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First-Order Deformations

We start by introducing the Hilbert scheme, which will be a model for the other situations, and which will provide us with examples as we go along. Then in Section 2 we discuss deformations over the dual numbers for Situations A, B, and C. In Section 3 we introduce the cotangent complex and the T^i functors, which are needed to discuss deformations of abstract schemes (Situation D) in Section 5. In Section 4 we examine the special role of nonsingular varieties, using the infinitesimal lifting property and the T^i functors. We also show that the relative notion of a smooth morphism is characterized by the vanishing of the relative T^1 functors.

1. The Hilbert Scheme

As motivation for all the local study of deformations we are about to embark on, we will introduce the Hilbert scheme of Grothendieck, as a typical example of the goals of this work. The Hilbert scheme gives a particularly satisfactory answer to the problem of describing families of closed subschemes of a given scheme. In fact, when I first lectured on this subject and wrote some preliminary notes that have grown into this book, my goal was to understand completely the proof of the following theorem.

Theorem 1.1. *Let Y be a closed subscheme of the projective space $X = \mathbb{P}^n_k$ over a field k. Then*

(a) *There exists a projective scheme H, called the* Hilbert scheme, *parametrizing closed subschemes of X with the same Hilbert polynomial P as Y, and there exists a universal subscheme $W \subseteq X \times H$, flat over H, such that the fibers of W over closed points $h \in H$ are all closed subschemes of X with the same Hilbert polynomial P. Furthermore, H is universal in the sense that if T is any other scheme, if $W' \subseteq X \times T$ is a closed subscheme, flat over T, all of whose fibers are subschemes of X with the same Hilbert*

polynomial P, then there exists a unique morphism $\varphi : T \to H$ such that $W' = W \times_H T$ as subschemes of $X \times T$.

(b) *The Zariski tangent space to H at the point $y \in H$ corresponding to Y is given by $H^0(Y, \mathcal{N})$, where $\mathcal{N}$ is the normal sheaf of Y in X.*

(c) *If Y is a locally complete intersection, and if $H^1(Y, \mathcal{N}) = 0$, then H is non-singular at the point y, of dimension equal to $h^0(Y, \mathcal{N}) = \dim_k H^0(Y, \mathcal{N})$.*

(d) *In any case, if Y is a locally complete intersection, the dimension of H at y is at least $h^0(Y, \mathcal{N}) - h^1(Y, \mathcal{N})$.*

Parts (a), (b), (c) of this theorem are due to Grothendieck [45]. For part (d) there are recent proofs due to Laudal [92] and Mori [109]. I do not know whether there is an earlier reference.

Since the main purpose of this book is to study the local theory, we will not prove the existence (a) of the Hilbert scheme. The proof of existence uses techniques quite different from those we consider here, and is not necessary for the comprehension of anything in this book. The reader who wishes to see a proof can consult any of many sources [45, exposé 221], [115], [151], [161], [152]. Parts (b), (c), (d) of the theorem will be proved in §2, §9, and §11, respectively.

Parts (b), (c), (d) of this theorem illustrate the benefit derived from Grothendieck's insistence on the systematic use of nilpotent elements. Let $D = k[t]/t^2$ be the ring of dual numbers. Taking D as our parameter scheme, we see from the universal property (a) that flat families $Y' \subseteq X \times D$ with closed fiber Y are in one-to-one correspondence with morphisms of schemes $\operatorname{Spec} D \to H$ that send the unique point to y. This set $\operatorname{Hom}_y(D, H)$ in turn can be interpreted as the Zariski tangent space to H at y [57, II, Ex. 2.8]. Thus to prove (b) of the theorem, we have only to classify schemes $Y' \subseteq X \times D$, flat over D, whose closed fiber is Y, which we will do in §2.

Part (c) of the theorem is related to *obstruction theory*. Given an infinitesimal deformation defined over an Artin ring A, to extend the deformation over a larger Artin ring there is usually some obstruction, whose vanishing is necessary and sufficient for the existence of an extended deformation. For closed subschemes with no local obstructions, such as locally complete intersection subschemes, the obstructions lie in $H^1(Y, \mathcal{N})$. If that group is zero, there are no obstructions, and the corresponding moduli space is nonsingular. The dimension estimate (d) comes out of obstruction theory.

Exercises.

1.1. Curves in $\mathbb{P}^2$.   Here we will verify the existence of the Hilbert scheme for curves in $\mathbb{P}^2$. Over an algebraically closed field k, we define a *curve* in $\mathbb{P}^2_k$ to be the closed subscheme defined by a homogeneous polynomial $f(x, y, z)$ of degree d in the coordinate ring $S = k[x, y, z]$. We can write f as $a_0 x^d + \cdots + a_n z^d$, $a_i \in k$, with $n = \binom{d+2}{2} - 1$ since f has that many terms. Consider $(a_0, \ldots, a_n)$ as a point in $\mathbb{P}^n_k$.

(a) Show that curves of degree d in $\mathbb{P}^2$ are in a one-to-one correspondence with points of $\mathbb{P}^n$ by this correspondence.

(b) Define $\mathcal{C} \subseteq \mathbb{P}^2 \times \mathbb{P}^n$ by the equation $f = a_0 x^d + \cdots + a_n z^d$ above, where the x, y, z are coordinates on $\mathbb{P}^2$ and $a_0, \ldots, a_n$ are coordinates on $\mathbb{P}^n$. Show that the correspondence of (a) is given by $a \in \mathbb{P}^n$ goes to the fiber $\mathcal{C}_a \subseteq \mathbb{P}^2$ over the point a. Therefore we call $\mathcal{C}$ a *tautological* family.

(c) For any finitely generated k-algebra A, we define a *family* of curves of degree d in $\mathbb{P}^2$ over A to be a closed subscheme $X \subseteq \mathbb{P}^2_A$, flat over A, whose fibers above closed points of $\operatorname{Spec} A$ are curves in $\mathbb{P}^2$. Show that the ideal $I_X \subseteq A[x, y, z]$ is generated by a single homogeneous polynomial f of degree d in $A[x, y, z]$.

(d) Conversely, if $f \in A[x, y, z]$ is homogeneous of degree d, what is the condition on f for the zero-scheme X defined by f to be flat over A? (Do not assume A reduced.)

(e) Show that the family $\mathcal{C}$ is *universal* in the sense that for any family $X \subseteq \mathbb{P}^2_A$ as in c), there is a unique morphism $\operatorname{Spec} A \to \mathbb{P}^n$ such that $X = \mathcal{C} \times_{\mathbb{P}^n} \operatorname{Spec} A$.

(f) For any curve $X \subseteq \mathbb{P}^2_k$ of degree d, show that $h^0(\mathcal{N}_X) = n$ and $h^1(\mathcal{N}_X) = 0$. (Do not assume X nonsingular.)

1.2. Curves on quadric surfaces in $\mathbb{P}^3$. Consider the family $\mathcal{C}$ of all nonsingular curves C that lie on some nonsingular quadric surface Q in $\mathbb{P}^3$ and have bidegree (a, b) with $a, b > 0$.

(a) By considering the linear system of curves C on a fixed Q, and then varying Q, show that if the total degree d is equal to $a + b \geq 5$, then the dimension of the family $\mathcal{C}$ is $ab + a + b + 9$.

(b) If $a, b \geq 3$, show that $H^0(C, \mathcal{N}_C)$ has the same dimension $ab + a + b + 9$, using the exact sequence of normal bundles

$$0 \to \mathcal{N}_{C/Q} \to \mathcal{N}_C \to \mathcal{N}_Q|_C \to 0.$$

Show that $\mathcal{N}_{C/Q} \cong \mathcal{O}_C(C^2)$ is nonspecial, i.e., its H^1 is zero, so you can compute its H^0 by Riemann–Roch. Then note that $\mathcal{N}_Q|_C \cong \mathcal{O}_C(2)$, and compute its H^0 using the exact sequence

$$0 \to \mathcal{O}_Q(2 - C) \to \mathcal{O}_Q(2) \to \mathcal{O}_C(2) \to 0$$

and the vanishing theorems for H^1 of line bundles on Q given in [57, III, Ex. 5.6].

(c) Conclude that for $a, b \geq 3$ the family $\mathcal{C}$ gives (an open subset of) an irreducible component of the Hilbert scheme of dimension $ab + a + b + 9$, which is smooth at each of its points.

(d) What goes wrong with this argument if $a = 2$ and $b \geq 4$? Cf. (Ex. 6.4).

1.3. Complete intersection curves in $\mathbb{P}^3$. A curve C in $\mathbb{P}^3_k$ is a *complete intersection* if its homogeneous ideal $I \subseteq k[x, y, z, w]$ is generated by two homogeneous polynomials. Let C be a complete intersection curve defined by polynomials of degrees $a, b \geq 1$.

(a) The complete intersection curve C has degree $d = ab$ and arithmetic genus $g = \frac{1}{2}ab(a + b - 4) + 1$. The dualizing sheaf ω_C is isomorphic to $\mathcal{O}_C(a + b - 4)$. For any $a, b \geq 1$, a general such complete intersection curve is nonsingular. The family of all such curves is irreducible and of dimension $2\binom{a+3}{3} - 2$ if $a = b$ or $\binom{a+3}{3} + \binom{b+3}{3} - \binom{b-a+3}{3} - 2$ if $a < b$.

(b) The normal sheaf is $\mathcal{N}_C \cong \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$. Using the resolution

$$0 \to \mathcal{O}_\mathbb{P}(-a-b) \to \mathcal{O}_\mathbb{P}(-a) \oplus \mathcal{O}_\mathbb{P}(-b) \to \mathcal{I}_C \to 0,$$

verify that $H^0(\mathcal{N}_C)$ has dimension equal to the dimension of the family above, so that the family of complete intersection curves defined by polynomials of degrees a and b is a nonsingular open subset of an irreducible component of the Hilbert scheme.

1.4.   The limit of a flat family of complete intersection curves in $\mathbb{P}^3$ need not be a complete intersection curve. In other words, the open set of the Hilbert scheme formed by complete intersection curves may not be closed. For an example, fix a $\lambda \in k$, $\lambda \neq 0, 1$, and consider the family of complete intersection curves over $k[t, t^{-1}]$ defined by the equations

$$\begin{cases} tyz - wx = 0, \\ yw - t(x-z)(x - \lambda z) = 0. \end{cases}$$

(a) Show that for any $t \neq 0$, these equations define a nonsingular cure C_t of degree 4 and genus 1.
(b) Now extend this family to a flat family over all of $k[t]$, and show that the special fiber C_0 over $t = 0$ is the union of a nonsingular plane cubic curve with a line not in that plane, but meeting the cubic curve at one point. Show also that C_0 is not a complete intersection. Since C_0 is a singular curve belonging to a flat family whose general member is nonsingular, we say that C_0 is a *smoothable* singular curve.

Note. What is happening in this example is that the curves C_t, as t approaches zero, are being pushed away from the point $P : (x, y, z, w) = (0, 0, 0, 1)$ of the curve toward the plane $w = 0$. In the end the irreducible curve C_t breaks into two pieces: the plane cubic curve plus a line through P.

1.5.   Show that the Hilbert scheme of degree 4 and genus 1 curves is still nonsingular of dimension 16 at the point corresponding to the curve C_0 of (Ex. 1.4).

(a) First show that if a curve Y is the union of two nonsingular curves C and D in $\mathbb{P}^3$, meeting transversally at a single point P, then there are exact sequences of normal sheaves

$$0 \to \mathcal{N}_Y \to \mathcal{N}_Y|_D \oplus \mathcal{N}_Y|_C \to \mathcal{N}_Y \otimes k_P \to 0$$

and

$$0 \to \mathcal{N}_C \to \mathcal{N}_Y|_C \to k_P \to 0,$$
$$0 \to \mathcal{N}_D \to \mathcal{N}_Y|_D \to k_P \to 0.$$

(b) Apply these sequences to the union of a plane cubic curve and a line C_0 as above, to show that $h^0(\mathcal{N}_{C_0}) = 16$. Since C_0 is contained in the closure of the complete intersection curves, which form a family of dimension 16, this shows that the Hilbert scheme is smooth at C_0. For another proof of this fact, see (Ex. 8.3).

1.6. Twisted cubic curves.   The twisted cubic curve in $\mathbb{P}^3$ is defined parametrically by $(x_0, x_1, x_2, x_3) = (u^3, tu^2, t^2u, t^3)$ for $(t, u) \in \mathbb{P}^1$. More generally we call any curve obtained from this one by a linear change of coordinates in $\mathbb{P}^3$ a *twisted cubic curve.*

(a) Show that any nonsingular curve of degree 3 and genus 0 in $\mathbb{P}^3$ is a twisted cubic curve. Show that these form a family of dimension 12, and that $H^0(C, \mathcal{N}_C) = 12$ for any such curve. Thus the twisted cubic curves form a nonsingular open subset of an irreducible component of the Hilbert scheme of curves with Hilbert polynomial $3z + 1$.

(b) Consider a subscheme $Y \subseteq \mathbb{P}^3$ that is a disjoint union of a plane cubic curve and a point. Show that these schemes form another nonsingular open subset of the Hilbert scheme of curves with Hilbert polynomial $3z + 1$. This component has dimension 15.

(c) There is a flat family of twisted cubic curves whose limit is a curve Y_0, supported on a plane nodal cubic curve, and having an embedded point at the node [57, III, 9.8.4]. Show that this curve is in the closure of both irreducible components mentioned above, hence corresponds to a singular point on the Hilbert scheme.

(d) Now show that $h^0(\mathcal{N}_{Y_0/\mathbb{P}^3}) = 16$, confirming that Y_0 is a singular point of the Hilbert scheme. *Hint:* Show that the homogeneous ideal of Y_0, $I = (z^2, yz, xz, y^2w - x^2(x + w))$, has a resolution over the polynomial ring $R = k[x, y, z, w]$ as follows:

$$R(-3)^3 \oplus R(-4) \to R(-2)^3 \oplus R(-3) \to I \to 0.$$

Tensor with $B = R/I$, then dualize and sheafify to get a resolution

$$0 \to \mathcal{N}_{Y/\mathbb{P}^3} \to \mathcal{O}_{Y_0}(2)^3 \oplus \mathcal{O}_{Y_0}(3) \to \mathcal{O}_{Y_0}(3)^3 \oplus \mathcal{O}_{Y_0}(4).$$

Compute explicitly with the sections of $\mathcal{O}_{Y_0}(2)$ and $\mathcal{O}_{Y_0}(3)$, which all come from polynomials in R, to show that $h^0(\mathcal{N}_{Y/\mathbb{P}^3}) = 16$.

Note. The structure of this Hilbert scheme is studied in detail in the paper [134].

1.7.   Let C be a nonsingular curve in $\mathbb{P}^n$ that is *nonspecial*, i.e., $H^1(\mathcal{O}_C(1)) = 0$. Show that the Hilbert scheme is nonsingular at the point corresponding to C. *Hint:* Use the Euler sequence for the tangent bundle on $\mathbb{P}^n$, restricted to C, and use the exact sequence relating the tangent bundle of C, the tangent bundle of $\mathbb{P}^n$, and the normal bundle of C.

2. Structures over the Dual Numbers

The very first deformation question to study is structures over the dual numbers $D = k[t]/t^2$. That is, one gives a structure (e.g., a scheme, or a scheme with a subscheme, or a scheme with a sheaf on it) over k and one seeks to classify extensions of this structure over the dual numbers. These are also called first-order deformations.

To ensure that our structure is evenly spread out over the base, we will always assume that the extended structure is *flat* over D. Flatness is the technical condition that corresponds to the intuitive idea of a deformation.

In this section we will apply this study to Situations A, B, and C.

Recall that a module M is *flat* over a ring A if the functor $N \mapsto N \otimes_A M$ is exact on the category of A-modules. A morphism of schemes $f : X \to Y$ is *flat* if for every point $x \in X$, the local ring $\mathcal{O}_{x,X}$ is flat over the ring $\mathcal{O}_{f(x),Y}$. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *flat* over Y if for every $x \in X$, its stalk $\mathcal{F}_x$ is flat over $\mathcal{O}_{f(x),Y}$.

Lemma 2.1. *A module M over a noetherian ring A is flat if and only if for every prime ideal $\mathfrak{p} \subseteq A$, $\mathrm{Tor}_1^A(M, A/\mathfrak{p}) = 0$.*

Proof. The exactness of the functor $N \mapsto N \otimes_A M$ is equivalent to $\mathrm{Tor}_1(M, N) = 0$ for all A-modules N. Since Tor commutes with direct limits, it is sufficient to require $\mathrm{Tor}_1(M, N) = 0$ for all finitely generated A-modules N. Now over a noetherian ring A, a finitely generated module N has a filtration whose quotients are of the form $A/\mathfrak{p}_i$ for various prime ideals $\mathfrak{p}_i \subseteq A$ [103, p. 51]. Thus, using the exact sequence of Tor, we see that $\mathrm{Tor}_1(M, A/\mathfrak{p}) = 0$ for all $\mathfrak{p}$ implies $\mathrm{Tor}_1(M, N) = 0$ for all N; hence M is flat.

In the sequel, we will often make use of the following result, which is a special case of the "local criterion of flatness."

Proposition 2.2. *Let $A' \to A$ be a surjective homomorphism of noetherian rings whose kernel J has square zero. Then an A'-module M' is flat over A' if and only if*

(1) $M = M' \otimes_{A'} A$ *is flat over A, and*
(2) *the natural map $M \otimes_A J \to M'$ is injective.*

Proof. Note that since J has square zero, it is an A-module and we can identify $M' \otimes_{A'} J$ with $M \otimes_A J$.
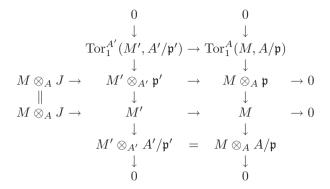
If M' is flat over A', then (1) follows by base extension, and (2) follows by tensoring M' with the exact sequence

$$0 \to J \to A' \to A \to 0.$$

Suppose conversely that M' satisfies conditions (1) and (2). By the lemma, it is sufficient to show that $\mathrm{Tor}_1^{A'}(M', A'/\mathfrak{p}') = 0$ for every prime ideal $\mathfrak{p}' \subseteq A'$. Since J is nilpotent, it is contained in $\mathfrak{p}'$. Letting $\mathfrak{p}$ be the prime ideal $\mathfrak{p}'/J$ of A, we can write a diagram of exact sequences

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & J \to & \mathfrak{p}' & \to & \mathfrak{p} & \to 0 \\
 & \| & \downarrow & & \downarrow & & \\
0 \to & J \to & A' & \to & A & \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & A'/\mathfrak{p}' & = & A/\mathfrak{p} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

Tensoring with M' we obtain

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Tor}_1^{A'}(M', A'/\mathfrak{p}') & \rightarrow & \mathrm{Tor}_1^A(M, A/\mathfrak{p}) \\
\downarrow & & \downarrow
\end{array}
$$

$$
\begin{array}{ccccccc}
M \otimes_A J \rightarrow & M' \otimes_{A'} \mathfrak{p}' & \rightarrow & M \otimes_A \mathfrak{p} & \rightarrow 0 \\
\| & \downarrow & & \downarrow \\
M \otimes_A J \rightarrow & M' & \rightarrow & M & \rightarrow 0 \\
& \downarrow & & \downarrow \\
& M' \otimes_{A'} A'/\mathfrak{p}' & = & M \otimes_A A/\mathfrak{p} \\
& \downarrow & & \downarrow \\
& 0 & & 0
\end{array}
$$

By hypothesis (2), the second (and therefore also the first) horizontal sequence is exact on the left. It follows from the snake lemma that the Tors at the top are isomorphic. The second is zero by hypothesis (1), so the first is also, as required.

Now we consider our first deformation problem, Situation A. Let X be a scheme over k and let Y be a closed subscheme of X. We define a *deformation of Y over D in X* to be a closed subscheme $Y' \subseteq X' = X \times D$, flat over D, such that $Y' \times_D k = Y$. We wish to classify all deformations of Y over D.

We consider the affine case first. Then X corresponds to a k-algebra B, and Y is defined by an ideal $I \subseteq B$. We are seeking ideals $I' \subseteq B' = B[t]/t^2$ with B'/I' flat over D and such that the image of I' in $B = B'/tB'$ is just I. Note that $(B'/I') \otimes_D k = B/I$. Since B is automatically flat over k, by (2.2) the flatness of B'/I' over D is equivalent to the exactness of the sequence

$$0 \to B/I \xrightarrow{t} B'/I' \to B/I \to 0.$$

Suppose I' is such an ideal, and consider the diagram

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 \to & I & \xrightarrow{t} & I' & \rightarrow & I & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to & B & \xrightarrow{t} & B' & \rightarrow & B & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to & B/I & \xrightarrow{t} & B'/I' & \rightarrow & B/I & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow \\
& 0 & & 0 & & 0
\end{array}
$$

where the exactness of the bottom row implies the exactness of the top row.

Proposition 2.3. *In the situation above, to give $I' \subseteq B'$ such that B'/I' is flat over D and the image of I' in B is I is equivalent to giving an element $\varphi \in \operatorname{Hom}_B(I, B/I)$. In particular, $\varphi = 0$ corresponds to the* trivial *deformation given by $I' = I \oplus tI$ inside $B' \cong B \oplus tB$.*

Proof. We will make use of the splitting $B' = B \oplus tB$ as B-modules, or, equivalently, of the section $\sigma : B \to B'$ given by $\sigma(b) = b + 0 \cdot t$, which makes B' into a B-module.

Take any element $x \in I$. Lift it to an element of I', which, using the splitting of B', can be written $x + ty$ for some $y \in B$. Two liftings differ by something of the form tz with $z \in I$. Thus y is not uniquely determined, but its image $\bar{y} \in B/I$ is. Now sending x to $\bar{y}$ defines a mapping $\varphi : I \to B/I$. It is clear from the construction that it is a B-module homomorphism.

Conversely, suppose $\varphi \in \operatorname{Hom}_B(I, B/I)$ is given. Define

$$I' = \{x + ty \mid x \in I, \ y \in B, \text{ and the image of } y \text{ in } B/I \text{ is equal to } \varphi(x)\}.$$

Then one checks easily that I' is an ideal of B', that the image of I' in B is I, and that there is an exact sequence

$$0 \to I \xrightarrow{t} I' \to I \to 0.$$

Therefore there is a diagram as before, where this time the exactness of the top row implies the exactness of the bottom row, and hence that B'/I' is flat over D.

These two constructions are inverse to each other, so we obtain a natural one-to-one correspondence between the set of such I' and the set $\operatorname{Hom}_B(I, B/I)$, whereby the trivial deformation $I' = I \oplus tI$ corresponds to the zero element.

Now we wish to globalize this argument to the case of a scheme X over k and a given closed subscheme Y. There are two ways to do this. One is to cover X with open affine subsets and use the above result. The construction is compatible with localization, and the correspondence is natural, so we get a one-to-one correspondence between the flat deformations $Y' \subseteq X' = X \times D$ and elements of the set $\operatorname{Hom}_X(\mathcal{I}, \mathcal{O}_Y)$, where $\mathcal{I}$ is the ideal sheaf of Y in X.

The other method is to repeat the above proof in the global case, simply dealing with sheaves of ideals and rings, on the topological space of X (which is equal to the topological space of X').

Before stating the conclusion, we will define the normal sheaf of Y in X. Note that the group $\operatorname{Hom}_X(\mathcal{I}, \mathcal{O}_Y)$ can be regarded as $H^0(X, \mathcal{H}om_X(\mathcal{I}, \mathcal{O}_Y))$. Furthermore, homomorphisms of $\mathcal{I}$ to $\mathcal{O}_Y$ factor through $\mathcal{I}/\mathcal{I}^2$, which is a sheaf on Y. So

$$\mathcal{H}om_X(\mathcal{I}, \mathcal{O}_Y) = \mathcal{H}om_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y),$$

and this latter sheaf is called the *normal sheaf* of Y in X, and is denoted by $\mathcal{N}_{Y/X}$. If X is nonsingular and Y is a locally complete intersection in X, then

$\mathcal{I}/\mathcal{I}^2$ is locally free, so $\mathcal{N}_{Y/X}$ is locally free also and can be called the *normal bundle* of Y in X. This terminology derives from the fact that if Y is also nonsingular, there is an exact sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0,$$

where $\mathcal{T}_Y$ and $\mathcal{T}_X$ denote the tangent sheaves to Y and X, respectively. In this case, therefore, $\mathcal{N}_{Y/X}$ is the usual normal bundle.

Summing up our results gives the following.

Theorem 2.4. *Let X be a scheme over a field k, and let Y be a closed subscheme of X. Then the deformations of Y over D in X are in natural one-to-one correspondence with elements of $H^0(Y, \mathcal{N}_{Y/X})$, the zero element corresponding to the trivial deformation.*

Corollary 2.5. *If Y is a closed subscheme of the projective space $X = \mathbb{P}^n_k$, then the Zariski tangent space of the Hilbert scheme H at the point y corresponding to Y is isomorphic to $H^0(Y, \mathcal{N}_{Y/X})$.*

Proof. The Zariski tangent space to H at y can be interpreted as the set of morphisms from the dual numbers D to H sending the closed point to y [57, II, Ex. 2.8]. Because of the universal property of the Hilbert scheme (1.1(a)), this set is in one-to-one correspondence with the set of deformations of Y over the dual numbers, which by (2.4) is $H^0(Y, \mathcal{N}_{Y/X})$.

Next we consider Situation B. Let X be a scheme over k and let $\mathcal{L}$ be an invertible sheaf on X. We will study the set of isomorphism classes of invertible sheaves $\mathcal{L}'$ on $X' = X \times D$ such that $\mathcal{L}' \otimes \mathcal{O}_X \cong \mathcal{L}$. In this case flatness is automatic, because $\mathcal{L}'$ is locally free and X' is flat over D.

Proposition 2.6. *Let X be a scheme over k, and $\mathcal{L}$ an invertible sheaf on X. The set of isomorphism classes of invertible sheaves $\mathcal{L}'$ on $X \times D$ such that $\mathcal{L}' \otimes \mathcal{O}_X \cong \mathcal{L}$ is in natural one-to-one correspondence with elements of the group $H^1(X, \mathcal{O}_X)$.*

Proof. We use the fact that on any ringed space X, the isomorphism classes of invertible sheaves are classified by $H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the sheaf of multiplicative groups of units in $\mathcal{O}_X$ [57, III, Ex. 4.5]. The exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

gives rise to an exact sequence of sheaves of abelian groups

$$0 \to \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_{X'}^* \to \mathcal{O}_X^* \to 0,$$

where $\alpha(x) = 1 + tx$. Here $\mathcal{O}_X$ is an additive group, while $\mathcal{O}_{X'}^*$ and $\mathcal{O}_X^*$ are multiplicative groups, and α is a truncated exponential map. Because the map of rings $D \to k$ has a section $k \to D$, it follows that this latter sequence is

a split exact sequence of sheaves of abelian groups. So taking cohomology we obtain a split exact sequence

$$0 \to H^1(X, \mathcal{O}_X) \to H^1(X', \mathcal{O}_{X'}^*) \to H^1(X, \mathcal{O}_X^*) \to 0.$$

This shows that the set of isomorphism classes of invertible sheaves on X' restricting to a given isomorphism class on X is a coset of the group $H^1(X, \mathcal{O}_X)$. Letting 0 correspond to the trivial extension $\mathcal{L}' = \mathcal{L} \times D$, we obtain the result.

Proceeding to Situation C, we will actually consider a slightly more general set-up. Let X be a scheme over k, and let $\mathcal{F}$ be a coherent sheaf on X. We define a *deformation* of $\mathcal{F}$ over D to be a coherent sheaf $\mathcal{F}'$ on $X' = X \times D$, flat over D, together with a homomorphism $\mathcal{F}' \to \mathcal{F}$ such that the induced map $\mathcal{F}' \otimes_D k \to \mathcal{F}$ is an isomorphism. We say that two such deformations $\mathcal{F}'_1 \to \mathcal{F}$ and $\mathcal{F}'_2 \to \mathcal{F}$ are *equivalent* if there is an isomorphism $\mathcal{F}'_1 \xrightarrow{\sim} \mathcal{F}'_2$ compatible with the given maps to $\mathcal{F}$.

Theorem 2.7. *Let X be a scheme over k, and let $\mathcal{F}$ be a coherent sheaf on X. The (equivalence classes of) deformations of $\mathcal{F}$ over D are in natural one-to-one correspondence with the elements of the group $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$, where the zero-element corresponds to the trivial deformation.*

Proof. By (2.2), the flatness of $\mathcal{F}'$ over D is equivalent to the exactness of the sequence

$$0 \to \mathcal{F} \xrightarrow{t} \mathcal{F}' \to \mathcal{F} \to 0$$

obtained by tensoring $\mathcal{F}'$ with $0 \to k \xrightarrow{t} D \to k \to 0$. Since the latter sequence splits, we have a splitting $\mathcal{O}_X \to \mathcal{O}_{X'}$, and thus we can regard this sequence of sheaves as an exact sequence of $\mathcal{O}_X$-modules. By Yoneda's interpretation of the Ext groups [24, Ex. A3.26], we obtain an element $\xi \in \operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$. Conversely, an element in that Ext group gives $\mathcal{F}'$ as an extension of $\mathcal{F}$ by $\mathcal{F}$ as $\mathcal{O}_X$-modules. To give a structure of an $\mathcal{O}_{X'}$-module on $\mathcal{F}'$ we have to specify multiplication by t. But this can be done in one and only one way compatible with the sequence above and the requirement that $\mathcal{F}' \otimes_D k \cong \mathcal{F}$, namely projection from $\mathcal{F}'$ to $\mathcal{F}$ followed by the injection $t : \mathcal{F} \to \mathcal{F}'$. Note finally that $\mathcal{F}' \to \mathcal{F}$ and $\mathcal{F}'' \to \mathcal{F}$ are equivalent as deformations of $\mathcal{F}$ if and only if the corresponding elements ξ, ξ' are equal. Thus the deformations $\mathcal{F}'$ are in natural one-to-one correspondence with elements of the group $\operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$.

Remark 2.7.1. Given $\mathcal{F}$ on X, we can also pose a different problem, like the one in (2.6), namely to classify isomorphism classes of coherent sheaves $\mathcal{F}'$ on X', flat over D, such that $\mathcal{F}' \otimes_D k$ is isomorphic to $\mathcal{F}$ (without specifying the isomorphism). This set need not be the same as the set of deformations of $\mathcal{F}$, but we can explain their relationship as follows. The group $\operatorname{Aut} \mathcal{F}$ of automorphisms of $\mathcal{F}$ acts on the set of deformations of $\mathcal{F}$ by letting $\alpha \in \operatorname{Aut} \mathcal{F}$

applied to $f : \mathcal{F}' \to \mathcal{F}$ be $\alpha f : \mathcal{F}' \to \mathcal{F}$. Now let $f : \mathcal{F}' \to \mathcal{F}$ and $g : \mathcal{F}'' \to \mathcal{F}$ be two deformations of $\mathcal{F}$. One sees easily that $\mathcal{F}'$ and $\mathcal{F}''$ are isomorphic as sheaves on X' if and only if there exists an $\alpha \in \operatorname{Aut} \mathcal{F}$ such that αf and g are equivalent as deformations of $\mathcal{F}$. Thus the set of $\mathcal{F}'$'s up to isomorphism as sheaves on X' is the orbit space of $\operatorname{Ext}^1_X(\mathcal{F}, \mathcal{F})$ under the action of $\operatorname{Aut} \mathcal{F}$. This kind of subtle distinction will play an important role in questions of pro-representability (Chapter 3).

Corollary 2.8. *If $\mathcal{E}$ is a vector bundle over X, then the deformations of $\mathcal{E}$ over D are in natural one-to-one correspondence with the elements of $H^1(X, \mathcal{E}nd\,\mathcal{E})$, where $\mathcal{E}nd\,\mathcal{E} = \mathcal{H}om(\mathcal{E}, \mathcal{E})$ is the sheaf of endomorphisms of $\mathcal{E}$. The trivial deformation corresponds to the zero element.*

Proof. In this case, since $\mathcal{E}$ is locally free, $\operatorname{Ext}^1(\mathcal{E}, \mathcal{E}) = \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{E}nd\,\mathcal{E}) = H^1(X, \mathcal{E}nd\,\mathcal{E})$.

Remark 2.8.1. If $\mathcal{E}$ is a line bundle, i.e., an invertible sheaf $\mathcal{L}$ on X, then $\mathcal{E}nd\,\mathcal{E} \cong \mathcal{O}_X$, and the deformations of $\mathcal{L}$ are classified by $H^1(\mathcal{O}_X)$. We get the same answer as in (2.6) because $\operatorname{Aut} \mathcal{L} = H^0(\mathcal{O}_X^*)$ and for any $\mathcal{L}'$ invertible on X', $\operatorname{Aut} \mathcal{L}' = H^0(\mathcal{O}_{X'}^*)$. Now $H^0(\mathcal{O}_{X'}^*) \to H^0(\mathcal{O}_X^*)$ is surjective because of the split exact sequence mentioned in the proof of (2.6), and from this it follows that two deformations $\mathcal{L}_1' \to \mathcal{L}$ and $\mathcal{L}_2' \to \mathcal{L}$ are equivalent as deformations of $\mathcal{L}$ if and only if $\mathcal{L}_1'$ and $\mathcal{L}_2'$ are isomorphic as invertible sheaves on X'.

Remark 2.8.2. *Use of the word "natural."* In each of the main results of this section, we have said that a certain set was in *natural* one-to-one correspondence with the set of elements of a certain group. We have not said exactly what we mean by this word natural. So for the time being, you may understand it something like this: If I say there is a natural mapping from one set to another, that means I have a particular construction in mind for that mapping, and if you see my construction, you will agree that it is natural. It does not involve any unnatural choices. Use of the word natural carries with it the expectation (but not the promise) that the same construction carried out in parallel situations will give compatible results. It should be compatible with localization, base-change, etc. However, natural does not mean unique. It is quite possible that someone else could find another mapping between these two sets, different from this one, but also natural from a different point of view.

In contrast to the natural correspondences of this section, we will see later situations in which there are nonnatural one-to-one correspondences. Having fixed one deformation, any other will define an element of a certain group, thus giving a one-to-one correspondence between the set of all deformations and the elements of the group, with the fixed deformation corresponding to the zero element. So there is a one-to-one correspondence, but it depends on the choice of a fixed deformation, and there may be no such choice that is natural, i.e., no one we can single out as a "trivial" deformation. In this case

we say that the set is a principal homogeneous space or torsor under the action of the group—cf. §6 for examples.

References for this section. The notion of flatness is due to Serre [153], who showed that there is a one-to-one correspondence between coherent algebraic sheaves on a projective variety over $\mathbb{C}$ and the coherent analytic sheaves on the associated complex analytic space. He observed that the algebraic and analytic local rings have the same completion, and that this makes them a "flat couple." The observation that localization and completion both enjoy this property, and that flat modules are those that are acyclic for the Tor functors, explained and simplified a number of situations by combining them into one concept. Then in the hands of Grothendieck, flatness became a central tool for managing families of structures of all kinds in algebraic geometry. The local criterion of flatness is developed in [47, IV, §5]. Our statement is [loc. cit., 5.5]. A note before [loc. cit. 5.2] says "La proposition suivante a été dégagée au moment du Séminaire par Serre; elle permet des simplifications substantielles dans le présent numéro."

The infinitesimal study of the Hilbert scheme is in Grothendieck's Bourbaki seminar [45, exposé 221].

Exercises.

2.1.   If X is a scheme with $H^1(\mathcal{O}_X) = 0$, then by (2.6) there are no nontrivial extensions of an invertible sheaf to a deformation of X over the dual numbers. This suggests that perhaps there are no global nontrivial families either. Indeed this is true with the following hypotheses. Let X be an integral projective scheme over k with $H^1(X, \mathcal{O}_X) = 0$. Let T be a connected scheme with a closed point t_0. Let $\mathcal{L}$ be an invertible sheaf on $X \times T$, and let $\mathcal{L}_0 = \mathcal{L} \otimes \mathcal{O}_{X_0}$ be the restriction of $\mathcal{L}$ to the fiber $X_0 = X \times k(t_0)$ over t_0. Show then that there is an invertible sheaf $\mathcal{M}$ on T such that $\mathcal{L} \cong p_1^* \mathcal{L}_0 \otimes p_2^* \mathcal{M}$. In particular, all the fibers of $\mathcal{L}$ over points of T are isomorphic. (*Hint:* Use [57, III, Ex. 12.6].)

2.2. The Jacobian of an elliptic curve.   Let C be an elliptic curve over k, that is, a nonsingular projective curve of genus 1 with a fixed point P_0. Then any invertible sheaf $\mathcal{L}$ of degree 0 on C is isomorphic to $\mathcal{O}_C(P - P_0)$ for a uniquely determined point $P \in C$. Thus the curve C itself acts as a parameter space for the group $\mathrm{Pic}^0(C)$ of invertible sheaves of degree 0, and as such is called the *Jacobian variety* J of C. Describe explicitly the functorial properties of J as a classifying space and thus justify the identification of the one-dimensional space $H^1(\mathcal{O}_C)$ with the Zariski tangent space to J at any point (cf. [57, III, §4]).

2.3. Vector bundles on $\mathbb{P}^1$.   One knows that every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles $\mathcal{O}(a_i)$ for various $a_i \in \mathbb{Z}$ [57, V, Ex. 2.6]. Thus the set of isomorphism classes of vector bundles of given rank and degree is a discrete set. Nevertheless, there are nontrivial deformations of bundles on $\mathbb{P}^1$. Let $\mathcal{E}_0 = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ and show that $H^1(\mathbb{P}^1, \mathcal{E}nd\,\mathcal{E}_0)$ has dimension one. A nontrivial family containing $\mathcal{E}_0$ is given by the extensions

$$0 \to \mathcal{O}(-1) \to \mathcal{E}_t \to \mathcal{O}(1) \to 0$$

for $t \in \mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(-1)) = H^1(\mathcal{O}(-2))$. Show that for $t \neq 0$, $\mathcal{E}_t \cong \mathcal{O} \oplus \mathcal{O}$, while for $t = 0$ we get $\mathcal{E}_0$.

2.4. Rank 2 bundles on an elliptic curve.   Let C be an elliptic curve. Let $\mathcal{E}$ be a rank 2 vector bundle obtained as a nonsplit extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C(P) \to 0$$

for some point $P \in C$.

(a) Show that $\mathcal{E}$ is *normalized* in the sense that $H^0(\mathcal{E}) \neq 0$, but for any invertible sheaf $\mathcal{L}$ with $\deg \mathcal{L} < 0$, $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$. Show also that $\mathcal{E}$ is uniquely determined by P, up to isomorphism.
(b) Show that $h^0(\mathcal{E}) = 1$ and $h^1(\mathcal{E}nd\,\mathcal{E}) = 1$.
(c) Show that any normalized rank 2 vector bundle of degree 1 on C is isomorphic to an $\mathcal{E}$ as above, for a uniquely determined point $P \in C$. Thus the family of all such bundles is parametrized by the curve C, consistent with the calculation $h^1(\mathcal{E}nd\,\mathcal{E}) = 1$.

2.5. A line bundle and its associated divisor.   Let X be an integral projective scheme. Let $\mathcal{L}$ be an invertible sheaf on X, let $s \in H^0(\mathcal{L})$ be a global section, and let $Y = (s)_0$ be the associated divisor on X. We wish to compare deformations of $\mathcal{L}$ as an invertible sheaf on X with deformations of Y as a closed subscheme of X.

(a) Show that the normal bundle of Y in X is isomorphic to $\mathcal{L}_Y = \mathcal{L} \otimes \mathcal{O}_Y$. Then use the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{L} \to \mathcal{L}_Y \to 0$$

to obtain a long exact sequence of cohomology

$$0 \to H^0(\mathcal{O}_X) \xrightarrow{s} H^0(\mathcal{L}) \xrightarrow{\alpha} H^0(\mathcal{L}_Y) \xrightarrow{\beta} H^1(\mathcal{O}_X) \xrightarrow{\gamma} H^1(\mathcal{L}) \to \cdots.$$

We interpret this as follows. The image of α corresponds to deformations of Y within the linear system $|Y|$. The map β gives the deformation of $\mathcal{L}$ associated to a deformation of Y. If the map γ is nonzero, then some deformations of $\mathcal{L}$ may not come from a deformation of Y, because the section s does not lift to the deformation of $\mathcal{L}$.
(b) For an example of this latter situation, let X be a nonsingular projective curve of genus $g \geq 2$, let $P \in X$ be a point, and let $\mathcal{L} = \mathcal{O}_X(P)$. If Q is another point, we can consider the family of invertible sheaves $\mathcal{L}_Q = \mathcal{O}_X(2P - Q)$. For $Q = P$ we recover $\mathcal{L}$. For $Q \neq P$, the sheaf $\mathcal{L}_Q$ has no global sections (assuming $2P$ is not in the linear system g_2^1 if X is hyperelliptic). In this case the sheaf deforms, but the section does not.
(c) The exact sequence in (a) shows that if $H^1(\mathcal{L}) = 0$, then for any lifting $\mathcal{L}'$ of $\mathcal{L}$ over the dual numbers, the section s lifts to a section of $\mathcal{L}'$. A corresponding global result also holds: Changing notation, let $\mathcal{L}$ be an invertible sheaf on $X \times T$ for some scheme T, let $\mathcal{L}_0$ be the restriction to the fiber over a point $t_0 \in T$, and assume that $H^1(X, \mathcal{L}_0) = 0$. Show that $p_{2*}\mathcal{L}$ is locally free on T, so that every section of $\mathcal{L}_0$ on X extends to a section of $\mathcal{L}$ over some neighborhood of $t_0 \in T$. (*Hint:* Use the theorem of cohomology and base change [57, III, 12.11].)

2.6. Rank 2 vector bundles on $\mathbb{P}^3$. Let $\mathcal{E}$ be a rank 2 vector bundle on $\mathbb{P}^3$, let s be a section of $H^0(\mathcal{E})$ that does not vanish on any divisor, and let $Y = (s)_0$ be the curve of zeros of s. Then there is an exact sequence

$$0 \to \mathcal{O} \xrightarrow{s} \mathcal{E} \to \mathcal{I}_Y(a) \to 0,$$

where $a = c_1(\mathcal{E})$ is the first Chern class of $\mathcal{E}$. We wish to compare deformations of $\mathcal{E}$ with deformations of the closed subscheme Y in $\mathbb{P}^3$.

(a) Show that the normal bundle of Y in $\mathbb{P}^3$ is $\mathcal{E}_Y = \mathcal{E} \otimes \mathcal{O}_Y$. (Note that since $\mathcal{E}$ has rank 2, its dual $\mathcal{E}^\vee$ is isomorphic to $\mathcal{E}(-a)$.)
(b) Show that there are exact sequences

$$0 \to \mathcal{E}^\vee \to \mathcal{E}nd\,\mathcal{E} \to \mathcal{E} \otimes \mathcal{I}_Y \to 0$$

and

$$0 \to \mathcal{E} \otimes \mathcal{I}_Y \to \mathcal{E} \to \mathcal{E}_Y \to 0$$

from which one can obtain exact sequences of cohomology

$$\to H^1(\mathcal{E}^\vee) \to H^1(\mathcal{E}nd\,\mathcal{E}) \to H^1(\mathcal{E} \otimes \mathcal{I}_Y) \to H^2(\mathcal{E}^\vee) \to \cdots$$
$$\|$$
$$\to\ H^0(\mathcal{E})\ \to\ H^0(\mathcal{E}_Y)\ \to H^1(\mathcal{E} \otimes \mathcal{I}_Y) \to\ H^1(\mathcal{E})\ \to \cdots .$$

Here $H^1(\mathcal{E}nd\,\mathcal{E})$ represents deformations of $\mathcal{E}$, and $H^0(\mathcal{E}_Y)$ represents deformations of Y in $\mathbb{P}^3$. In general a deformation of one may not correspond to a deformation of the other.
(c) Now consider a particular case, the so-called *null-correlation bundle* on $\mathbb{P}^3$. It belongs to a sequence

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_Y(2) \to 0,$$

where Y is a disjoint union of two lines in $\mathbb{P}^3$. For existence of such bundles, show that $\mathrm{Ext}^1(\mathcal{I}_Y(2), \mathcal{O}) \cong \mathrm{Ext}^2(\mathcal{O}_Y(2), \mathcal{O}) \cong H^0(\mathcal{O}_Y)$, so that an extension as above may be determined by choosing two scalars, one for each of the two lines in Y.
(d) For the bundles in (c) verify that $h^0(\mathcal{E}nd\,\mathcal{E}) = 1$, $h^1(\mathcal{E}nd\,\mathcal{E}) = 5$; $h^0(\mathcal{E}) = 5$, $h^0(\mathcal{E}_Y) = 8$, $h^1(\mathcal{E} \otimes \mathcal{I}_Y) = 4$ and $h^1(\mathcal{E}) = h^2(\mathcal{E}^\vee) = 0$. So in this case, any deformation of $\mathcal{E}$ corresponds to a deformation of Y and vice versa. In fact, there is a 5-dimensional global family of such bundles, parametrized by $\mathbb{P}^5$ minus the four-dimensional Grassmann variety $G(1, 3)$ of lines in $\mathbb{P}^3$ [58, 8.4.1], consistent with the calculation that $h^1(\mathcal{E}nd\,\mathcal{E}) = 5$.

3. The T^i Functors

In this section we will present the construction and main properties of the T^i functors introduced by Lichtenbaum and Schlessinger [96]. For any ring homomorphism $A \to B$ and any B-module M they define functors $T^i(B/A, M)$, for $i = 0, 1, 2$. With A and B fixed these form a cohomological functor in M, giving a nine-term exact sequence associated to a short exact

sequence of modules $0 \to M' \to M \to M'' \to 0$. On the other hand, if $A \to B \to C$ are three rings and homomorphisms, and if M is a C-module, then there is a nine-term exact sequence of T^i functors associated with the three ring homomorphisms $A \to B$, $A \to C$, and $B \to C$. The principal application of these functors for us is the study of deformations of rings and schemes (Situation D). We will see that deformations of a ring are classified by a certain T^1 group (§5), and that obstructions lie in a certain T^2 group (§10). We will also see in §4 that the vanishing of the T^1 functor characterizes smooth morphisms and the vanishing of the T^2 functor characterizes locally complete intersection morphisms.

Construction 3.1. Let $A \to B$ be a homomorphism of rings and let M be a B-module. Here we will construct the groups $T^i(B/A, M)$ for $i = 0, 1, 2$. The rings are assumed to be commutative with identity, but we do not impose any finiteness conditions yet.

First choose a polynomial ring $R = A[x]$ in a set of variables $x = \{x_i\}$ (possibly infinite) such that B can be written as a quotient of R as an A-algebra. Let I be the ideal defining B, so that there is an exact sequence

$$0 \to I \to R \to B \to 0.$$

Second choose a free R-module F and a surjection $j : F \to I \to 0$ and let Q be the kernel:

$$0 \to Q \to F \xrightarrow{j} I \to 0.$$

Having chosen R and F as above, the construction proceeds with no further choices. Let F_0 be the submodule of F generated by all "Koszul relations" of the form $j(a)b - j(b)a$ for $a, b \in F$. Note that $j(F_0) = 0$ so $F_0 \subseteq Q$.

We define a complex of B-modules, called the *cotangent complex*,

$$L_2 \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0$$

as follows. Take $L_2 = Q/F_0$. Why is L_2 a B-module? A priori it is an R-module. But if $x \in I$ and $a \in Q$, we can write $x = j(x')$ for some $x' \in F$ and then $xa = j(x')a \equiv j(a)x' \pmod{F_0}$. But $j(a) = 0$, since $a \in Q$, so we see that $xa = 0$. Therefore L_2 is a B-module.

Take $L_1 = F \otimes_R B = F/IF$, and let $d_2 : L_2 \to L_1$ be the map induced from the inclusion $Q \to F$.

Take $L_0 = \Omega_{R/A} \otimes_R B$, where $\Omega_{R/A}$ is the module of relative differentials. To define d_1 just map L_1 to I/I^2, then apply the derivation $d : R \to \Omega_{R/A}$, which induces a B-module homomorphism $I/I^2 \to L_0$.

Clearly $d_1 d_2 = 0$, so we have defined a complex of B-modules. Note also that L_1 and L_0 are *free* B-modules: L_1 is free because it is defined from the free R-module F; L_0 is free because R is a polynomial ring over A and so $\Omega_{R/A}$ is a free R-module.

For any B-module M we now define the modules

$$T^i(B/A, M) = h^i(\operatorname{Hom}_B(L_\bullet, M))$$

as the cohomology modules of the complex of homomorphisms of the complex $L_\bullet$ into M.

To show that these modules are well-defined (up to isomorphism), we must verify that they are independent of the choices made in the construction.

Lemma 3.2. *The modules $T^i(B/A, M)$ constructed above are independent of the choice of F (keeping R fixed).*

Proof. If F and F' are two choices of free R-modules mapping onto I, then $F \oplus F'$ is a third choice, so by symmetry it is sufficient to compare F with $F \oplus F'$. Since F' is free, the map $j' : F' \to I$ factors through F, i.e., $j' = jp$ for some map $p : F' \to F$. Changing bases in $F \oplus F'$, replacing each generator e' of F' by $e' - p(e')$, we may assume that the map $F \oplus F' \to I$ is just j on the first factor and 0 on the second factor. Thus we have the diagram

$$
\begin{array}{ccccccccc}
0 \to & Q \oplus F' & \to & F \oplus F' & \overset{(j,0)}{\to} & I & & \to 0 \\
& \downarrow & & \downarrow pr_1 & & \downarrow id & & \\
0 \to & Q & \to & F & \overset{j}{\to} & I & & \to 0
\end{array}
$$

showing that the kernel of $(j,0) : F \oplus F' \to I$ is just $Q \oplus F'$. Then clearly $(F \oplus F')_0 = F_0 + IF'$. Denoting by $L'_\bullet$ the complex obtained from the new construction, we see that $L'_2 = L_2 \oplus F'/IF'$, $L'_1 = L_1 \oplus (F' \otimes_R B)$, and $L'_0 = L_0$. Since $F' \otimes_R B = F'/IF'$ is a free B-module, the complex $L'_\bullet$ is obtained by taking the direct sum of $L_\bullet$ with the free acyclic complex $F' \otimes_R B \to F' \otimes_R B$. Hence when we take Hom of these complexes into M and then cohomology, the result is the same.

Lemma 3.3. *The modules $T^i(B/A, M)$ are independent of the choice of R.*

Proof. Let $R = A[x]$ and $R' = A[y]$ be two choices of polynomial rings with surjections to B. As in the previous proof, it will be sufficient to compare R with $R'' = A[x, y]$. Furthermore, the map $A[y] \to B$ can be factored through $A[x]$ by a homomorphism $p : A[y] \to A[x]$. Then, changing variables in $A[x, y]$, replacing each y_i by $y_i - p(y_i)$, we may assume that all the y_i go to zero in the ring homomorphism $A[x, y] \to B$. Then we have the diagram

$$
\begin{array}{ccccccc}
0 \to & IR'' + yR'' & \to & R'' & \to & B & \to 0 \\
& \uparrow & & \uparrow\downarrow p & & \uparrow id & \\
0 \to & I & & \to R & \to & B & \to 0
\end{array}
$$

showing that the kernel of $R'' \to B$ is generated by I and all the y-variables.

Since we have already shown that the construction is independent of the choice of F, we may use any F's we like in the present proof. Take any free

R-module F mapping surjectively to I. Take F' a free R''-module on the same number of generators as F, and take G' a free R''-module on the index set of the y variables. Then we have

$$
\begin{array}{ccccccccc}
0 & \to & Q' & \to & F' \oplus G' & \to & IR'' + yR'' & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & Q & \to & F & \to & I & \to & 0
\end{array}
$$

Observe that since the y_i are independent variables, and G' has a basis e_i going to y_i, the kernel Q' in the upper row must be generated by (1) things in Q, (2) things of the form $y_i a - j(a)e_i$ with $a \in F$, and (3) things of the form $y_i e_j - y_j e_i$. Clearly the elements of types (2) and (3) are in $(F' \oplus G')_0$. Therefore $Q'/(F' \oplus G')_0$ is a B-module generated by the image of Q, so $L_2 = L_2'$.

On the other hand, $L_1' = L_1 \oplus (G' \otimes_{R'} B)$, and $L_0' = L_0 \oplus (\Omega_{A[y]/A} \otimes B)$. Thus L_1' has an extra free B-module generated by the e_i, L_0' has an extra free B-module generated by the dy_i, and the map d_1 takes e_i to dy_i. As in the previous proof we see that $L_\bullet'$ is obtained from $L_\bullet$ by adding a free acyclic complex, and hence the modules $T^i(B/A, M)$ are the same.

Remark 3.3.1. Even though the complex $L_\bullet$ is not unique, the proofs of (3.2) and (3.3) show that it gives a well-defined element of the derived category of the category of B-modules.

Theorem 3.4. *Let $A \to B$ be a homomorphism of rings. Then for $i = 0, 1, 2$, $T^i(B/A, \cdot)$ is a covariant, additive functor from the category of B-modules to itself. If*

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of B-modules, then there is a long exact sequence

$$
\begin{aligned}
0 \to T^0(B/A, M') &\to T^0(B/A, M) \to T^0(B/A, M'') \to \\
\to T^1(B/A, M') &\to T^1(B/A, M) \to T^1(B/A, M'') \to \\
\to T^2(B/A, M') &\to T^2(B/A, M) \to T^2(B/A, M'').
\end{aligned}
$$

In the language of [57, III, §1], the T^i's form a truncated δ-functor.

Proof. We have seen that the $T^i(B/A, M)$ are well-defined. By construction they are covariant additive functors. Given a short exact sequence of modules as above, since the terms L_1 and L_0 of the complex $L_\bullet$ are free, we get a sequence of complexes

$$0 \to \mathrm{Hom}_B(L_\bullet, M') \to \mathrm{Hom}_B(L_\bullet, M) \to \mathrm{Hom}_B(L_\bullet, M'') \to 0$$

that is exact except possibly for the map

$$\mathrm{Hom}_B(L_2, M) \to \mathrm{Hom}_B(L_2, M''),$$

which may not be surjective. This sequence of complexes gives the long exact sequence of cohomology above. Note that since the complex $L_\bullet$ is unique up to adding free acyclic complexes, the coboundary maps of the long exact sequence are also functorial.

Theorem 3.5. *Let $A \to B \to C$ be rings and homomorphisms, and let M be a C-module. Then there is an exact sequence of C-modules*

$$0 \to T^0(C/B, M) \to T^0(C/A, M) \to T^0(B/A, M) \to$$
$$\to T^1(C/B, M) \to T^1(C/A, M) \to T^1(B/A, M) \to$$
$$\to T^2(C/B, M) \to T^2(C/A, M) \to T^2(B/A, M).$$

Proof. To prove this theorem, we will show that for suitable choices in the construction (3.1), the resulting complexes form a sequence

$$0 \to L_\bullet(B/A) \otimes_B C \to L_\bullet(C/A) \to L_\bullet(C/B) \to 0$$

that is split exact on the degree 0 and 1 terms, and right exact on the degree 2 terms. Given this, taking $\mathrm{Hom}(\cdot, M)$ will give a sequence of complexes that is exact on the degree 0 and 1 terms, and left exact on the degree 2 terms. Taking cohomology will give the nine-term exact sequence above.

First choose a surjection $A[x] \to B \to 0$ with kernel I, and a surjection $F \to I \to 0$ with kernel Q, where F is a free $A[x]$-module, to calculate the functors $T^i(B/A, M)$.

Next choose a surjection $B[y] \to C \to 0$ with kernel J, and a surjection $G \to J \to 0$ of a free $B[y]$-module G with kernel P, to calculate $T^i(C/B, M)$.

To calculate the functors T^i for C/A, take a polynomial ring $A[x, y]$ in the x-variables and the y-variables. Then $A[x, y] \to B[y] \to C$ gives a surjection of $A[x, y] \to C$. If K is its kernel then there is an exact sequence

$$0 \to I[y] \to K \to J \to 0,$$

where $I[y]$ denotes polynomials in y with coefficients in I. Take F' and G' to be free $A[x, y]$-modules on the same index sets as F and G respectively. Choose a lifting of the map $G \to J$ to a map $G' \to K$. Then adding the natural map $F' \to K$ we get a surjection $F' \oplus G' \to K$. Let S be its kernel:

$$0 \to S \to F' \oplus G' \to K \to 0.$$

Now we are ready to calculate. Out of the choices thus made there are induced maps of complexes

$$L_\bullet(B/A) \otimes_B C \to L_\bullet(C/A) \to L_\bullet(C/B).$$

On the degree 0 level we have

$$\Omega_{A[x]/A} \otimes C \to \Omega_{A[x,y]/A} \otimes C \to \Omega_{B[y]/B} \otimes C.$$

These are free C-modules with bases $\{dx_i\}$ on the left, $\{dy_i\}$ on the right, and $\{dx_i, dy_i\}$ in the middle. So this sequence is clearly split exact.

On the degree 1 level we have

$$F \otimes C \to (F' \oplus G') \otimes C \to G \otimes C,$$

which is split exact by construction.

On the degree 2 level we have

$$(Q/F_0) \otimes_B C \to S/(F' \oplus G')_0 \to P/G_0.$$

The right-hand map is surjective because the map $S \to P$ is surjective. Clearly the composition of the two maps is 0. We make no claim of injectivity for the left-hand map. So to complete our proof it remains only to show exactness in the middle.

Let $s = f' + g'$ be an element of S, and assume that its image in P is contained in G_0. We must show that s can be written as a sum of something in $(F' \oplus G')_0$ and something in the image of $Q[y]$. In the map $S \to P$, the element f' goes to 0. Let g be the image of g'. Then $g \in G_0$, so g can be written as a linear combination of expressions $j(a)b - j(b)a$ with $a, b \in G$. Lift a, b to elements a', b' in G'. Then the expressions $j(a')b' - j(b')a'$ are in S. Let g'' be g' minus a linear combination of these expressions $j(a')b' - j(b')a'$. We get a new element $s' = f' + g''$ in S, differing from s by something in $(F' \oplus G')_0$, and where now g'' is in the kernel of the map $G' \to G$, which is IG'. So we can write g'' as a sum of elements xh with $x \in I$ and $h \in G'$. Let $x' \in F$ map to x by j. Then $xh = j(x')h \equiv j(h)x' (\bmod F_0)$. Therefore $s' \equiv f' + \Sigma j(h)x' (\bmod(F' \oplus G')_0)$, and this last expression is in $F' \cap S$, and therefore is in $Q[y]$.

Now we will give some special cases and remarks concerning these functors.

Proposition 3.6. *For any* $A \to B$ *and any* M, $T^0(B/A, M) = \operatorname{Hom}_B(\Omega_{B/A}, M) = \operatorname{Der}_A(B, M)$. *In particular,* $T^0(B/A, B) = \operatorname{Hom}_B(\Omega_{B/A}, B)$ *is the tangent module* $T_{B/A}$ *of* B *over* A.

Proof. Write B as a quotient of a polynomial ring R, with kernel I. Then there is an exact sequence [57, II, 8.4A]

$$I/I^2 \overset{d}{\to} \Omega_{R/A} \otimes_R B \to \Omega_{B/A} \to 0.$$

Since $F \to I$ is surjective, there is an induced surjective map $L_1 \to I/I^2 \to 0$. Thus the sequence

$$L_1 \to L_0 \to \Omega_{B/A} \to 0$$

is exact. Taking $\operatorname{Hom}(\cdot, M)$, which is left exact, we see that $T^0(B/A, M) = \operatorname{Hom}_B(\Omega_{B/A}, M)$.

Proposition 3.7. *If B is a polynomial ring over A, then $T^i(B/A, M) = 0$ for $i = 1, 2$ and for all M.*

Proof. In this case we can take $R = B$ in the construction. Then $I = 0$, $F = 0$, so $L_2 = L_1 = 0$, and the complex $L_\bullet$ is reduced to the L_0 term. Therefore $T^i = 0$ for $i = 1, 2$ and any M.

Remark 3.7.1. We will see later that the vanishing of the T^1 functor characterizes smooth morphisms (4.11).

Proposition 3.8. *If $A \to B$ is a surjective ring homomorphism with kernel I, then $T^0(B/A, M) = 0$ for all M, and $T^1(B/A, M) = \operatorname{Hom}_B(I/I^2, M)$. In particular, $T^1(B/A, B) = \operatorname{Hom}_B(I/I^2, B)$ is the normal module $N_{B/A}$ of $\operatorname{Spec} B$ in $\operatorname{Spec} A$.*

Proof. In this case we can take $R = A$, so that $L_0 = 0$. Thus $T^0 = 0$ for any M. Furthermore, the exact sequence

$$0 \to Q \to F \to I \to 0,$$

tensored with B, gives an exact sequence

$$Q \otimes_A B \to F \otimes_A B \to I/I^2 \to 0.$$

There is also a surjective map $Q \otimes_A B \to Q/F_0$, since the latter is a B-module, so we have an exact sequence

$$L_2 \to L_1 \to I/I^2 \to 0.$$

Taking $\operatorname{Hom}(\cdot, M)$ shows that $T^1(B/A, M) = \operatorname{Hom}_B(I/I^2, M)$.

A useful special case is the following.

Corollary 3.9. *If A is a local ring and B is a quotient A/I, where I is generated by a regular sequence $a_1, \dots, a_r$, then $T^2(B/A, M) = 0$ for all M.*

Proof. Indeed, in this case, since the Koszul complex of a regular sequence is exact [104, 16.5], we find $Q = F_0$ in the construction of the T^i-functors. Thus $L_2 = 0$ and $T^2(B/A, M) = 0$ for all M.

Remark 3.9.1. We will see later that the vanishing of the T^2 functor characterizes relative local complete intersection morphisms (4.13).

Another useful special case is given by the following proposition.

Proposition 3.10. *Suppose $A = k[x_1, \dots, x_n]$ and $B = A/I$. Then for any M there is an exact sequence*

$$0 \to T^0(B/k, M) \to \operatorname{Hom}(\Omega_{A/k}, M) \to \operatorname{Hom}(I/I^2, M) \to T^1(B/k, M) \to 0$$

and an isomorphism

$$T^2(B/A, M) \xrightarrow{\sim} T^2(B/k, M).$$

Proof. Write the long exact sequence of T^i-functors for the composition $k \to A \to B$ and use (3.6), (3.7), and (3.8). The same works for any base ring k, not necessarily a field.

Remark 3.10.1. Throughout this section we have not made any finiteness assumptions on the rings and modules. However, it is easy to see that if A is a noetherian ring, B a finitely generated A-algebra, and M a finitely generated B-module, then the B-modules $T^i(B/A, M)$ are also finitely generated. Indeed, we can take R to be a polynomial ring in finitely many variables over A, which is therefore noetherian. Then I is finitely generated and we can take F to be a finitely generated R-module. Thus the complex $L_\bullet$ consists of finitely generated B-modules, whence the result.

Notation. In the sequel we will often denote the modules $T^i(B/A, B)$ and $T^i(B/k, B)$ by $T^i_{B/A}$ and $T^i_{B/k}$, or even T^i_B, if there is no confusion as to the base. Furthermore $T^0_{B/A}$ will be written $T_{B/A}$, the tangent module of B over A. Similarly for the sheaves $\mathcal{T}^i(X/Y, \mathcal{O}_X)$ and $\mathcal{T}^i(X/k, \mathcal{O}_X)$ (see (Ex. 3.5)), we will write $\mathcal{T}^i_{X/Y}$ and $\mathcal{T}^i_{X/k}$, or even $\mathcal{T}^i_X$. The sheaf $\mathcal{T}^0_X$ will be written $\mathcal{T}_X$, the tangent sheaf of X.

References for this section. The development of the T^i functors presented here is due to Lichtenbaum and Schlessinger [96]. A more general cohomology theory for commutative rings, extending the definition to functors T^i for all $i \geq 0$ has been developed independently by André and Quillen. Quillen states [138] that the T^1 and T^2 functors of Lichtenbaum and Schlessinger are the same as those defined more generally, though I have not seen a direct proof of this fact. Later Illusie [73] globalized those theories by constructing the cotangent complex of a morphism of schemes. This has been extended to stacks in [93]. Independently, Laudal [92] gave another globalization of André's cohomology of algebras. For a computational approach with many examples, see [160].

Exercises.

3.1. Let $B = k[x, y]/(xy)$. Show that $T^1(B/k, M) = M \otimes k$ and $T^2(B/k, M) = 0$ for any B-module M.

3.2. More generally, if $B = k[x, y]/(f)$, then $T^1(B/k, M) = M/(f_x, f_y)M$ for any M, where f_x and f_y are the partial derivatives of f with respect to x and y.

3.3. Let $B = k[x, y]/(x^2, xy, y^2)$. Show that $T^0(B/k, B) = k^4$, $T^1(B/k, B) = k^4$, and $T^2(B/k, B) = k$.

3.4. Let B be a finitely generated integral domain over an algebraically closed field k, and let M be a torsion-free B-module. Show that $T^1(B/k, M) = \mathrm{Ext}^1_B(\Omega^1_{B/k}, M)$. *Hint:* Compare the exact sequence of (3.10) with an exact sequence arising from the cotangent sequence [57, II, 8.4A], and use the fact that $\mathrm{Spec}\, B$ has a dense open subset that is nonsingular.

3.5. Localization.   Show that the construction of the T^i functors is compatible with localization, and thus define sheaves $\mathcal{T}^i(X/Y, \mathcal{F})$ for any morphism of schemes $f : X \to Y$ and any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, such that for any open affine $V \subseteq Y$ and any open affine $U \subseteq f^{-1}(V)$, where $\mathcal{F} = \tilde{M}$, the sections of $\mathcal{T}^i(X/Y, \mathcal{F})$ over U give $T^i(U/V, M)$. Show that the $\mathcal{T}^i$ sheaves satisfy a long exact sequence for change of $\mathcal{F}$ analogous to (3.4) and a long exact sequence analogous to (3.5) for change of schemes. Note that since the complexes $L_\bullet$ used to define the T^i functors are not unique, one cannot in general define an analogous complex of sheaves $\mathcal{L}_\bullet$ on X by this method.

3.6. Global construction of $\mathcal{T}^i$ sheaves.   Let $f : X \to Y$ be a projective morphism, so that X can be realized as a closed subscheme of the projective space $\mathbb{P}_Y^n$ over Y for some n. Show that in this case one can define a global complex $\mathcal{L}_\bullet$ of sheaves on X such that for any $\mathcal{O}_X$-module $\mathcal{F}$, the sheaf $\mathcal{T}^i$ functors can be computed as $h^i(\mathcal{H}om_X(\mathcal{L}_\bullet, \mathcal{F}))$.

3.7. Base change I.   Assume A noetherian, B a finitely generated A-algebra, and M a B-module. Let $A \to A'$ be a flat morphism, and let $B' = B \otimes_A A'$ and $M' = M \otimes_B B'$ be obtained by base extension. Show that $T^i(B/A, M) \otimes_A A' \cong T^i(B'/A', M')$ for each i.

3.8. Base change II.   Again with A noetherian, B finitely generated, and $A \to A'$ a base extension, this time assume that B is flat over A. Let $B' = B \otimes_A A'$, and let M' be a B'-module. Show that $T^i(B/A, M') = T^i(B'/A', M')$ for each i.

4. The Infinitesimal Lifting Property

In this section we first review the properties of nonsingular varieties. Then we show that nonsingularity can be characterized by an "infinitesimal lifting property" that is closely related to deformation theory. We also show that nonsingular varieties and smooth morphisms are characterized by the vanishing of the T^1 functors, and that local complete intersections are characterized by the vanishing of the T^2 functors. As a matter of terminology, we will use the word "nonsingular" only for varieties over an algebraically closed field. Otherwise we talk of a "smooth morphism," or a scheme "smooth" over a base scheme. If the base scheme is an algebraically closed field, the two notions coincide.

Let us consider a scheme X of finite type over an algebraically closed ground field k. After the affine space $\mathbb{A}_k^n$ and the projective space $\mathbb{P}_k^n$, the nicest kind of scheme is a nonsingular one. The property of being nonsingular can be defined extrinsically on open affine pieces by the Jacobian criterion [57, I, §5]. Let Y be a closed subscheme of $\mathbb{A}^n$, with $\dim Y = r$. Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ be a set of generators for the ideal I_Y of Y. Then Y is *nonsingular* at a closed point $P \in Y$ if the rank of the Jacobian matrix $\|(\partial f_i/\partial x_j)(P)\|$ is equal to $n - r$. We say that Y is nonsingular if it is nonsingular at every closed point. A scheme X is *nonsingular* if it can be covered by open affine subsets that are nonsingular.

This definition is awkward, because it is not obvious that the property of being nonsingular is independent of the affine embedding used in the definition. For this reason it is useful to have an intrinsic criterion for nonsingularity.

Proposition 4.1. *A scheme X of finite type over an algebraically closed field k is nonsingular if and only if the local ring $\mathcal{O}_{P,X}$ is a regular local ring for every point $P \in X$ [57, I, 5.1; II, 8.14A].*

Using differentials we have another characterization of nonsingular varieties.

Proposition 4.2. *Let X be a scheme over k algebraically closed. Then X is nonsingular if and only if the sheaf of differentials $\Omega^1_{X/k}$ is locally free of rank $n = \dim X$ at every point of X [57, II, 8.15].*

This result is closely related to the original definition using the Jacobian criterion. The generalization of the Jacobian criterion describes when a closed subscheme Y of a nonsingular scheme X over k is nonsingular.

Proposition 4.3. *Let Y be an irreducible closed subscheme of a nonsingular scheme X over k algebraically closed, defined by a sheaf of ideals $\mathcal{I}$. Then Y is nonsingular if and only if*

(1) $\Omega_{Y/k}$ is locally free, and
(2) the sequence of differentials [57, II, 8.12]

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{X/k} \otimes \mathcal{O}_Y \to \Omega^1_{Y/k} \to 0$$

is exact on the left.
    Furthermore, in this case $\mathcal{I}$ is locally generated by $n - r = \dim X - \dim Y$ elements, and $\mathcal{I}/\mathcal{I}^2$ is locally free on Y of rank $n - r$ [57, II, 8.17].

In this section we will see that nonsingular schemes have a special property related to deformation theory, called the infinitesimal lifting property. The general question is this. Suppose we are given a morphism $f : Y \to X$ of schemes and an *infinitesimal thickening* $Y \subseteq Y'$. This means that Y is a closed subscheme of another scheme Y', and that the ideal $\mathcal{I}$ defining Y inside Y' is nilpotent. Then the question is, does there exist a *lifting* $g : Y' \to X$, i.e., a morphism such that g restricted to Y is f? Of course, there is no reason for this to hold in general, but we will see that if Y and X are affine, and X is nonsingular, then it does hold, and this property of X, for all such morphisms $f : Y \to X$, characterizes nonsingular schemes.

Proposition 4.4 (Infinitesimal Lifting Property). *Let X be a nonsingular affine scheme of finite type over k, let $f : Y \to X$ be a morphism from an affine scheme Y over k, and let $Y \subseteq Y'$ be an infinitesimal thickening of Y. Then the morphism f lifts to a morphism $g : Y' \to X$ such that $g|_Y = f$.*

Proof (cf. [57, II, Ex. 8.6]). First we note that Y' is also affine [57, III, Ex. 3.1], so we can rephrase the problem in algebraic terms. Let $X = \operatorname{Spec} A$, let $Y = \operatorname{Spec} B$, and let $Y' = \operatorname{Spec} B'$. Then f corresponds to a ring homomorphism, which (by abuse of notation) we call $f : A \to B$. On the other hand, B is a quotient of B' by an ideal I with $I^n = 0$ for some n. The problem is to find a homomorphism $g : A \to B'$ lifting f, i.e., such that g followed by the projection $B' \to B$ is f.

If we filter I by its powers and consider the sequence $B' = B'/I^n \to B'/I^{n-1} \to \cdots \to B'/I^2 \to B'/I$, it will be sufficient to lift one step at a time. Thus (changing notation) we reduce to the case $I^2 = 0$.

Since X is of finite type over k, we can write A as a quotient of a polynomial ring $P = k[x_1, \ldots, x_n]$ by an ideal J. Composing the projection $P \to A$ with f we get a homomorphism $P \to B$, which we can lift to a homomorphism $h : P \to B'$, since one can send the variables x_i to any liftings of their images in B (this corresponds to the fact that the polynomial ring is a free object in the category of k-algebras):

$$
\begin{array}{ccccccccc}
0 & \to & J & \to & P & \to & A & \to & 0 \\
& & & & \downarrow h & & \downarrow f & & \\
0 & \to & I & \to & B' & \to & B & \to & 0
\end{array}
$$

Now h induces a map $h : J \to I$, and since $I^2 = 0$, this gives a map $\bar{h} : J/J^2 \to I$.

Next we note that the homomorphism $P \to A$ gives an embedding of X in an affine n-space $\mathbb{A}_k^n$. By (4.3), we obtain an exact sequence

$$
0 \to J/J^2 \to \Omega^1_{P/k} \otimes_P A \to \Omega^1_{A/k} \to 0,
$$

and note that these modules correspond to locally free sheaves on X, hence are projective A-modules. Via the maps h, f, we get a P-module structure on B', and A-module structures on B, I. Applying the functor $\operatorname{Hom}_A(\cdot, I)$ to the above sequence gives another exact sequence

$$
0 \to \operatorname{Hom}_A(\Omega^1_{A/k}, I) \to \operatorname{Hom}_P(\Omega^1_{P/k}, I) \to \operatorname{Hom}_A(J/J^2, I) \to 0.
$$

Let $\theta \in \operatorname{Hom}_P(\Omega^1_{P/k}, I)$ be an element whose image is $\bar{h} \in \operatorname{Hom}_A(J/J^2, I)$. We can regard θ as a k-derivation of P to the module I. Then we define a new map $h' : P \to B'$ by $h' = h - \theta$. I claim that h' is a ring homomorphism lifting f and with $h'(J) = 0$. The first statement is a consequence of the lemma (4.5) below. To see that $h'(J) = 0$, let $y \in J$. Then $h'(y) = h(y) - \theta(y)$. We need only consider $y \bmod J^2$, and then $h(y) = \theta(y)$ by choice of θ, so $h'(y) = 0$. Now since $h'(J) = 0$, h' descends to give the desired homomorphism $g : A \to B'$ lifting f.

Lemma 4.5. *Let $B' \to B$ be a surjective homomorphism of k-algebras with kernel I of square zero. Let $R \to B$ be a homomorphism of k-algebras.*

(a) *If $f, g : R \to B'$ are two liftings of the map $R \to B$ to B', then $\theta = g - f$ is a k-derivation of R to I.*
(b) *Conversely, if $f : R \to B'$ is one lifting, and $\theta : R \to I$ a derivation, then $g = f + \theta$ is another homomorphism of R to B' lifting the given map $R \to B$.*

In other words, if it is nonempty, the set of liftings $R \to B$ to k-algebra homomorphisms of R to B' is a principal homogeneous space under the action by addition of the group $\mathrm{Der}_k(R, I) = \mathrm{Hom}_R(\Omega_{R/k}, I)$. (Note that since $I^2 = 0$, I has a natural structure of a B-module and hence also of an R-module.)

Proof. (a) Let $f, g : R \to B'$ and let $\theta = g - f$. As a k-linear map, θ followed by the projection $B' \to B$ is zero, so θ sends R to I. Let $x, y \in R$. Then

$$\begin{aligned}
\theta(xy) &= g(xy) - f(xy) \\
&= g(x)g(y) - f(x)f(y) \\
&= g(x)(g(y) - f(y)) + f(y)(g(x) - f(x)) \\
&= g(x)\theta(y) + f(y)\theta(x) \\
&= x\theta(y) + y\theta(x),
\end{aligned}$$

the last step being because $g(x)$ and $f(y)$ act in I just like x, y. Thus θ is a k-derivation of R to I.

(b) Conversely, given f and θ as above, let $g = f + \theta$. Then

$$\begin{aligned}
g(xy) &= f(xy) + \theta(xy) \\
&= f(x)f(y) + x\theta(y) + y\theta(x) \\
&= (f(x) + \theta(x))(f(y) + \theta(y)) \\
&= g(x)g(y),
\end{aligned}$$

where we note that $\theta(x)\theta(y) = 0$, since $I^2 = 0$. Thus g is a homomorphism of $R \to B'$ lifting R to B.

For a converse to (4.4), we need only a special case of the infinitesimal lifting property.

Proposition 4.6. *Let X be a scheme of finite type over k algebraically closed. Suppose that for every morphism $f : Y \to X$ of a punctual scheme Y (meaning Y is the Spec of a local Artin ring), finite over k, and for every infinitesimal thickening $Y \subseteq Y'$ with ideal sheaf of square zero, there is a lifting $g : Y' \to X$. Then X is nonsingular.*

Proof. It is sufficient (4.1) to show that the local ring $\mathcal{O}_{P,X}$ is a regular local ring for every closed point $P \in X$. So again we reduce to an algebraic question, namely, let $A, \mathfrak{m}$ be a local k-algebra, essentially of finite type over k, and with residue field k. Assume that for every homomorphism $f : A \to B$, where B

is a local artinian k-algebra and for every thickening $0 \to I \to B' \to B \to 0$ with $I^2 = 0$, there is a lifting $g : A \to B'$. Then A is a regular local ring.

Let $a_1, \ldots, a_n$ be a minimal set of generators for the maximal ideal $\mathfrak{m}$ of A. Then there is a surjective homomorphism f of the formal power series ring $P = k[[x_1, \ldots, x_n]]$ to $\hat{A}$, the completion of A, sending x_i to a_i, and creating an isomorphism of $P/\mathfrak{n}^2$ to $A/\mathfrak{m}^2$, where $\mathfrak{n} = (x_1, \ldots, x_n)$ is the maximal ideal of P.

Consider the surjections $P/\mathfrak{n}^{i+1} \to P/\mathfrak{n}^i$, each defined by an ideal of square zero. Starting with the map $A \to A/\mathfrak{m}^2 \cong P/\mathfrak{n}^2$, we can lift step by step to get maps of $A \to P/\mathfrak{n}^i$ for each i, and hence a map g into the inverse limit, which is P. Passing to $\hat{A}$, we have maps $P \xrightarrow{f} \hat{A} \xrightarrow{g} P$ with the property that $g \circ f$ is an isomorphism on $P/\mathfrak{n}^2$. It follows that $g \circ f$ is an automorphism of P (Ex. 4.1). Hence $g \circ f$ has no kernel, so f is injective. But f was surjective by construction, so f is an isomorphism, and $\hat{A}$ is regular. From this it follows that A is regular, as required.

Corollary 4.7. *Let A be a local ring, essentially of finite type over an algebraically closed field k, with residue field k. Then A is a regular local ring if and only if it has the infinitesimal lifting property for local Artin rings $B' \to B$ finite over k.*

Proof. Just localize (4.4) and (4.6).

The following result shows that infinitesimal deformations of nonsingular affine schemes are trivial.

Corollary 4.8. *Let X be a nonsingular affine scheme over k. Let A be a local Artin ring over k, and let X' be a scheme, flat over $\operatorname{Spec} A$, such that $X' \times_A k$ (where by abuse of notation we mean $X' \times_{\operatorname{Spec} A} \operatorname{Spec} k$) is isomorphic to X. Then X' is isomorphic to the trivial deformation $X \times_k A$ of X over A.*

Proof. We apply (4.4) to the identity map of X to X and the infinitesimal thickening $i : X \hookrightarrow X'$ defined by the isomorphism $X' \times_A k \cong X$. Therefore there is a lifting $p : X' \to X$ such that $p \circ i = \operatorname{id}_X$. The maps of X' to X and to $\operatorname{Spec} A$ define a map to the product: $X' \to X \times_k A$. Both of these schemes are flat over A, and this map restricts to the identity on X, so it is an isomorphism (Ex. 4.2).

Remark 4.8.1. The infinitesimal lifting property for nonsingular varieties over an algebraically closed field that we have explained here can be generalized to the relative case of a morphism of schemes, giving a characterization of smooth morphisms (Ex. 4.7, Ex. 4.8). In fact, Grothendieck takes the infinitesimal lifting property as one of the equivalent definitions of smooth morphisms. See [48, IV, §17].

Next we investigate the relation between nonsingularity and the T^i functors.

Theorem 4.9. *Let* $X = \operatorname{Spec} B$ *be an affine scheme over* k *algebraically closed. Then* X *is nonsingular if and only if* $T^1(B/k, M) = 0$ *for all* B*-modules* M. *Furthermore, if* X *is nonsingular, then also* $T^2(B/k, M) = 0$ *for all* M.

Proof. Write B as a quotient of a polynomial ring $A = k[x_1, \ldots, x_n]$ over k. Then $\operatorname{Spec} A$ is nonsingular, and we can use the criterion of (4.3), which shows that X is nonsingular if and only if the conormal sequence

$$0 \to I/I^2 \to \Omega_{A/k} \otimes_A B \to \Omega_{B/k} \to 0$$

is exact and $\Omega_{B/k}$ is locally free, i.e., a projective B-module. Since $\Omega_{A/k}$ is a free A-module, the sequence will split, so we see that X is nonsingular if and only if this sequence is split exact. By (3.10), $T^1(B/k, M) = 0$ for all M if and only if the map

$$\operatorname{Hom}(\Omega_{A/k}, M) \to \operatorname{Hom}(I/I^2, M)$$

is surjective for all M, and this is equivalent to the splitting of the sequence above (just consider the case $M = I/I^2$). Thus X is nonsingular if and only if $T^1(B/k, M) = 0$ for all M.

For the vanishing of $T^2(B/k, M)$, suppose X is nonsingular. By (3.10) again, $T^2(B/k, M) = T^2(B/A, M)$. Localizing at any point $x \in X$, by (4.3) the ideal I_x is generated by $n - r = \dim A - \dim B$ elements in the regular local ring A_x. Hence these generators form a regular sequence, and (3.9) shows that $T^2(B_x/A_x, M) = 0$ for all B_x-modules M. Thus $T^2(B/A, M) = 0$ by localization (Ex. 3.5).

Corollary 4.10. *Let* B *be a local* k*-algebra with residue field* k *algebraically closed. Then* B *is a regular local ring if and only if* $T^1(B/k, M) = 0$ *for all* B*-modules* M, *and in this case* $T^2(B/k, M) = 0$ *for all* M.

Proof. By localization, using (4.1) and (4.9).

From this theorem we can deduce a relative version. We say that a morphism of finite type $f : X \to Y$ of noetherian schemes is *smooth* if f is flat, and for every point $y \in Y$, the *geometric fiber* $X_y \otimes_{k(y)} \overline{k(y)}$ is nonsingular over $\overline{k(y)}$, where $\overline{k(y)}$ is the algebraic closure of $k(y)$ (cf. [57, III, 10.2]).

Theorem 4.11. *A morphism of finite type* $f : X \to Y$ *of noetherian schemes is smooth if and only if it is flat, and* $\mathcal{T}^1(X/Y, \mathcal{F}) = 0$ *for all coherent sheaves* $\mathcal{F}$ *on* X. *Furthermore, if* f *is smooth, then also* $\mathcal{T}^2(X/Y, \mathcal{F}) = 0$ *for all* $\mathcal{F}$.

Proof. The question is local, so we may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine and that f is given by a ring homomorphism $A \to B$.

First suppose B is flat over A and $T^1(B/A, M) = 0$ for all B-modules M. Let $y \in Y$ be a point, corresponding to a prime ideal $\mathfrak{p} \subseteq A$, and let $k = k(y)$

be its residue field. Let $A' = A/\mathfrak{p}$ and $B' = B \otimes_A A' = B/\mathfrak{p}B$. Then for any B'-module M we obtain $T^1(B'/A', M) = T^1(B/A, M) = 0$ by base change II (Ex. 3.8). Write B' as a quotient of a polynomial ring $R = A'[x_1, \ldots, x_n]$, with kernel I. Then in particular $T^1(B'/A', I/I^2) = 0$.

Now consider the flat base extension from A' to $\bar{k}$, where $\bar{k}$ is the algebraic closure of $k(y)$, which is the quotient field of A'. By base change I (Ex. 3.7), $T^1(B' \otimes \bar{k}/\bar{k}, (I/I^2) \otimes \bar{k}) = 0$. But since the base change is flat, it follows that $B' \otimes \bar{k}$ is the quotient of the polynomial ring $R \otimes \bar{k} = \bar{k}[x_1, \ldots, x_n]$ with kernel $\bar{I} = I \otimes \bar{k}$, and $(I/I^2) \otimes \bar{k} = \bar{I}/\bar{I}^2$. Then from the proof of (4.9) it follows that $\operatorname{Spec} B' \otimes \bar{k}$ is nonsingular over $\bar{k}$. Thus the geometric fibers of the morphism f are nonsingular, and f is smooth.

For the converse, suppose that B is smooth over A. First we will show that $T^1(B/A, B/\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m} \subseteq B$. Let $\mathfrak{m}$ correspond to the point $x \in \operatorname{Spec} B$, let $f(x) = y$, and let k be the residue field of y. Then $B/\mathfrak{m}$ is a module over the ring $B \otimes_A k$, so by base change II (Ex. 3.8), we obtain $T^1(B/A, B/\mathfrak{m}) = T^1(B \otimes_A k/k, B/\mathfrak{m})$. Then by base change I (Ex. 3.7), this latter module, tensored with $\bar{k}$, the algebraic closure of k, is equal to $T^1(B \otimes \bar{k}/\bar{k}, (B/\mathfrak{m}) \otimes \bar{k})$, and this one is zero, since the geometric fibers are nonsingular. Since $k \to \bar{k}$ is a faithfully flat extension, it follows that $T^1(B \otimes k/k, B/\mathfrak{m}) = 0$ and hence $T^1(B/A, B/\mathfrak{m}) = 0$.

We observe that the functor $T^1(B/A, \cdot)$ is an additive functor from finitely generated B-modules to finitely generated B-modules, and is *semi-exact* in the sense that to each short exact sequence of modules it gives a sequence of three modules that is exact in the middle. It follows from the lemma of Dévissage below (4.12) that $T^1(B/A, M) = 0$ for all finitely generated B-modules, and hence for all B-modules, since the T^i functors commute with direct limits. The same argument shows also that $T^2(B/A, M) = 0$ for all M.

Lemma 4.12 (Dévissage). *Let B be a noetherian ring, and let F be a semiexact additive functor from finitely generated B-modules to finitely generated B-modules. Assume that $F(B/\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ of B. Then $F(M) = 0$ for all finitely generated B-modules.*

Proof. Any finitely generated B-module M has a composition series whose quotients are $B/\mathfrak{p}_i$ for various prime ideals $\mathfrak{p}_i$. By semi-exactness, it is sufficient to show that F vanishes on each of these. Thus we may assume $M = B/\mathfrak{p}$.

We proceed by induction on the dimension of the support of M. If $\dim \operatorname{Supp} M = 0$, then M is just $B/\mathfrak{m}$ for some maximal ideal, and $F(M) = 0$ by hypothesis. For the general case, let $M = B/\mathfrak{p}$ have some dimension r. For any maximal ideal $\mathfrak{m} \supseteq \mathfrak{p}$, choose an element $t \in \mathfrak{m} - \mathfrak{p}$. Then t is a non-zero-divisor for M and we can write

$$0 \to M \xrightarrow{t} M \to M' \to 0,$$

where M' is a module with support of dimension $< r$. Hence by the induction hypothesis, $F(M') = 0$ and we get a surjection $F(M) \xrightarrow{t} F(M) \to 0$. It follows

from Nakayama's lemma that $F(M)$ localized at $\mathfrak{m}$ is zero. This holds for any $\mathfrak{m} \supseteq \mathfrak{p}$, i.e., any point of $\operatorname{Spec} B/\mathfrak{p}$, and so $F(M) = 0$.

If A is a regular local ring and $B = A/I$ is a quotient, we say that B is a *local complete intersection in A* if the ideal I can be generated by $\dim A - \dim B$ elements.

Theorem 4.13. *Let A be a regular local k-algebra with residue field k algebraically closed, and let $B = A/I$ be a quotient of A. Then B is a local complete intersection in A if and only if $T^2(B/k, M) = 0$ for all B-modules M.*

Proof. Since A is regular, we have $T^1(A/k, M) = 0$ for $i = 1, 2$ and all M by (4.10). Then from the exact sequence (3.5) we obtain $T^2(B/k, M) = T^2(B/A, M)$ for all M. If B is a local complete intersection in A, then the vanishing of T^2 follows from (3.9).

Conversely, suppose that $T^2(B/k, M) = 0$ for all M. As above, this implies $T^2(B/A, M) = 0$ for all M. To compute this group, in (3.1) we can take $R = A$, $I = I$, and let F map to a minimal set of generators $(a_1, \ldots, a_s)$ of I, with kernel Q. Then the hypothesis $T^2(B/A, M) = 0$ for all M implies that

$$\operatorname{Hom}(F/IF, M) \to \operatorname{Hom}(Q/F_0, M)$$

is surjective for all M, and this in turn (taking $M = Q/F_0$) implies that the mapping $d_2 : Q/F_0 \to F/IF$ has a splitting, i.e., a map $p : F/IF \to Q/F_0$ such that $p \circ d_2 = \operatorname{id}_{Q/F_0}$. Since we chose a minimal set of generators for I, it follows that $Q \subseteq \mathfrak{m}F$, where $\mathfrak{m}$ is the maximal ideal of A. Thus the identity map $p \circ d_2$ sends Q/F_0 into $\mathfrak{m}(Q/F_0)$, and so by Nakayama's lemma, $Q/F_0 = 0$. But Q/F_0 is just the first homology group of the Koszul complex $K_\bullet(a_1, \ldots, a_s)$ over A, and the vanishing of this group is equivalent to $a_1, \ldots, a_s$ being a regular sequence [104, 16.5]. Thus B is a local complete intersection in A.

Remark 4.13.1. Since the condition $T^2(B/k, M) = 0$ for all B-modules M depends only on B, and not on A, it follows that if B is a local complete intersection in one regular local ring, then it will be a local complete intersection in any regular local ring of which it is a quotient. Thus we can say simply that B is a *local complete intersection ring* without mentioning A.

Example 4.13.2. The node of (Ex. 3.1) is a local complete intersection and correspondingly has $T^2 = 0$ for all M. The thick point of (Ex. 3.3) is not a local complete intersection and has $T^2(B/k, B) \neq 0$.

Remark 4.13.3. If we define a *relative local complete intersection* morphism $f : X \to Y$ to be one that is flat and whose geometric fibers are local complete intersection schemes, then an argument similar to the proof of (4.13) shows that f is a relative local complete intersection morphism if and only if $\mathcal{T}^2(X/Y, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$ on X.

References for this section. The definition of smooth morphisms and their characterization by the infinitesimal lifting property are due to Grothendieck [47]. The characterizations of smooth and local complete intersection morphisms using the T^i functors are due to Lichtenbaum and Schlessinger [96].

Exercises.

4.1. Let A be a local k-algebra with residue field k. Let $f : A \to A$ be a k-algebra homomorphism inducing an isomorphism $A/\mathfrak{m}^2 \to A/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of A. Show that f itself is an isomorphism, i.e., an automorphism of A.

4.2. Let A be a local artinian k-algebra, let X_1 and X_2 be schemes of finite type, flat over A, and let $f : X_1 \to X_2$ be an A-morphism that induces an isomorphism of closed fibers $f \otimes_A k : X_1 \times_A k \to X_2 \times_A k$. Show that f itself is an isomorphism.

4.3. If X is any scheme of finite type over k, the sheaves $\mathcal{T}^i(X/k, \mathcal{F})$ for $i = 1, 2$ and any coherent $\mathcal{O}_X$-module $\mathcal{F}$ have support in the singular locus of X.

4.4. Let B and B' be local rings, essentially of finite type over k, having isolated singularities at the closed points, and assume that B and B' are *analytically isomorphic*, i.e., the completions $\hat{B}$ and $\hat{B}'$ are isomorphic. Show that the modules $T^i_{B/k}$ and $T^i_{B'/k}$ for $i = 1, 2$ are isomorphic as modules over the isomorphic completions of B and B'. In particular, they have the same lengths.

4.5. Let Y be a closed subscheme of a nonsingular scheme X over k with ideal sheaf of $\mathcal{I}$. Then for any coherent $\mathcal{O}_Y$-module $\mathcal{F}$ there is an exact sequence of sheaves

$$0 \to \mathcal{T}^0(Y/k, \mathcal{F}) \to \mathcal{H}om_X(\Omega_{X/k}, \mathcal{F}) \to \mathcal{H}om_Y(\mathcal{I}/\mathcal{I}^2, \mathcal{F}) \to \mathcal{T}^1(Y/k, \mathcal{F}) \to 0.$$

4.6. Let A be a regular local k-algebra with residue field k, let $B = A/I$ for some ideal I, and assume that $I \subseteq \mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of A. Show that $T^1(B/k, k)$ is a k-vector space of dimension equal to the minimal number of generators of I. Conclude that a local ring B is regular if and only if $T^1(B/k, k) = 0$.

4.7. Prove the relative version of the infinitesimal lifting property: Let X be smooth over a scheme S, let $f : Y \to X$ be a morphism of an affine scheme over S to X, and let $Y \subseteq Y'$ be an infinitesimal thickening of S-schemes. Then f lifts to a morphism $g : Y' \to X$ such that $g/y = f$. *Hint:* Imitate the proof of (4.4), using (4.11) to characterize smoothness.

4.8. Prove the converse to (Ex. 4.7): assuming X/S flat, and that the infinitesimal lifting property holds for punctual schemes Y over S, as in (4.6), show that X/S is smooth. *Hint:* For any point $s \in S$, consider the base extension $\text{Spec } \overline{k(s)} \to S$.

4.9. Affine elliptic curves. Let k be an algebraically closed field, fix $\lambda \in k$, $\lambda \neq 0, 1$, and consider the family of affine elliptic curves over $k[t]$ defined by the equation $y^2 = x(x - 1)(x - (\lambda + t))$.

(a) This family is not trivial over any neighborhood of $t = 0$ because over a field, the j-invariant is already determined by any open affine piece of an elliptic curve, and the j-invariant varies in this family—cf. [57, IV, §4].

(b) This family is still not trivial over the complete local ring $k[[t]]$ at the origin, because one can look at the j-invariant over the field of fractions of this ring.

(c) The projective completion of this family in $\mathbb{P}^2$ is not trivial even over the Artin ring $C = k[t]/t^n$ for any $n \geq 2$, because the computation of the j-invariant can be made to work over the ring C.

(d) However, the affine family over the Artin ring C is trivial for any n because of (4.8). Your task, should you choose to accept it, is to find an explicit isomorphism of this family over C with the trivial family $y^2 = x(x-1)(x-\lambda)$. Hints follow.

(e) Find $a, b, c, d \in C$ such that the transformation $x' = (ax+b)/(cx+d)$ sends $0, 1, \lambda$ to $0, 1, \lambda + t$.

(f) Substitute for x' in the equation

$$y'^2 = x'(x'-1)(x'-(\lambda+t))$$

and show that the result can be written as

$$y'^2 \frac{(cx+d)^3}{a(a-c)(a-c(\lambda+t))} = x(x-1)(x-\lambda).$$

(g) Now, using the fact that t is nilpotent, show that one can find $f(x,t)$ and $g(x,t)$ in $C[x]$ such that the substitutions

$$x' = x + tf(x,t),$$
$$y' = y(1 + tg(x,t)),$$

bring the equation into the form $y^2 = x(x-1)(x-\lambda)$.

(h) Show that the transformation $(x,y) \mapsto (x',y')$ is an automorphism of the ring $C[x,y]$, so the two families are isomorphic over C.

5. Deformations of Rings

In this section we use the T^i functors to study deformations of arbitrary schemes (Situation D). We will see that the deformations of an affine scheme $X = \operatorname{Spec} B$ over k are given by $T^1(B/k, B)$, and that the deformations of a nonsingular scheme X are given by $H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf. As an application, we study deformations of cones.

Definition. If X is a scheme over k, and A an Artin ring over k, we define a *deformation of X over A* to be a scheme X', flat over A, together with a closed immersion $i : X \hookrightarrow X'$ such that the induced map $i \times_A k : X \to X' \times_A k$ is an isomorphism. Two such deformations X'_1, i_1 and X'_2, i_2 are *equivalent* if there is an isomorphism $f : X'_1 \to X'_2$ over A compatible with i_1 and i_2, i.e., such that $i_2 = f \circ i_1$.

Remark 5.0.1. Why don't we simply define a deformation of X to be a scheme X' flat over A for which $X' \times_A k$ is isomorphic to X, without specifying the isomorphism? The reason is that the set of these is less well behaved functorially than the definition we have given, and in any case, this latter set

can be recovered from the former by dividing by the action of the group of automorphisms of X over k. This matter of automorphisms makes most deformation problems more complicated than Situation A, where an isomorphism of closed subschemes is just equality, and there are no automorphisms. (See §18, where automorphisms play an important role.)

We start by considering deformations of affine schemes. Let B be a k-algebra. A deformation of $\operatorname{Spec} B$ over the dual numbers D is then a D-algebra B', flat over D, together with a homomorphism $B' \to B$ inducing an isomorphism $B' \otimes_D k \to B$. Because of (2.2) the flatness of B' is equivalent to the exactness of the sequence

$$0 \to B \xrightarrow{t} B' \to B \to 0. \tag{$*$}$$

Here we think of B' and B on the right as rings, and B on the left as an ideal of square 0, which is a B-module. Furthermore, B' is a D-algebra and B is a k-algebra. On the other hand, we can forget the D-algebra structure of B' and regard it simply as a k-algebra via the inclusion $k \subseteq D$. Then, as in (2.7), we see that the D-algebra structure of B' can be recovered in a unique way compatible with the exact sequence $(*)$. We need only specify multiplication by t, and this must be done by passing from B' to B on the right, followed by the inclusion $B \to B'$ on the left.

Thus we see that equivalence classes of deformations of B over D are in one-to-one correspondence with equivalence classes of exact sequences $(*)$, where B' and B are regarded only as k-algebras. We say in that case that B' is an extension as k-algebras of the k-algebra B by the B-module B.

This discussion suggests that we consider a more general situation. Let A be a ring, let B be an A-algebra, and let M be a B-module. We define an *extension* of B by M as A-algebras to be an exact sequence

$$0 \to M \to B' \to B \to 0,$$

where $B' \to B$ is a homomorphism of A-algebras, and M is an ideal in B' with $M^2 = 0$. Two such extensions B', B'' are *equivalent* if there is an isomorphism $B' \to B''$ compatible in the exact sequences with the identity maps on B and M. The *trivial* extension is given by $B' = B \oplus M$ made into a ring by the rule $(b, m) \cdot (b_1, m_1) = (bb_1, bm_1 + b_1 m)$.

Theorem 5.1. *Let A be a ring, B an A-algebra, and M a B-module. Then equivalence classes of extensions of B by M as A-algebras are in natural one-to-one correspondence with elements of the group $T^1(B/A, M)$. The trivial extension corresponds to the zero element.*

Proof. Let $A[x] \to B$ be a surjective map of a polynomial ring over A to B, let $\{e_i\}$ be a set of generators of the B-module M, and let $y = \{y_i\}$ be a set of indeterminates with the same index set as $\{e_i\}$. We consider the polynomial ring $A[x, y]$, and note that if B' is any extension of B by M, then one can find

a surjective ring homomorphism $f : A[x, y] \to B'$, not unique, that makes a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & (y) & \to & A[x, y] & \to & A[x] & \to & 0 \\
& & \downarrow & & \downarrow f & & \downarrow & & \\
0 & \to & M & \to & B' & \to & B & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

where the two outer vertical arrows are determined by the construction. Here (y) denotes the ideal in $A[x, y]$ generated by the y_i, and the map $(y) \to M$ sends y_i to e_i.

Now we proceed in two steps. First we classify quotients $f : A[x, y] \to B'$ that form a diagram as above. Then we ask, for a given extension B', how many different ways are there to express B' as a quotient of $A[x, y]$? Dividing out by this ambiguity will give us a description of the set of extensions B'.

For the first step, we complete the above diagram by adding a top row consisting of the kernels of the vertical arrows:

$$ 0 \to Q \to I' \to I \to 0. $$

Giving B' as a quotient of $A[x, y]$ is equivalent to giving the ideal I' in $A[x, y]$. Since we have a splitting of the middle row given by the ring inclusion $A[x] \to A[x, y]$, the argument used in the proof of (2.3) shows that the set of such diagrams is in natural one-to-one correspondence with the group $\mathrm{Hom}_{A[x]}(I, M) = \mathrm{Hom}_B(I/I^2, M)$.

For the second step, we use (4.5), whose proof works over any ring A in place of k, taking $R = A[x]$, to see that the set of possible choices for $f : A[x, y] \to B'$ forms a principal homogeneous space under the action of $\mathrm{Der}_A(A[x], M)$. Note that because of the inclusion $A[x] \to A[x, y]$, any map $f : A[x, y] \to B'$ determines a map $g : A[x] \to B'$ and is uniquely determined by it.

Now write the long exact sequence of T^i functors (3.5) for the three rings $A \to A[x] \to B$ and the module M. The part that interests us is

$$ T^0(A[x]/A, M) \to T^1(B/A[x], M) \to T^1(B/A, M) \to T^1(A[x]/A, M). $$

The first term here, by (3.6), is $\mathrm{Hom}_{A[x]}(\Omega_{A[x]/A}, M)$, which is just the module of derivations $\mathrm{Der}_A(A[x], M)$. The second term, by (3.8), is $\mathrm{Hom}_B(I/I^2, M)$. The fourth term, by (3.7), is 0. Therefore $T^1(B/A, M)$ appears as the cokernel of a natural map

$$ \mathrm{Der}_A(A[x], M) \to \mathrm{Hom}_B(I/I^2, M). $$

Under the interpretations above we see that this cokernel is the set of diagrams $A[x, y] \to B'$ as above, modulo the ambiguity of choice of the map $A[x, y] \to B'$, and so $T^1(B/A, M)$ is in one-to-one correspondence with the set of extensions B', as required.

Corollary 5.2. *Let k be a field and let B be a k-algebra. Then the set of deformations of B over the dual numbers is in natural one-to-one correspondence with the group $T^1(B/k, B)$.*

Proof. This follows from the theorem and the discussion at the beginning of this section, which showed that such deformations are in one-to-one correspondence with the k-algebra extensions of B by B.

Next we consider deformations of a nonsingular variety. We use Čech cohomology of an open covering, knowing that deformations of affine nonsingular varieties are trivial.

Theorem 5.3. *Let X be a nonsingular variety over k. Then the deformations of X over the dual numbers are in natural one-to-one correspondence with the elements of the group $H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X = \mathcal{H}om_X(\Omega_{X/k}, \mathcal{O}_X)$ is the tangent sheaf of X.*

Proof (cf. [57, III, 9.13.2]). Let X' be a deformation of X, and let $\mathcal{U} = (U_i)$ be an open affine covering of X. Over each U_i the induced deformation U_i' is trivial by (4.8), or by (4.9) combined with (5.2), so we can choose an isomorphism $\varphi_i : U_i \times_k D \xrightarrow{\sim} U_i'$. Then on $U_{ij} = U_i \cap U_j$ we get an automorphism $\psi_{ij} = \varphi_j^{-1} \varphi_i$ of $U_{ij} \times_k D$, which corresponds to an element $\theta_{ij} \in H^0(U_{ij}, \mathcal{T}_X)$ by (Ex. 5.2). By construction, on U_{ijk} we have $\theta_{ij} + \theta_{jk} + \theta_{ki} = 0$, since composition of automorphisms corresponds to addition of derivations. Therefore (θ_{ij}) is a Čech 1-cocycle for the covering $\mathcal{U}$ and the sheaf $\mathcal{T}_X$. If we replace the original chosen isomorphisms $\varphi_i : U_i \times_k D \xrightarrow{\sim} U_i'$ by some others φ_i', then $\varphi_i'^{-1} \varphi_i$ will be an automorphism of $U_i \times_k D$ coming from a section $\alpha_i \in H^0(U_i, \mathcal{T}_X)$, and the new $\theta_{ij}' = \theta_{ij} + \alpha_i - \alpha_j$. So the new 1-cocycle θ_{ij}' differs from θ_{ij} by a coboundary, and we obtain a well-defined element θ in the Čech cohomology group $\check{H}^1(\mathcal{U}, \mathcal{T}_X)$. Since $\mathcal{U}$ is an open affine covering and $\mathcal{T}_X$ is a coherent sheaf, this is equal to the usual cohomology group $H^1(X, \mathcal{T}_X)$. Clearly θ is independent of the covering chosen.

Reversing this process, an element $\theta \in H^1(X, \mathcal{T}_X)$ is represented on $\mathcal{U}$ by a 1-cocycle θ_{ij}, and these θ_{ij} define automorphisms of the trivial deformations $U_{ij} \times_k D$ that can be glued together to make a global deformation X' of X. So we see that the deformations of X over D are given by $H^1(X, \mathcal{T}_X)$.

Example 5.3.1. If $X = \mathbb{P}_k^n$ for $n \geq 1$, then $H^1(\mathcal{T}_X) = 0$, so every deformation of X over the dual numbers is trivial. Thus X is an example of a *rigid* scheme, by which we mean a scheme all of whose deformations over the dual numbers are trivial. We have already seen that any affine nonsingular scheme is rigid (4.8). This result also follows from (5.3), since an affine scheme has no higher cohomology. We will see examples of singular rigid schemes in (5.5.1) and the exercises of this section.

One needs to exercise some caution around this notion of rigid scheme, lest intuition lead one into error. One might think, for example, that a rigid

scheme has no nontrivial deformations. We will show indeed (Ex. 10.3) that every deformation over an Artin ring is trivial. However, there can be non-trivial global deformations of a rigid affine scheme (Ex. 4.9). For an even more striking example, consider a family C_t of nonsingular plane cubic curves degenerating to C_0, a union of two lines L, M, with M counted twice. Now pass to the affine plane by removing M. Then we have a family of affine elliptic curves whose limit is the affine line $\mathbb{A}^1$, which is rigid.

Another intuition might say that a singular rigid scheme cannot be embedded in a flat family whose general member is smooth, i.e., is not smoothable. This is true, but the proof is not obvious (29.6).

For projective varieties, the situation is somewhat better. If X_0 is a rigid projective scheme, then one can show that nearby fibers in a flat family of projective schemes are isomorphic to X_0 (Ex. 24.7c). If X_0 is an affine rigid scheme with an isolated singularity, the best one can hope for is that nearby fibers in a flat family have analytically isomorphic singularities (Ex. 18.8).

Example 5.3.2. Let C be a nonsingular projective curve of genus g. Then by Serre duality $H^1(\mathcal{T}_C)$ is dual to $H^0(\Omega_C^{\otimes 2})$, which has degree $4g - 4$. For $g \geq 2$ this is nonspecial, so by Riemann–Roch, $H^1(\mathcal{T}_C)$ has dimension $3g - 3$.

Now, as an application, we will study deformations of cones. Let Y be a nonsingular subvariety of $P = \mathbb{P}_k^n$, and let $X = \operatorname{Spec} B$ be the affine cone over Y inside $\mathbb{A}^{n+1} = \operatorname{Spec} R$, where $R = k[x_0, \ldots, x_n]$ is the homogeneous coordinate ring of P. We wish to study the deformations of X, i.e., the module $T^1_{B/k} = T^1(B/k, B)$, in terms of properties of Y. To relate the two we will compare each of them to the open subset $U = X - \{x\}$, where $x = (x_0, \ldots, x_n)$ is the vertex of the cone.

Theorem 5.4. *In the situation above, if* $\operatorname{depth}_x B \geq 2$, *then there is an exact sequence*

$$0 \to T^1_{B/k} \to H^1(U, \mathcal{T}_U) \to H^1(U, T_R|_U).$$

If furthermore, $\operatorname{depth}_x B \geq 3$, *then* $T^1_{B/k} \cong H^1(U, \mathcal{T}_U)$ *and there is an injection*

$$0 \to T^1_{B/k} \to \bigoplus_{\nu \in \mathbb{Z}} H^1(Y, \mathcal{T}_Y(\nu)).$$

Proof. Since U is nonsingular, we have an exact conormal sequence of sheaves

$$0 \to \mathcal{T}_U \to T_R|_U \to \mathcal{N}_{U/R} \to 0.$$

We consider the following diagram, where the second row is the cohomology sequence of this exact sequence of sheaves, and the first row comes from (3.10) applied to $\operatorname{Spec} B$ in $\operatorname{Spec} R$. The vertical arrows are restriction maps from X to U. Note that each of the modules $T^0_{B/k}, T_R \otimes B, N_{B/R}$ is the dual of some B-module. Because of the hypothesis $\operatorname{depth}_x B \geq 2$, these modules are reflexive, and hence of depth ≥ 2 [63, 1.8, 1.9]. Hence by the exact sequence

of local cohomology with support at x and the cohomological interpretation of depth [57, III, Ex. 3.4], the restriction maps are isomorphisms:

$$
\begin{array}{ccccccccc}
0 \to & T^0_{B/k} & \to & T_R \otimes B & \to & N_{B/R} & \to & T^1_{B/k} & \to & 0 \\
 & \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} & & & & \\
0 \to & H^0(U, \mathcal{T}_U) & \to H^0(U, T_R|_U) & \to H^0(U, \mathcal{N}_{U/R}) & \to H^1(U, \mathcal{T}_U) & \to H^1(U, T_R|_U) & \to \cdots
\end{array}
$$

From this we obtain the first exact sequence of the theorem.

Now suppose that $\mathrm{depth}_x B \geq 3$. Since $T_R|_U$ is a free $\mathcal{O}_U$-sheaf, and $H^1(U, \mathcal{O}_U) \cong H^2_x(B) = 0$ by the depth condition, we obtain $T^1_{B/k} \cong H^1(U, \mathcal{T}_U)$.

To compare this to Y, we use the exact sequence of relative tangent sheaves (Ex. 5.3)

$$ 0 \to \mathcal{O}_U \to \mathcal{T}_U \to \pi^* \mathcal{T}_Y \to 0. $$

Since $\mathrm{depth}_x B \geq 3$, we have $H^1(U, \mathcal{O}_U) = 0$. On the other hand, $U \to Y$ is an affine morphism with fibers that are punctured affine lines $\mathbb{A}^1 - \{0\}$. So $H^1(U, \pi^* \mathcal{T}_Y) \cong \bigoplus_{\nu \in \mathbb{Z}} H^1(Y, \mathcal{T}_Y(\nu))$. Thus $H^1(U, \mathcal{T}_U)$ injects into this latter group.

Remark 5.4.1. The depth conditions on B can be expressed in terms of Y. Thus $\mathrm{depth}_x B \geq 2$ is equivalent to saying that Y is projectively normal, which in turn is equivalent to $H^0(\mathcal{O}_P(\nu)) \to H^0(\mathcal{O}_Y(\nu))$ being surjective for all ν. And $\mathrm{depth}_x B \geq 3$ if and only if in addition, $H^1(\mathcal{O}_Y(\nu)) = 0$ for all ν.

Corollary 5.5. *If Y is a nonsingular projectively normal subvariety of $P = \mathbb{P}^n_k$, and if $H^1(\mathcal{O}_Y(\nu)) = H^1(\mathcal{T}_Y(\nu)) = 0$ for all $\nu \in \mathbb{Z}$, then the affine cone X over Y is a rigid scheme.*

Proof. Indeed, taking into account (5.4.1), the theorem implies $T^1_{B/k} = 0$.

Example 5.5.1. Let Y be the Veronese surface in $\mathbb{P}^5$, which is the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. It is easy to see that Y is projectively normal and that $H^1(\mathcal{O}_Y(\nu)) = 0$ for all ν. There is just one twist of the tangent sheaf $\mathcal{T}_{\mathbb{P}^2}$ that has a nonzero H^1, namely $H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(-3)) = k$. However, since we are dealing with the 2-uple embedding, $H^1(Y, \mathcal{T}_Y(\nu)) = H^1(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}(2\nu))$ for each ν, and this will be 0 for all ν. Thus the cone over the Veronese surface is rigid.

References for this section. The applications of the T^1 functors to deformations of rings appear in [96]. The deformations of cones are treated in two papers by Schlessinger [146], [147]. See also [8]. For a further study of deformations of cones, see §29.

Exercises.

5.1. Show that

(a) A node $k[x, y]/(xy)$ has a 1-dimensional space of deformations over the dual numbers.

(b) A cusp $k[x, y]/(y^2 - x^3)$ has a 2-dimensional space of deformations.

(c) An ordinary double point of a surface $k[x, y, z]/(xy - z^2)$ has a 1-dimensional space of deformations.

5.2.   Automorphisms. Examining the proof of (5.1) more carefully, show that automorphisms of extensions B' of B by M are given by $T^0(B/A, M)$. Hence if B' is a deformation of B/k over D as in (5.2), the automorphisms of B' are given by $T^0(B/k, B)$, which is the tangent module of B/k.

5.3.   (a) Let $P = \mathbb{P}_k^n$, let $R = k[x_0, \ldots, x_n]$ be its homogeneous coordinate ring, and let $V = \operatorname{Spec} R - \{x\}$, where x is the closed point $(x_0, \ldots, x_n)$. Using the projection $\pi : V \to P$, and comparing the Euler sequences on P and $\mathbb{P}^{n+1}$, which contains V, show that there is an exact sequence

$$0 \to \mathcal{O}_V \to \mathcal{T}_V \to \pi^* \mathcal{T}_P \to 0.$$

(b) Now let Y be a nonsingular closed subscheme of P, let X be the affine cone over Y in $\operatorname{Spec} R$, and let $U = X - \{x\}$. Show similarly that there is an exact sequence

$$0 \to \mathcal{O}_U \to \mathcal{T}_U \to \pi^* \mathcal{T}_Y \to 0.$$

**5.4.   **Using the criterion of (5.4), show that the cone X in $\mathbb{A}^6$ over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$ is rigid.

**5.5.   **Let $X \subseteq \mathbb{A}^4$ be the union of two planes meeting at a point. This can be regarded as the cone over two skew lines in $\mathbb{P}^3$. Use the method of proof of (5.4) to show that X is rigid. *Hints:* Be careful, because in this case, $\operatorname{depth}_x B$ is only 1! However, two special features of this example save the day. One is that U is a disjoint union of two punctured affine planes, so that the conormal sequence for U is split exact. The other is that $H_x^1(B) = k$, and one can show by a direct analysis of the situation that the composed map $H^0(U, \mathcal{T}_U) \to H_x^1(T_R \otimes B)$ is surjective. A surprising consequence of this is that $N_{B/R}$ has depth 2, even though B only has depth 1.

5.6. Abstract versus embedded deformations.   The question is, when Y is a closed subscheme of X, can every abstract deformation of Y be realized as an embedded deformation of Y in X? Cf. (Ex. 10.1) for higher-order deformations, and §20 for further study of this question.

(a) If Y is affine and X is nonsingular, then every abstract deformation of Y over the dual numbers can be realized as an embedded deformation.

(b) On the other hand, if C is a nonsingular projective plane curve of degree $d \geq 5$, then there are abstract deformations of C over the dual numbers that cannot be realized as embedded deformations in $\mathbb{P}^2$.

5.7. Deformations of nonaffine schemes.   Let X be a scheme over k, and let X' be a deformation of X over the dual numbers. For each open affine subset $U_i \subseteq X$, the restriction of X' to U_i is a deformation of U_i, so determines an element α_i in $T^1(U, \mathcal{O}_U)$. These glue to make an element $\alpha \in H^0(X, \mathcal{T}_X^1)$, where $\mathcal{T}_X^1$ is the sheaf $\mathcal{T}^1(X/k, \mathcal{O}_X)$. If α is zero, we say that X' is locally trivial. Show that the locally trivial deformations are classified by $H^1(X, \mathcal{T}_X)$. On the other hand, given an $\alpha \in H^0(\mathcal{T}_X^1)$, show how to construct an element δ_α in $H^2(\mathcal{T}_X)$ with the property

that $\delta_\alpha = 0$ if and only if α comes from a global deformation of X. Thus, if we denote by $\mathrm{Def}(X/k, D)$ the set of global deformations of X over D, there is an exact sequence

$$0 \to H^1(X, \mathcal{T}_X) \to \mathrm{Def}(X/k, D) \to H^0(X, \mathcal{T}_X^1) \xrightarrow{\delta} H^2(X, \mathcal{T}_X).$$

(Perhaps some astute reader will recognize this as the exact sequence of terms of low degree of a suitable spectral sequence.)

5.8. Hilb^8($\mathbb{P}^4$) is not irreducible. Consider the Hilbert scheme of zero-dimensional closed subschemes of $\mathbb{P}_k^4$ of length 8. There is one component of dimension 32 that has a nonsingular open subset corresponding to sets of eight distinct points. We will exhibit another component containing a nonsingular open subset of dimension 25.

(a) Let $R = k[x, y, z, w]$, let $\mathfrak{m}$ be a maximal ideal, and let $I = V + \mathfrak{m}^3$, where V is a 7-dimensional subvector space of $\mathfrak{m}^2/\mathfrak{m}^3$. Let $B = R/I$, and let Z be the associated closed subscheme of $\mathbb{A}^4 \subseteq \mathbb{P}^4$. Show that the set of all such Z, as the point of its support ranges over $\mathbb{P}^4$, forms an irreducible 25-dimensional subset of the Hilbert scheme $H = \mathrm{Hilb}^8(\mathbb{P}^4)$.

(b) Now look at the particular case

$$I = (x^2, xy, y^2, z^2, zw, w^2, xz - yw).$$

Show that $\dim_k \mathrm{Hom}(I/I^2, B) = 25$ as follows. First show that any homomorphism $\varphi : I/I^2 \to B$ has image contained in $\mathfrak{m}B$. Second, observe that homomorphisms φ with image in $\mathfrak{m}^2 B$ form a vector space of dimension 21. Third, show that if the image of φ in $\mathfrak{m}B/\mathfrak{m}^2 B$ is nonzero, then φ is completely determined, modulo those ψ mapping I/I^2 to $\mathfrak{m}^2 B$, by $\varphi(xz - yw)$, and for this one there is a four-dimensional vector space of choices.

(c) Conclude that the family of all Z's described in (a) forms an irreducible component of H of dimension 25 that is nonsingular at the point studied in (b). In particular, by reason of dimension, the zero-scheme described in (b) is not in the closure of the component corresponding to sets of eight distinct points. It is therefore a *nonsmoothable* subscheme of $\mathbb{P}^4$.

(d) Show that the image of the natural map $\mathrm{Hom}(\Omega_R, B) \to \mathrm{Hom}(I/I^2, B)$ has dimension 12, generated by the four homomorphisms that can be described as $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, $\partial/\partial w$, so that $T^1(B/k, B)$ has dimension 13. Thus this singularity is not rigid.

Note. This is a slight variant of an example discovered by Iarrobino and Emsalem [72].

5.9. Reduced locally complete intersection curves. (a) Let C be a reduced locally complete intersection curve in $P = \mathbb{P}^n$, with ideal sheaf $\mathcal{I}$. Then $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $n - 1$ and there is an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_P \otimes \mathcal{O}_C \to \Omega_C \to 0.$$

From this, deduce a map

$$\bigwedge^{n-1} (\mathcal{I}/\mathcal{I}^2) \otimes \Omega_C \to \omega_P \otimes C$$

and then by tensoring with $\bigwedge^{n-1}(\mathcal{I}/\mathcal{I}^2)^\vee$, a map

$$\varphi : \Omega_C \to \omega_C,$$

where ω_C is the dualizing sheaf of C (cf. [57, III.7.11]).

(b) Now suppose C is contained in a nonsingular surface X. Tensoring the exact sequence

$$0 \to \mathcal{T}_{C/k} \to \mathcal{T}_{X/k} \otimes \mathcal{O}_C \to \mathcal{N}_{C/X} \to \mathcal{T}^1_{C/k} \to 0$$

with ω_X, we get a sequence

$$0 \to \mathcal{T}_{C/k} \otimes \omega_X \to \Omega_{X/k} \otimes \mathcal{O}_C \xrightarrow{\psi} \omega_C \to \mathcal{T}^1_{C/k} \otimes \omega_X \to 0.$$

Show that the map ψ factors through the projection of $\Omega_X \otimes \mathcal{O}_C$ to Ω_C and the map φ above, so that the cokernel of φ is $\mathcal{T}^1_C \otimes \omega_X$. Show also that the kernel of φ is the torsion submodule of Ω_C, which is isomorphic to $\mathcal{T}^{1*}_C \otimes \omega_X^\vee$, where $*$ denotes the dual vector space with the appropriate module structure.

(c) Conclude that if C is a reduced locally complete intersection curve in $\mathbb{P}^n$ whose singularities all have embedding dimension 2, then φ sits in an exact sequence

$$0 \to \mathcal{R} \to \Omega_C \xrightarrow{\varphi} \omega_C \to \mathcal{S} \to 0,$$

where $\mathcal{S}$ is locally isomorphic to $\mathcal{T}^1_C$ and $\mathcal{R}$ is locally isomorphic to $\mathcal{T}^{1*}_C$.

(d) With C a curve as in (c), let $\Delta = \text{length } \mathcal{T}^1_C$. Show that the tangent sheaf $\mathcal{T}_C$ has degree $2 - 2p_a + \Delta$, where p_a is the arithmetic genus of C. Conclude from Riemann–Roch that if $H^0(\mathcal{T}_C) = 0$ (which is the case, for example, if C is integral and $\Delta < 2p_a - 2$), then $H^1(\mathcal{T}_C)$, which gives the locally trivial deformations of C over the dual numbers, has dimension $3p_a - 3 - \Delta$. Taking into account (Ex. 5.7) show that the total space of (abstract) deformations of C over the dual numbers has dimension $3p_a - 3$.

5.10. Deformations of a double line. Let L be a line on a nonsingular cubic surface X in $\mathbb{P}^3$, and let Y be the scheme associated to the divisor $2L$ on X. Then Y is a curve of degree 2, supported on the line L, and there is an exact sequence

$$0 \to \mathcal{O}_L(1) \to \mathcal{O}_Y \to \mathcal{O}_L \to 0.$$

The curve Y is obviously not smoothable in $\mathbb{P}^3$, because it has arithmetic genus $p_a = -2$, and there are no nonsingular curves of that degree and genus in $\mathbb{P}^3$. The only nonsingular curves of degree 2 are the conic, with $p_a = 0$, and a disjoint union of two lines, with $p_a = -1$.

(a) Show that the family of all such double lines in $\mathbb{P}^3$ has dimension 9; also show that $H^0(\mathcal{N}_{Y/\mathbb{P}^3}) = 9$, so these curves correspond to a nonsingular open subset of an irreducible component of the Hilbert scheme for Hilbert polynomial $2z - 1$.

(b) By looking at an affine piece, show that the sheaf $\mathcal{T}^1_Y$ is locally isomorphic to $\mathcal{O}_L$. Then, by looking at the sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X \otimes \mathcal{O}_Y \to \mathcal{N}_{Y/X} \to \mathcal{T}^1_Y \to 0$$

conclude that $\mathcal{T}^1_Y \cong \mathcal{O}_L(-2)$. It follows that even though there are many local deformations of Y, the sheaf $\mathcal{T}^1_Y$ has no global sections, so by (Ex. 5.7) every global abstract deformation of Y is locally trivial.

(c) Now show that $\mathcal{T}_Y$ belongs to an exact sequence

$$0 \to \mathcal{O}_L(-1) \oplus \mathcal{O}_L(2) \to \mathcal{T}_Y \to \mathcal{O}_L(1) \to 0.$$

Conclude that $H^1(\mathcal{T}_Y) = 0$, so as an abstract scheme, Y is rigid.

5.11.   Use the method of (5.5.1) to show that for any $n \geq 2$ and any $d \geq 2$ the cone over the Veronesean d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ is rigid. (Watch out for the case $n = 2$, $d = 3$!)

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