nabla\sqrt{u+1}|^2\,dx$$
$$+\int_\Omega [(u+1)\log(u+1)-u] \le p_1(U_0)\big[\|u(t)\|_{L_2}^2 + \|\rho(t)\|_{H^2}^2 + 1\big].$$

As we already know that (12.29) and (12.33) are valid, it is possible to solve this inequality by using Proposition 1.4. We then deduce that

$$N_{1,\log}(u(t)) \le e^{-t}N_{1,\log}(u_0) + p_1(U_0), \quad 0 \le t \le T_U. \tag{12.34}$$

Step 4. In this step, we use the notation

$$p_2(U_0) = p(N_{1,\log}(u_0) + \|\rho_0\|_{H^1}).$$

Multiply the first equation of (12.24) by u and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + a\int_\Omega |\nabla u|^2\,dx = \mu\int_\Omega u\nabla u\cdot\nabla\chi(\rho)\,dx + \int_\Omega f(u)u\,dx.$$

Here,

$$\int_\Omega u\nabla u\cdot\nabla\chi(\rho)\,dx = -\frac{1}{2}\int_\Omega u^2\Delta\chi(\rho)\,dx,$$
$$\Delta\chi(\rho) = \chi'(\rho)\Delta\rho + \chi''(\rho)|\nabla\rho|^2.$$

Furthermore, with the aid of (1.98) ($p=2,\ q=1$, and $r=3$), we verify that

$$\|u\|_{L_3} \le \zeta\|u\|_{H^1}^{\frac{2}{3}} N_{1,\log}(u)^{\frac{1}{3}} + C_\zeta\|u\|_{L_1}$$

with any $\zeta > 0$, and, by the moment inequality (2.117), we observe that

$$\|\Delta\rho\|_{L_3} \le C\|\Delta\rho\|_{H^1}^{\frac{1}{3}}\|\Delta\rho\|_{L_2}^{\frac{2}{3}} \le C\|A_2\rho\|_{H^1}^{\frac{1}{3}}\|A_2\rho\|_{L_2}^{\frac{2}{3}} \le C\left\|A_2^{\frac{3}{2}}\rho\right\|_{L_2}^{\frac{2}{3}}\left\|A_2^{\frac{1}{2}}\rho\right\|_{L_2}^{\frac{1}{3}}.$$

Therefore, on account of (12.4), (12.31), and (12.34),

$$-\int_\Omega \chi'(\rho)u^2\Delta\rho\,dx \le C\|u\|_{L_3}^2\|\Delta\rho\|_{L_3} \le p_2(U_0)\Big[\zeta\Big(\|u\|_{H^1}^2 + \big\|A_2^{\frac{3}{2}}\rho\big\|_{L_2}^2\Big) + C_\zeta\Big].$$

Similarly,

$$-\int_\Omega \chi''(\rho)u^2|\nabla\rho|^2\,dx \le C\|u\|_{L_3}^2\|\nabla\rho\|_{L_6}^2 \le C\|u\|_{L_3}^2\|\rho\|_{H^2}^{\frac{4}{3}}\|\rho\|_{H^1}^{\frac{2}{3}}$$
$$\le p_2(U_0)\Big[\zeta\Big(\|u\|_{H^1}^2 + \big\|A_2^{\frac{3}{2}}\rho\big\|_{L_2}^2\Big) + C_\zeta\Big]$$

with any $\zeta > 0$. Thus, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + a\int_\Omega |\nabla u|^2\,dx + \gamma\int_\Omega u^3\,dx$$
$$\le p_2(U_0)\Big[\zeta\Big(\|u\|_{H^1}^2 + \big\|A_2^{\frac{3}{2}}\rho\big\|_{L_2}^2\Big) + C_\zeta\Big]. \tag{12.35}$$

In this step, it is necessary to combine two inequalities for u and ρ. Multiply the second equation of (12.24) by $A_2^2\rho$ and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A_2\rho|^2\,dx + \int_\Omega \big|A_2^{\frac{3}{2}}\rho\big|^2\,dx = \nu\int_\Omega A_2^{\frac{1}{2}}u\cdot A_2^{\frac{3}{2}}\rho\,dx$$
$$\le \frac{1}{2}\big\|A_2^{\frac{3}{2}}\rho\big\|_{L_2}^2 + C\|u\|_{H^1}^2,$$

i.e.,

$$\frac{d}{dt}\int_\Omega |A_2\rho|^2\,dx + \int_\Omega \big|A_2^{\frac{3}{2}}\rho\big|^2\,dx \le C\|u(t)\|_{H^1}^2.$$

After multiplying a parameter $2\xi > 0$ to (12.35), we add the product to this inequality to obtain that

$$\frac{d}{dt}\int_\Omega \big(\xi u^2 + |A_2\rho|^2\big)\,dx + \int_\Omega \big((2a\xi - C)|\nabla u|^2 + 2\gamma\xi u^3\big)\,dx + \int_\Omega \big|A_2^{\frac{3}{2}}\rho\big|^2\,dx$$
$$\le p_2(U_0)\xi\Big[\zeta\Big(\|u\|_{H^1}^2 + \big\|A_2^{\frac{3}{2}}\rho\big\|_{L_2}^2\Big) + C_\zeta\Big].$$

Now, fix the parameters ξ and ζ so that $2a\xi - C \ge 1$ and $p_2(U_0)\xi\zeta \le \frac{1}{2}$. Then, we arrive at the differential inequality

$$\frac{d\psi}{dt} + \delta\psi + \frac{1}{2}\Big(\|u(t)\|_{H^1}^2 + \big\|A_2^{\frac{3}{2}}\rho(t)\big\|_{L_2}^2\Big) \le p_2(U_0)$$

for $\psi(t) = \xi\|u(t)\|_{L_2}^2 + \|A_2\rho(t)\|_{L_2}^2$ with some exponent $\delta > 0$. We can now employ Proposition 1.4 to conclude that

$$\xi\|u(t)\|_{L_2}^2 + \|A_2\rho(t)\|_{L_2}^2 \le e^{-\delta t}\big(\xi\|u_0\|_{L_2}^2 + \|A_2\rho_0\|_{H^2}^2\big) + p_2(U_0), \quad 0 \le t \le T_U,$$

i.e.,

$$\|u(t)\|_{L_2}^2 + \|\rho(t)\|_{H^2}^2 \le p_2(U_0)\big[e^{-\delta t}\big(\|u_0\|_{L_2}^2 + \|\rho_0\|_{H^2}^2\big) + 1\big], \quad 0 \le t \le T_U. \tag{12.36}$$

As well,

$$\int_s^t \Big[\|u(\tau)\|_{H^1}^2 + \big\|A_2^{\frac{3}{2}}\rho(\tau)\big\|_{L_2}^2\Big]d\tau \le p_2(U_0)\big[(t-s) + \|u_0\|_{L_2}^2 + \|\rho_0\|_{H^2}^2 + 1\big],$$
$$0 \le s < t \le T_U.$$

We have in this way established the desired a priori estimate (12.27). □

3.3 Global Solutions

We can now apply Corollary 4.3 to conclude the global existence of solutions. As a result, for any $U_0 \in \mathcal{K}$, (12.24) possesses a unique global solution in the function space:

$$\begin{cases} 0 \le u \in \mathcal{C}((0,\infty); H_N^2(\Omega)) \cap \mathcal{C}([0,\infty); L_2(\Omega)) \cap \mathcal{C}^1((0,\infty); L_2(\Omega)), \\ 0 \le \rho \in \mathcal{C}((0,\infty); \mathcal{H}_{N^2}^4(\Omega)) \cap \mathcal{C}([0,\infty); H_N^2(\Omega)) \cap \mathcal{C}^1((0,\infty); H_N^2(\Omega)). \end{cases} \tag{12.37}$$

For $U_0 \in \mathcal{K}$, let $U(t; U_0)$ be the global solution of (12.24) with the initial value U_0 in the function space (12.37). Recalling that Proposition 12.3 was established by four steps, let us apply (12.28) to $U(t; U_0)$ in the interval $\big[0, \frac{t}{4}\big]$, (12.31) in the interval $\big[\frac{t}{4}, \frac{2t}{4}\big]$, (12.34) in the interval $\big[\frac{2t}{4}, \frac{3t}{4}\big]$, and (12.36) in the interval $\big[\frac{3t}{4}, t\big]$, respectively. Then, we obtain the (more refined than (12.27)) estimate

$$\|U(t; U_0)\|_X \le p(e^{-\delta t} p(\|U_0\|_X) + 1), \quad 0 \le t < \infty,\ U_0 \in \mathcal{K}, \tag{12.38}$$

with some suitable exponent $\delta > 0$.

3.4 Dynamical System

Let us construct a dynamical system determined from problem (12.24) in the universal space X by following the general methods in Chap. 6, Sect. 5. Indeed, we can take $\beta = 0$. Furthermore, (12.38) shows that (6.56) is fulfilled. Consequently, it is deduced that problem (12.24) determines a dynamical system $(S(t), \mathcal{K}, X)$.

For any exponent $0 < \theta < 1$, (12.24) equally determines a dynamical system in a universal space $\mathcal{D}_\theta = \mathcal{D}(A^\theta)$ with phase space $\mathcal{K}_\theta = \mathcal{K} \cap \mathcal{D}_\theta$. Indeed, the nonlinear operator F defined by (12.25) fulfills (4.21) and a fortiori (4.2) with $\beta = \theta$. Let $\mathcal{K}_{\theta,R} = \mathcal{K} \cap \overline{B}^{\mathcal{D}_\theta}(0; R)$. Applying Theorem 4.1, we conclude that there exist $\tilde{\tau}_R > 0$ and a constant $C_{\theta,R} > 0$ such that

$$\|A^\theta S(t) U_0\|_X \le C_{\theta,R}, \quad 0 \le t \le \tilde{\tau}_R,\ U_0 \in \mathcal{K}_{\theta,R}.$$

Meanwhile, by Proposition 6.1,

$$\|A^\theta S(t)U_0\|_X \le C_{\theta,R} t^{-\theta}, \quad 0 < t < \infty, \ U_0 \in \mathcal{K}_{\theta,R}.$$

These two estimates then yield that

$$\|A^\theta S(t)U_0\|_X \le C_{\theta,R}, \quad 0 \le t < \infty, \ U_0 \in \mathcal{K}_{\theta,R}.$$

This means that (6.56) is valid in the space $\mathcal{D}_\theta$, too. So, $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$ defines a dynamical system.

3.5 *Exponential Attractors*

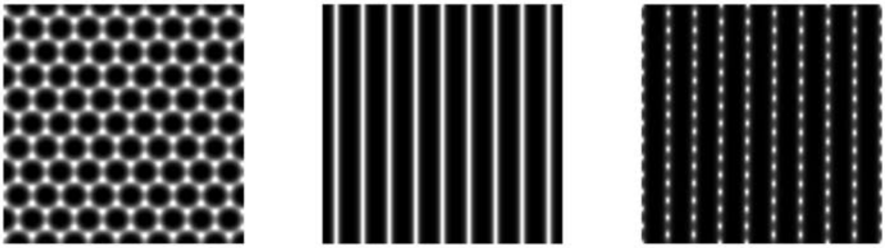
Let us next construct exponential attractors for $(S(t), \mathcal{K}, X)$. To this end, it suffices to verify (6.59) and (6.60). However, the space condition (6.59) is fulfilled in the present case. Meanwhile, (12.38) shows that the dissipative condition (6.60) is also fulfilled. Hence, $(S(t), \mathcal{K}, X)$ possesses a family of exponential attractors. For any exponent $0 < \theta < 1$, these exponential attractors are seen to keep their properties as exponential attractors in the universal space $\mathcal{D}_\theta$, too.

Remark 12.1 As the linear operator A given by (12.7) is not symmetric, A does not satisfy (6.62). So, it is unclear whether $S(t)$ enjoys the squeezing property (6.45)–(6.46) or not. However, if Ω is of $\mathcal{C}^3$ class, then one can use the shift property ($\Delta u \in H^1(\Omega)$, together with $u \in H^2_N(\Omega)$, implies $u \in H^3(\Omega)$) to rewrite (12.24) into the form (12.6) in an underlying space different from (12.5) in which the linear operator is a self-adjoint operator satisfying (6.62). For details, see Osaki–Tsujikawa–Yagi–Mimura [OTYM02]. In such an abstract formulation, the semigroup enjoys the squeezing property, and therefore the dimension of exponential attractor is estimated precisely by (6.50).

3.6 *Numerical Examples*

In this subsection, we present some numerical examples. Let $\Omega = [-8, 8] \times [-8, 8]$ be a quadratic domain. We consider the linear sensitivity function $\chi(\rho) = \rho$ and the growth function $f(u) = u^2(1-u)$. More precisely, we consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} = 0.0625\Delta u - \mu\nabla\cdot[u\nabla\rho] + u^2(1-u) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial \rho}{\partial t} = \Delta\rho - 32\rho + u & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n} = \frac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \qquad \rho(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases} \tag{12.39}$$



(a) $\mu = 6.2$ (b) $\mu = 7.2$ (c) $\mu = 8.5$

Fig. 1 Stationary patterns for $\mu = 6.2$, $\mu = 7.2$, and $\mu = 8.5$

We treat $\mu > 0$ as a control parameter. As the growth function in (12.39) is a cubic function, we cannot apply the results obtained in this section to (12.39) directly, but, as a matter of fact, we can repeat the same arguments in order to conclude for any $0 < \mu < \infty$ that (12.39) equally defines a dynamical system $(S(t), \mathcal{K}, X)$ which possesses exponential attractors, cf. Aida–Efendiev–Yagi [AEY05]. Note also that (12.39) has a homogeneous stationary solution $(\overline{u}, \overline{\rho}) = (1, \frac{1}{32})$ whatever $0 < \mu < \infty$ is.

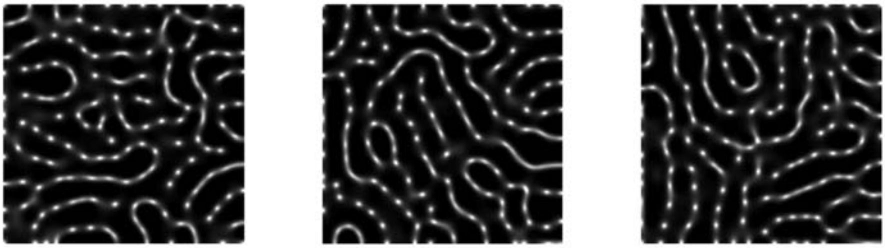
The initial functions $(u_0(x), \rho_0(x))$ are taken as $(1 + \varepsilon(x), \frac{1}{32})$, where $\varepsilon(x)$ is a small perturbation which vanishes outside of a small disk centered at the origin. So, (u_0, ρ_0) is taken in a neighborhood of the stationary solution $(\overline{u}, \overline{\rho})$.

When μ is small enough, the homogeneous stationary solution $(\overline{u}, \overline{\rho})$ is asymptotically stable (cf. (12.48)). The numerical solution is also observed to tend to the stationary solution $(\overline{u}, \overline{\rho})$. When $\mu = 6.2$, the homogeneous stationary solution is no longer stable (cf. (12.49)), and the numerical solution is observed to tend to an inhomogeneous stationary solution with honeycomb structure as in Fig. 1(a). (In the figure, white indicates high concentration of biological individuals, and black oppositely shows the graph of u.) As μ increases from 6.2 to 7.2 and 8.5, the inhomogeneous stationary solution changes its types from honeycomb to stripe and perforated stripe as Fig. 1 shows.

When μ still increases, such an inhomogeneous and ordered stationary solution is no longer found. That may lose its stability or completely vanish. Instead, the numerical solution is observed to tend to some moving pattern. When $\mu = 9.0$, a moving perforated labyrinthine pattern is observed as in Fig. 2.

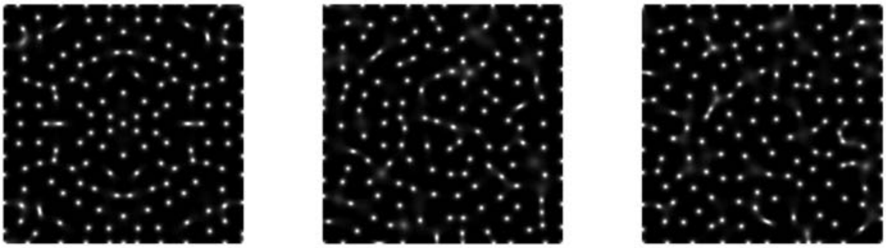
When $\mu = 11.0$, a chaotic spot pattern is observed as in Fig. 3. Each spot continues to move in a chaotic manner, a few spots are combined here and there in Ω; on the other hand, some new spots are generated to conserve the total number of spots at every moment.

When μ becomes larger, moving patterns as in Figs. 2 and 3 disappear. Instead, the numerical solution is observed to tend to a stationary pattern again. A number of steady spots are regularly located in Ω as in Fig. 4, but these stationary patterns seem to be quite different from those in Fig. 1.



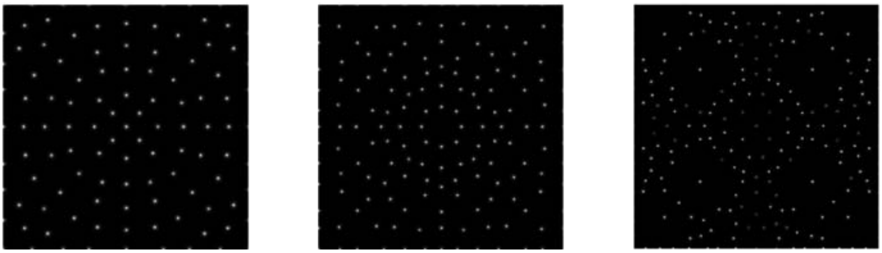
(a) $t = 1024$ (b) $t = 2048$ (c) $t = 4096$

Fig. 2 Moving perforated labyrinthine pattern for $\mu = 9.0$



(a) $t = 64$ (b) $t = 128$ (c) $t = 256$

Fig. 3 Chaotic spot pattern for $\mu = 11.0$



(a) $\mu = 25.0$ (b) $\mu = 40.0$ (c) $\mu = 70.0$

Fig. 4 Stationary patterns for $\mu = 25.0$, $\mu = 40.0$, and $\mu = 70.0$

These numerical results seem to be very interesting when we interpret them in accordance with the analytical result that the dimension of exponential attractors of the dynamical system for (12.39) increases as the parameter μ increases (cf. (12.50)).

4 Instability of Homogeneous Stationary Solution of (12.24)

We are concerned with homogeneous stationary solutions to (12.24). The homogeneous solution is given as a solution to

$$\begin{cases} cu - \gamma u^2 = 0, \\ -d\rho + \nu u = 0, \end{cases}$$

namely, $(u, \rho) = (0, 0)$ and $\left(\frac{c}{\gamma}, \frac{c\nu}{d\gamma}\right)$. We want to investigate the stability and instability of the nonzero homogeneous stationary solution $(\overline{u}, \overline{\rho}) = \left(\frac{c}{\gamma}, \frac{c\nu}{d\gamma}\right)$.

In this section, we make the new assumption on $\chi(\rho)$ that

$$\chi(\rho) \text{ is real analytic in a neighborhood of } \overline{\rho}. \tag{12.40}$$

4.1 *Localized Problem*

From (12.40) we can extend the sensitivity function as an analytic function in a complex neighborhood $|\rho - \overline{\rho}| < R$; furthermore, this analytic function can be extended over the whole complex plane in such a way that the extended function, denoted by $\tilde{\chi}(\rho)$, is a $\mathcal{C}^2$ function with respect to the real variables $(\rho', \rho'') \in \mathbb{R}^2$ such that $\rho = \rho' + i\rho'' \in \mathbb{C}$ with uniformly bounded partial derivatives in $\mathbb{C}$ up to the second order. Similarly, let $\varphi(u)$ be an extension of the function u such that $\varphi(u) \equiv u$ in a complex neighborhood $|u - \overline{u}| < R$ and $\varphi(u)$ is a $\mathcal{C}^1$ function with respect to the real variable $(u', u'') \in \mathbb{R}^2$ such that $u = u' + iu'' \in \mathbb{C}$ with uniformly bounded partial derivatives up to the first order. Using these functions, we introduce the localized problem

$$\begin{cases} \frac{\partial u}{\partial t} = a\Delta u - u - \mu\nabla \cdot [\varphi(u)\nabla\tilde{\chi}(\rho)] + (c+1)\varphi(u) - \gamma\varphi(u)^2 & \text{in } \Omega \times (0, \infty), \\ \frac{\partial \rho}{\partial t} = b\Delta\rho - d\rho + \nu u & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial n} = \frac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \qquad \rho(x, 0) = \rho_0(x) & \text{in } \Omega. \end{cases} \tag{12.41}$$

We can handle this localized problem in a quite analogous way as for (12.24). In fact, the problem is formulated as the Cauchy problem for an abstract equation in X of the form (12.6) in which the nonlinear operator $\tilde{F} : \mathcal{D}(A^\eta) \to X$, where $\frac{1+\varepsilon}{2} \le \eta < 1$ and $0 < \varepsilon < \frac{1}{2}$, is given by

$$\tilde{F}(U) = \begin{pmatrix} -\mu\nabla \cdot [\varphi(u)\nabla\tilde{\chi}(\rho)] + (c+1)\varphi(u) - \gamma\varphi(u)^2 \\ 0 \end{pmatrix}, \quad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in \mathcal{D}(A^\eta).$$

Obviously, $\tilde{F}$ is a nonlinear operator from $\mathcal{D}(A^\eta)$ into X. Let $0 \le \theta < \eta$. It is then possible to construct local solutions and global solutions to (12.41) for any initial value $U_0 \in \mathcal{D}(A^\theta)$ in the function space:

$$U \in \mathcal{C}([0,\infty); \mathcal{D}(A^\theta)) \cap \mathcal{C}((0,\infty); \mathcal{D}(A)) \cap C^1((0,\infty); X).$$

4.2 Complexified Dynamical System

Problem (12.41) then defines a nonlinear semigroup $\tilde{S}(t)$ acting on $\mathcal{D}_\theta = \mathcal{D}(A^\theta)$, $0 \le \theta < \eta$. Furthermore, (12.41) defines a complexified dynamical system $(\tilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$ in $\mathcal{D}_\theta$.

We are concerned with the case where $\frac{1}{2} < \theta < \eta$. Since $\mathcal{D}_\theta \subset \mathcal{C}(\overline{\Omega})$ for $\theta > \frac{1}{2}$, any solution to (12.24) is a solution of (12.41) in a suitable neighborhood of $\overline{U}$ in $\mathcal{D}_\theta$. Naturally, in such a neighborhood, any trajectory of $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$ is that of $(\tilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$. In particular, $\overline{U}$ is an equilibrium of $(\tilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$. If $\overline{U}$ is stable as an equilibrium of $(\tilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$, then it is the same as that of $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$. On the other hand, even if $\overline{U}$ is unstable as an equilibrium of $(\tilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$, we cannot say that $\overline{U}$ is automatically unstable as that of $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$. However, as before, the instability of $\overline{U}$ in $(\tilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$ provides us an important information which indicates the instability even in the original system $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$, too.

We can follow the general methods described in Chap. 6, Sect. 6. It is indeed sufficient to verify the differentiability conditions (6.68) and (6.69) for the nonlinear operator $\tilde{F}(U)$ and the spectrum separation condition (6.76) for the linearized operator $A - \tilde{F}'(\overline{U})$.

4.3 Differentiability of $\tilde{F}(U)$

If $r > 0$ is sufficiently small so that $\|A^\theta(U - \overline{U})\| < r$ implies $\|U - \overline{U}\|_{\mathcal{C}} < R$, then, in the open ball $B^{\mathcal{D}(A^\theta)}(\overline{U}; r)$, $\tilde{F} : \mathcal{D}(A^\eta) \to X$ is seen to be Fréchet differentiable with the derivative

$$\tilde{F}'(U)H = \begin{pmatrix} -\mu\nabla\cdot[h\nabla\chi(\rho)] - \mu\nabla\cdot\{u\nabla[\chi'(\rho)\varrho]\} + (c+1)h - 2\gamma uh \\ 0 \end{pmatrix},$$

$$U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in \mathcal{D}(A^\eta) \cap B^{\mathcal{D}(A^\beta)}(\overline{U}; r), \;\; H = \begin{pmatrix} h \\ \varrho \end{pmatrix} \in \mathcal{D}(A^\eta). \tag{12.42}$$

Furthermore, it is immediate to check that $\tilde{F}(U)$ fulfills conditions (6.68) and (6.69).

4.4 Spectrum Separation Condition of $\overline{A}$

For $\overline{A} = A - \tilde{F}'(\overline{U})$, let us verify condition (6.76).

Let Λ be the realization of $-\Delta$ in $L_2(\Omega)$ under the homogeneous Neumann boundary conditions. Let

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \to \infty$$

be the eigenvalues of Λ in $L_2(\Omega)$, and let $\phi_0 = |\Omega|^{-\frac{1}{2}}$, ϕ_1, $\phi_2, \ldots,$ be the corresponding real-valued eigenfunctions which constitute an orthonormal basis of $L_2(\Omega)$ (cf. Corollary 2.1). Using Λ, we can characterize the space $H^2_N(\Omega)$ by

$$H^2_N(\Omega) = \mathcal{D}(\Lambda) = \left\{ \rho = \sum_{k=0}^{\infty} \eta_k \phi_k;\ \sum_{k=0}^{\infty} (\mu_k + 1)^2 |\eta_k|^2 < \infty \right\}$$

with the inner product

$$(\rho, \rho')_{H^2_N} = \sum_{k=0}^{\infty} (\mu_k + 1)^2 \eta_k \overline{\eta'_k}, \quad \rho = \sum_{k=0}^{\infty} \eta_k \phi_k,\ \rho' = \sum_{k=0}^{\infty} \eta'_k \phi_k.$$

Then, we notice that our underlying space X given by (12.5) is obtained as the infinite sum of two-dimensional orthogonal subspaces

$$X_k = \left\{ U = \xi_k \begin{pmatrix} \phi_k \\ 0 \end{pmatrix} + \eta_k \begin{pmatrix} 0 \\ \phi_k \end{pmatrix};\ \xi_k, \eta_k \in \mathbb{C} \right\}, \quad k = 0, 1, 2, \ldots,$$

equipped with the norm

$$\|U\|^2_{X_k} = |\xi_k|^2 + (\mu_k + 1)^2 |\eta_k|^2.$$

It is then obvious that

$$\Phi_k = \begin{pmatrix} \phi_k \\ 0 \end{pmatrix} \quad \text{and} \quad \Psi_k = (\mu_k + 1)^{-1} \begin{pmatrix} 0 \\ \phi_k \end{pmatrix}$$

are orthonormal bases of X_k.

According to (12.42), we have

$$\tilde{F}'(\overline{U}) = \begin{pmatrix} c + 1 - 2\gamma \overline{u} & -\mu \overline{u} \chi'(\overline{\rho}) \Delta \\ 0 & 0 \end{pmatrix}.$$

Note that $\overline{u} = \frac{c}{\gamma}$ and put $\chi_1 = \chi'(\overline{\rho}) > 0$. Since

$$A = \begin{pmatrix} a\Lambda + 1 & 0 \\ -\nu & b\Lambda + d \end{pmatrix} \quad \text{and} \quad \tilde{F}'(\overline{U}) = \begin{pmatrix} -c + 1 & \mu \overline{u} \chi_1 \Lambda \\ 0 & 0 \end{pmatrix},$$

we observe that $\overline{A} = A - \tilde{F}'(\overline{U})$ maps the subspace X_k into itself, that is, X_k is invariant under $\overline{A}$ for every k. Consequently, the operator $\overline{A}$ can also be decomposed as $\overline{A} = \sum_{k=0}^{\infty} \overline{A}_k$, where $\overline{A}_k$ is the part of $\overline{A}$ in X_k. Obviously,

$$\overline{A}_k \Phi_k = (a\mu_k + c)\Phi_k - \nu(\mu_k + 1)\Psi_k,$$

$$\overline{A}_k \Psi_k = -\mu \overline{u} \chi_1 \mu_k (\mu_k + 1)^{-1} \Phi_k + (b\mu_k + d)\Psi_k.$$

The transformation matrix $\overline{M}_k$ of $\overline{A}_k$ with respect to the basis Φ_k, Ψ_k is then given by

$$\overline{M}_k = \begin{pmatrix} a\mu_k + c & -\nu(\mu_k + 1) \\ -\mu \overline{u} \chi_1 \mu_k (\mu_k + 1)^{-1} & b\mu_k + d \end{pmatrix}.$$

Since

$$|\lambda - \overline{M}_k| = [\lambda - (a\mu_k + c)][\lambda - (b\mu_k + d)] - \mu\nu\overline{u}\chi_1\mu_k,$$

the characteristic equation of $\overline{M}_k$ is written by

$$\lambda^2 - [(a+b)\mu_k + c + d]\lambda + (a\mu_k + c)(b\mu_k + d) - \mu\nu\overline{u}\chi_1\mu_k = 0.$$

So, the discriminant satisfies

$$D_k = [(a-b)\mu_k + c - d]^2 + 4\mu\nu\overline{u}\chi_1\mu_k \geq 0.$$

That is, for every $k = 0, 1, 2, \ldots$, the matrix $\overline{M}_k$ has two real eigenvalues $\lambda_k \leq \lambda'_k$. Furthermore, for $\lambda \neq \lambda'_k$, λ''_k,

$$(\lambda - \overline{M}_k)^{-1} = \frac{1}{(\lambda - \lambda'_k)(\lambda - \lambda''_k)} \begin{pmatrix} \lambda - (b\mu_k + d) & -\nu(\mu_k + 1) \\ -\mu\overline{u}\chi_1\mu_k(\mu_k + 1)^{-1} & \lambda - (a\mu_k + c) \end{pmatrix}. \tag{12.43}$$

For verifying (6.76), we here make the assumption that

$$ab\mu_k^2 + (ad + bc - \mu\nu\overline{u}\chi_1)\mu_k + cd \neq 0 \quad \text{for all } k = 0, 1, 2, \ldots. \tag{12.44}$$

If

$$(a\mu_k + c)(b\mu_k + d) - \mu\nu\overline{u}\chi_1\mu_k = ab\mu_k^2 + (ad + bc - \mu\nu\overline{u}\chi_1)\mu_k + cd > 0, \tag{12.45}$$

then the eigenvalues of $\overline{M}_k$ are both positive, i.e., $0 < \lambda'_k \leq \lambda''_k$. Furthermore, the following estimate

$$\begin{aligned} \frac{ab\mu_k^2 + (ad + bc - \mu\nu\overline{u}\chi_1)\mu_k + cd}{(a+b)\mu_k + c + d} &< \lambda'_k \leq \lambda''_k \\ &< (a+b)\mu_k + c + d - \frac{ab\mu_k^2 + (ad + bc - \mu\nu\overline{u}\chi_1)\mu_k + cd}{(a+b)\mu_k + c + d} \end{aligned} \tag{12.46}$$

holds for the eigenvalues. In the meantime, if

$$ab\mu_k^2 + (ad + bc - \mu\nu\overline{u}\chi_1)\mu_k + cd < 0, \tag{12.47}$$

then one of the eigenvalues is negative, and the other is positive, i.e., $\lambda'_k < 0 < \lambda''_k$.

Let $\lambda \in i\mathbb{R}$. Consider the parts $\overline{A}_k$ on X_k. Let $\psi_k : X_k \to \mathbb{C}^2$ be a natural isomorphism such that $\psi_k(\xi_k \Phi_k + \eta_k \Psi_k) = {}^t(\xi_k, \eta_k)$ ($\|\psi_k\| = \|\psi_k^{-1}\| = 1$); then $\overline{A}_k$ is written as $\overline{A}_k = \psi_k^{-1}\overline{M}_k\psi_k$. Consequently,

$$(\lambda - \overline{A}_k)^{-1} = \psi_k^{-1}(\lambda - \overline{M}_k)^{-1}\psi_k.$$

Therefore, if k are sufficiently large so that (12.45) take place, then (12.43) provides, with the aid of (12.46) and the obvious inequalities $|\lambda - \lambda_k'| \geq \lambda_k'$ and $|\lambda - \lambda_k''| \geq \lambda_k''$, that

$$\left\|(\lambda - \overline{A}_k)^{-1}\right\|_{\mathcal{L}(X_k)} \leq C_\lambda,$$

$C_\lambda > 0$ being independent of k. This fact means that $\lambda \in i\mathbb{R}$ belongs to $\rho(\overline{A})$ if and only if $\lambda \in \rho(\overline{A}_k)$ for every $k = 0, 1, 2, \ldots$. In other words, $\lambda \notin \sigma(\overline{A})$ if and only $\lambda \notin \sigma(\overline{A}_k) = \{\lambda_k', \lambda_k''\}$ for every k.

We already know that, under (12.44), $i\mathbb{R} \cap \sigma(\overline{A}_k) = \emptyset$ for every $k = 0, 1, 2, \ldots$. Hence, $i\mathbb{R} \cap \sigma(\overline{A}) = \emptyset$, that is, the separation condition (6.76) is fulfilled.

4.5 Stability or Instability Conditions

Assume the condition

$$\sqrt{\mu\nu\overline{u}\chi_1} < \sqrt{ad} + \sqrt{bc}. \tag{12.48}$$

Then, since

$$ab\mu^2 + (ad + bc)\mu + cd \geq (\sqrt{ad} + \sqrt{bc})^2\mu \quad \text{for all } \mu \geq 0,$$

it follows from (12.48) that (12.45) takes place, that is, $\overline{A}$ fulfills (6.67), and $\overline{U}$ is a stable equilibrium of $(\tilde{S}(t), \mathcal{D}_\theta, D_\theta)$.

On the other hand, under (12.44), if the condition

$$N(\overline{U}) = \#\left\{\mu_k;\ ab\mu_k^2 + (ad + bc - \mu\nu\overline{u}\chi_1)\mu_k + cd < 0\right\} \geq 1 \tag{12.49}$$

takes place, then $\sigma(\overline{A}) \cap \{\lambda;\ \mathrm{Re}\,\lambda < 0\} \neq \emptyset$; therefore, $\overline{U}$ has a nontrivial local unstable manifold $\tilde{\mathcal{W}}_+(\overline{U}; \mathcal{O})$ and is an unstable equilibrium of $(\tilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$.

For μ_k satisfying (12.47), let $\lambda_k' < 0 < \lambda_k''$. Since $\overline{M}_k$ is a real matrix, there exists a real eigenvector $\boldsymbol{u}_k \in \mathbb{R}^2$ corresponding to λ_k'. Meanwhile, $\psi_k^{-1}\boldsymbol{u}_k$ is an eigenfunction of $\overline{A}_k$ corresponding to λ_k'. Since ϕ_k is a real eigenfunction of Λ, Φ_k and Ψ_k are real valued. Therefore, $\overline{A}$ has an eigenvector for $\lambda_k' < 0$ whose components are real-valued functions. Let us consider the instability of $\overline{U}$ as an equilibrium of the original dynamical system $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$. We already know that $\overline{U}$ is an unstable equilibrium of $(\tilde{S}(t), \mathcal{D}_\theta, \mathcal{D}_\theta)$ having a local unstable manifold $\tilde{\mathcal{W}}_+(\overline{U}; \mathcal{O})$ with dimension $N(\overline{U})$ which is tangential to $\overline{U} + X_-$ at $\overline{U}$. Furthermore, the sum X_- of eigenspaces of $\overline{A}$ corresponding to the negative eigenvalues contain an orthogonal basis which consists of real-valued functions. These facts indicate that the local unstable manifold $\mathcal{W}_+(\overline{U}; \mathcal{O})$ for the original dynamical system is neither trivial, that is, $\overline{U}$ is an unstable equilibrium of $(S(t), \mathcal{K}_\theta, \mathcal{D}_\theta)$.

It is already known that $\dim \tilde{\mathcal{W}}_+(\overline{U}, \mathcal{O}) = \dim X_-$ is equal to the number $N(\overline{U})$ given by (12.49). We easily verify that, if $\mu\nu \to \infty$, the other parameters being all fixed, then $N(\overline{U}) \to \infty$. In this sense, as $\mu\nu \to \infty$,

$$\dim \tilde{\mathcal{W}}_+(\overline{U}; \mathcal{O}) = N(\overline{U}) \to \infty. \tag{12.50}$$

Notes and Further Researches

The diffusion–advection model for chemotactic phenomenon was first introduced by Keller–Segel [KS70]. Afterward, this modeling was developed further by Nanjundiah [Nan73] and others [AC98, AL85, Den77, LK83, MM92]. More detailed considerations on the sensitivity functions were made by [FL91]. On the basis of these works, several models were presented for understanding theoretically the pattern formation by chemotactic bacteria discovered by Budrene–Berg [BB91, BB95]. The model (12.1) was introduced by Woodward–Tyson–Myerscough–Murray–Budrene–Berg [WTMB] (cf. also [Mur03, Chap. 5]). Meanwhile, the model (12.2) was presented by Mimura–Tsujikawa [MT96]. We want to quote also [KS93].

The evolutional problem for the model (12.3) (without proliferation) was first studied by Childress–Percus [CP81]. When $\chi(\rho) = \rho$, the global existence for initial functions having small L_1 norm $\|u_0\|_{L_1}$ was obtained by Ryu–Yagi [Yag97b, RY01]. The Lyapunov function (12.23) was constructed by Nagai–Senba–Yosida [NSY97] using techniques due to Biler–Hebish–Nadzieja [BHN94]. Recently, Feireisl–Laurençot–Petzeltová [FLP07] showed the convergence of global solutions to equilibria as $t \to \infty$ by using the Simon and Łojasiewicz method, see [Sim83] and [Loj63, Loj65]. The blowup of solutions of (12.3) was proved by Herrero–Velázquez [HV96, HV97]. Some related results were shown by Nagai–Senba–Suzuki [NSS00] and Horstmann–Wang [HW01]. The stationary problem for (12.3) was studied by [LNT88, MW06, Sch85, SS00]. For one-dimensional problem, we cite Osaki–Yagi [OY01] and Kang–Kolokolnikov–Ward [KKW07]. Finally, we quote reviewing papers due to Horstmann [Hor03, Hor04] and the references therein.

For the model (12.24) (with proliferation), a dynamical system with not only global solutions was constructed by Osaki–Tsujikawa–Yagi–Mimura [OTYM02]. Under $\mathcal{C}^3$ regularity of Ω, they proved the existence of exponential attractors by showing the squeezing property of the semigroup. When Ω is a convex domain, the similar results were obtained by Aida–Efendiev–Yagi [AEY05] by using the compact perturbation. The construction of a smooth unstable manifold for the homogeneous stationary solution $\overline{U}$ was done by Aida–Tsujikawa–Efendiev–Yagi–Mimura [ATEYM]. In these papers, $\chi(\rho)$ is assumed to satisfy (12.4). Aida–Osaki–Tsujikawa–Yagi–Mimura [AOTYM] constructed global solutions for (12.24) in the case where $\chi(\rho)$ has a singularity at $\rho = 0$. But, in such a case, it still remains to study the asymptotic behavior of solutions. In the case where $\chi(\rho) = \rho^2$, even the existence of global solutions is unknown. Although various stationary solutions to (12.39) were found numerically in Sect. 3.6, little is known about the stationary problem for (12.39) and (12.24). The numerical results in Sect. 3.6 were obtained by Aida [Aid03], Aida–Yagi [AY04a], and Hai–Yagi [HY09].

The techniques described in this chapter are available to other similar chemotaxis models. For example, we can treat the more complicated model

$$\begin{cases} \frac{\partial u}{\partial t} = a\Delta u - \mu\nabla\cdot\left[\frac{u}{(\rho+1)^2}\nabla\rho\right] + fu\left(\delta\frac{s^2}{1+s^2} - u\right) & \text{in } \Omega\times(0,\infty), \\ \frac{\partial \rho}{\partial t} = b\Delta\rho - u\rho + \nu s\frac{u^2}{g+u^2} & \text{in } \Omega\times(0,\infty), \\ \frac{\partial s}{\partial t} = c\Delta s - hu\frac{s^2}{1+s^2} & \text{in } \Omega\times(0,\infty), \end{cases}$$

which was presented by Murray in [Mur03, Chap. 5]. Here, u and ρ are the same as in (12.1), and s denotes the concentration of stimulant in Ω. We shall consider in the subsequent chapter a chemotaxis model arising in ecology.

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