

# Chapter 2

## Distributed-Order Linear Time-Invariant System (DOLTIS) and Its Stability Analysis

### 2.1 Introduction

By using distributed-order concept, we can describe the dynamical properties of real world system more accurately, so distributed-order system identification problem was studied in Hartley and Lorenzo (1999, 2003, 2004). In the following sections, the stability analysis of distributed-order linear time-invariant systems in four cases are first studied, then the frequency-domain responses are presented, and time-domain responses on the basis of numerical inverse Laplace transform technique are shown in details.

### 2.2 Stability Analysis of DOLTIS in Four Cases

Consider a distributed-order system described by the following linear time-invariant (LTI) distributed-order differential equation (DODE) and algebraic output equation:

$$\begin{aligned} {}_0D_t^{w(\alpha)}x(t) &= \int_0^1 w(\alpha) {}_0D_t^\alpha x(t) d\alpha = Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \quad (2.1)$$

where  $w(\alpha)$  is the function of distribution of order  $\alpha \in [0, 1]$ ,  ${}_0D_t^\alpha$  denotes the Caputo fractional-order derivative operator,  $A, B, C, D$  are matrices with appropriate dimensions.

*Remark 2.1* Since any interval  $(\gamma_1, \gamma_2)$  can be converted to  $(0, 1)$  through variable substitution, without loss of generality, the integral interval in (2.1) is considered to be  $(0, 1)$ .

For the distributed-order derivative operator  $D^{w(\alpha)}x(t)$ , the Laplace transform is

$$L \left\{ D^{w(\alpha)}x(t) \right\} (s) = \tilde{x}(s) \int_0^1 w(\alpha) s^\alpha d\alpha - x(0) \frac{1}{s} \int_0^1 w(\alpha) s^\alpha d\alpha, \quad s \in \mathbb{C} \setminus (-\infty, 0]$$

where  $\tilde{x}(s) = L \{x(t)\} (s) := \int_0^\infty x(t) e^{-st} dt$ . By applying the Laplace transform to (2.1) with the assumptions that  $x(0) = 0$ ,  $u(t) = \delta(t)$  ( $\delta(t)$  is the Dirac delta distribution), one obtains

$$\tilde{x}(s) \int_0^1 w(\alpha) s^\alpha d\alpha = A \tilde{x}(s) + B$$

i.e.,

$$\tilde{x}(s) = \left( \left( \int_0^1 w(\alpha) s^\alpha d\alpha \right) I - A \right)^{-1} B$$

where  $I$  is the identity matrix. Application of the inverse Laplace transform to the previous expression yields

$$x(t) = L^{-1} \left[ \left( \left( \int_0^1 w(\alpha) s^\alpha d\alpha \right) I - A \right)^{-1} B \right] (t), \quad t > 0. \quad (2.2)$$

In the following, four different cases of the weighting function of order are discussed respectively.

*Case 1*  $w(\alpha) = 1$

In this case, it can be followed by (2.2) that

$$\begin{aligned} x(t) &= L^{-1} \left[ \left( \left( \int_0^1 s^\alpha d\alpha \right) I - A \right)^{-1} B \right] (t) \\ &= L^{-1} \left[ \left( \frac{s-1}{\ln s} I - A \right)^{-1} B \right] (t) \\ &= L^{-1} \left[ \ln s (sI - (I + \ln s A))^{-1} B \right] (t). \end{aligned} \quad (2.3)$$

*Remark 2.2* It is well known from complex analysis (Asmar and Jones 2002) that complex logarithm  $\ln z = \ln |z| + i \arg z$  ( $z \neq 0$ ) defines a multiple-valued function, because  $\arg z$  is multiple-valued. For term  $\ln s$  in (2.3), we know that it is a multi-valued function of the complex variable  $s$  whose domain can be seen as a Riemann surface (Cuadrado and Cabanes 1989; Westerlund and Ekstam 1994) of a number of sheets which is infinite. Note that in multiple-valued functions only the first Riemann sheet has its physical significance (Gross and Braga 1961), so we can make  $\ln s$  a single-valued function by specifying a single-valued  $-\pi < \arg s < \pi$ . Because

$s = 0$  and  $s$  on the negative real axis are nonremovable discontinuities, the branch cut of  $\ln s$  is  $(-\infty, 0]$ .

**Definition 2.1** A distributed-order system  $H(s)$  defined by its impulse response  $h(t) = L^{-1}\{H(s)\}$  is BIBO stable if and only if  $\forall u \in L^\infty(R^+)$ ,  $h * u \in L^\infty(R^+)$ .  $*$  stands for the convolution product and  $L^\infty(R^+)$  stands for the Lebesgue space of measurable function  $h$  such that  $\sup_{t \in R^+} |h(t)| < \infty$ .

Based on Definition 2.1 and the above analysis, the following theorem can be established.

**Theorem 2.1** The distributed-order linear time-invariant system (2.1) with transfer function  $G_1(s) = C \ln s (sI - (I + A \ln s))^{-1} B + D$  is BIBO stable, if and only if all the eigenvalues of  $A$  lie on the left of curve  $l_1 := l_a \cup l_b$  in the complex plane, where  $l_a$  and  $l_b$  are symmetrical with respect to the real axis, and

$$l_a := \left\{ x - iy \mid x = \frac{2\pi\omega - 4 \ln \omega}{4(\ln \omega)^2 + \pi^2}, y = \frac{4\omega \ln \omega + 2\pi}{4(\ln \omega)^2 + \pi^2} \right\}$$

with  $\omega \in [0, \infty)$ .

*Proof* (if part) Note that the final value theorem implies that  $\lim_{t \rightarrow \infty} g(t) = sG_1(s) \rightarrow 0$ , if all poles of  $sG_1(s)$  are in the left half-plane when  $s \rightarrow 0$ . It can be easily known that all the poles of  $sG_1(s)$  satisfy the transcendental characteristic equation of the form

$$|(s - 1)I - A \ln s| = 0. \quad (2.4)$$

From (2.4) we know that  $\frac{s-1}{\ln s} = \sigma_i(A)$  ( $i = 1, \dots, n$ ), where  $\sigma(A)$  denotes the set of eigenvalues of  $A$ . As all the zeros of (2.4) should lie in the left half-plane to ensure the BIBO stability of distributed-order system  $G_1(s)$ , it is necessary to derive the range of  $\lambda = \frac{s-1}{\ln s}$  when  $s$  belongs to the left half-plane.

It is natural to determine the range of  $\lambda = \frac{s-1}{\ln s}$  when  $s$  lies on the imaginary axis. Then, for  $s = j\omega$ ,  $(-\infty < \omega < 0)$ , we have

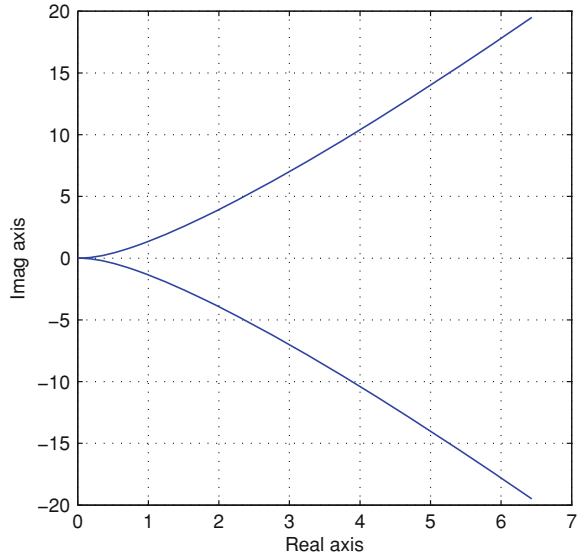
$$\lambda = \frac{(2\pi(-\omega) - 4 \ln(-\omega)) + j(4(-\omega) \ln(-\omega) + 2\pi)}{4(\ln(-\omega))^2 + \pi^2}$$

while for  $s = j\omega$ ,  $(0 \leq \omega < \infty)$ , we have

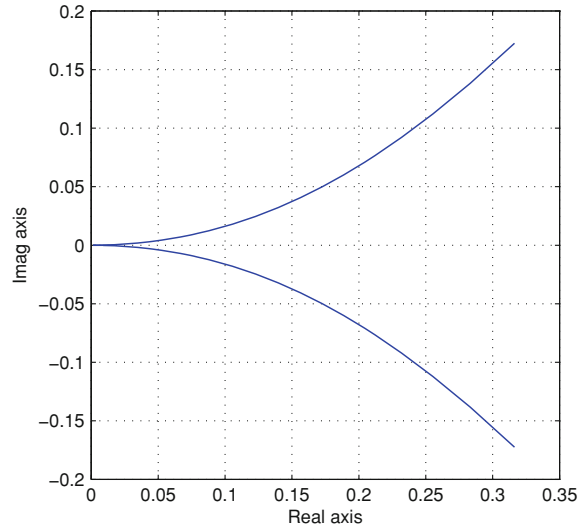
$$\lambda = \frac{(2\pi\omega - 4 \ln \omega) - j(4\omega \ln \omega + 2\pi)}{4(\ln \omega)^2 + \pi^2}$$

which means that the imaginary axis is mapped to a curve denoted by  $l_1$ , which is symmetrical with respect to the real axis. By choosing a point  $s$  randomly which lies on the left of the imaginary axis, the range of  $\lambda = \frac{s-1}{\ln s}$  lies on the left of curve  $l_1$ , which means that the stable region of distributed-order system (2.1) is the left region

**Fig. 2.1** The stable boundary of the distributed-order system (2.1)  $G_1(s)$



**Fig. 2.2** The stable boundary of the distributed-order system (2.1)  $G_1(s)$  (Zoomed)



of curve  $l_1$ . In the following,  $l_1$  is plotted in Fig. 2.1, with the local property around 0 zoomed in Fig. 2.2.

It can be easily known from the above analysis that if all the eigenvalues of  $A$  lie on the left of curve  $l_1$ , all the poles of  $sG_1(s)$  lie on the left half-plane. From the final value theorem, we further know that  $\lim_{t \rightarrow \infty} g(t) = 0$ , which means the

distributed-order system with transfer function  $G_1(s) = C \ln s (sI - (I + A \ln s))^{-1} B + D$  is BIBO stable.

(only if part) It is obviously known from Definition 2.1 that  $G_1(s)$  lies in  $H_\infty$ , the space of bounded analytic functions on the right half plane of the complex plane, which means that all the poles of  $G_1(s)$  lie in the left half plane of the complex plane. From the proof of (if part), it is known that  $\{s_k\}_{k=1,2,\dots,n}$  lie in the open left half plane, which is equivalent to that all the eigenvalues of  $A$  lie in the left region with respect to  $l_1$ .

**Remark 2.3** It is easy to conclude that the slope of the curve  $l_1$  at the original point is 0, and is infinity at the infinite point, which means that any ray in the first quadrant starts at point 0 will have point of intersection with the curve  $l_1$ . This means any constant fractional-order approximation of DODS is problematic, since the stability domains are different.

**Case 2**  $w(\alpha) = \alpha$

In this case, the following can be obtained under the similar analysis procedure in Case 1,

$$\begin{aligned} x(t) &= L^{-1} \left[ \left( \left( \int_0^1 \alpha s^\alpha d\alpha \right) I - A \right)^{-1} B \right] \\ &= L^{-1} \left[ \left( \frac{1-s+s \ln s}{\ln^2 s} I - A \right)^{-1} B \right] \\ &= L^{-1} \left[ \ln^2 s \left( (1-s+s \ln s) I - \ln^2 s A \right)^{-1} B \right]. \end{aligned}$$

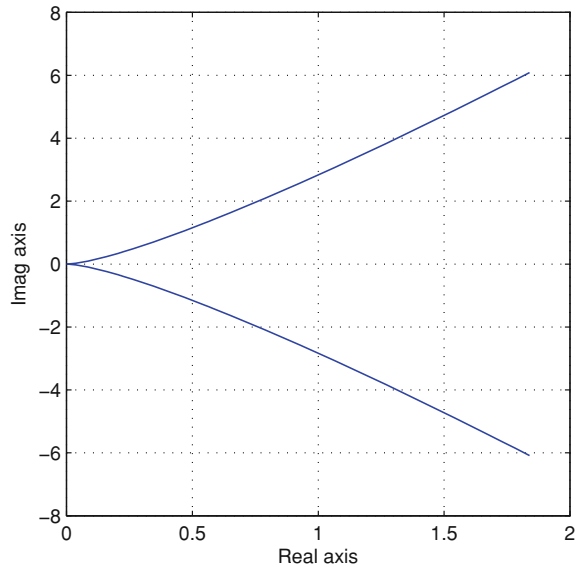
**Theorem 2.2** *The distributed-order linear time-invariant system (2.1) with transfer function  $G_2(s) = C \ln^2 s ((1-s+s \ln s)I - A \ln^2 s)^{-1} B + D$  is BIBO stable, if and only if all the eigenvalues of  $A$  lie on the left of curve  $l_2 := l_c \cup l_d$ , where  $l_c$  and  $l_d$  are symmetrical with respect to the real axis, and  $l_c := \{x + iy \mid x = x_\omega, y = y_\omega, \omega \in (0, \infty)\}$ , with notations*

$$x_\omega = \frac{\left( \ln \omega - \frac{\pi^2}{4} \right) \left( 1 - \frac{\pi}{2} \omega \right) - \pi \ln \omega (\omega - \omega \ln \omega)}{\left( \ln^2 \omega + \frac{\pi^2}{4} \right)^2}$$

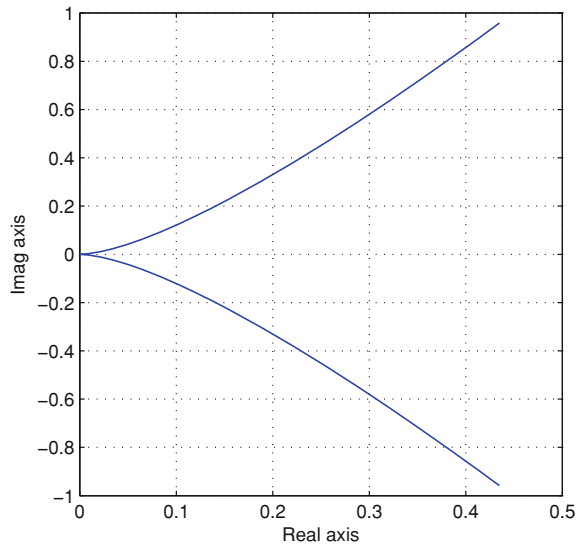
and

$$y_\omega = \frac{\left( \ln \omega - \frac{\pi^2}{4} \right) (\omega - \omega \ln \omega) + \pi \ln \omega \left( 1 - \frac{\pi}{2} \omega \right)}{\left( \ln^2 \omega + \frac{\pi^2}{4} \right)^2}.$$

**Fig. 2.3** The stable boundary of distributed-order system (2.1)  $G_2(s)$



**Fig. 2.4** The stable boundary of distributed-order system (2.1)  $G_2(s)$  (Zoomed)



The proof of Theorem 2.2 can be given by the similar procedures in Theorem 2.1, the stable boundary for distributed-order system  $G_2(s)$  is shown in Fig. 2.3, with the local property around 0 shown in Fig. 2.4.

*Case 3*  $w(\alpha) = \delta(\alpha - \beta)$ ,  $(0 < \beta < 1)$

In this case, the DODE (2.1) converts to a constant-order fractional-order system described by

$$\begin{aligned} {}_0D_t^\beta x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \quad (2.5)$$

Using Laplace transform, the irrational transfer function of fractional-order system (2.5) with null initial conditions is

$$G_3(s) = C(s^\beta I - A)^{-1}B + D. \quad (2.6)$$

*Remark 2.4* Note that term  $s^\beta$  in (2.6) defines a multi-valued function of the complex variable  $s$  whose domain can be seen as a Riemann surface (Cuadrado and Cabanes 1989; Westerlund and Ekstam 1994) of a number of sheets which is finite in the case of  $\beta \in \mathbb{Q}^+$ , and infinite in the case of  $\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ . It is well known that in multiple-valued functions only the principal sheet defined by  $-\pi < \arg s < \pi$  has its physical significance (Gross and Braga 1961).

The following can be obtained under the similar analysis procedure in the previous cases,

$$\begin{aligned} x(t) &= L^{-1} \left[ \left( \left( \int_0^1 \delta(\alpha - \beta) s^\alpha d\alpha \right) I - A \right)^{-1} B \right] (t) \\ &= L^{-1} \left[ (s^\beta I - A)^{-1} B \right] (t). \end{aligned}$$

The following theorem which corresponds to the stability condition of fractional-order system obtained in Matignon (1996) can be given.

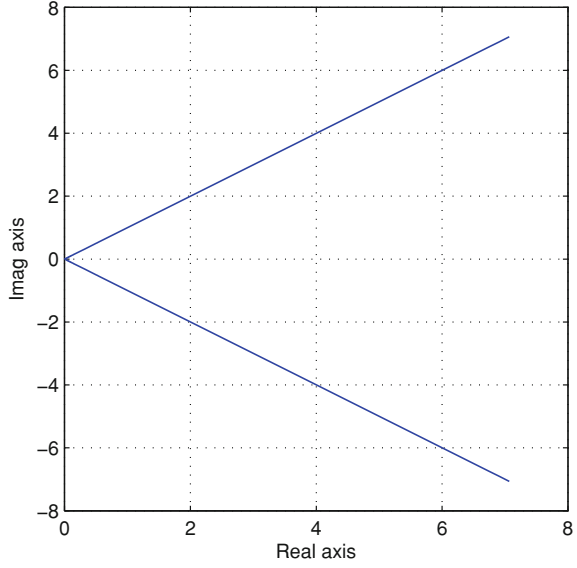
**Theorem 2.3** *The fractional-order linear time-invariant system with transfer function  $G_3(s) = C(s^\beta I - A)^{-1}B + D$  is BIBO stable, if and only if all the eigenvalues of  $A$  lie on the left of curve  $l_3 := l_e \cup l_f$ , where  $l_e$  and  $l_f$  are symmetrical with respect to the real axis, and  $l_e := \{re^{i\theta} \mid r = \omega^\beta, \theta = \pi\beta/2, \omega \in (0, \infty)\}$ .*

The proof of Theorem 2.3 can be given by the similar procedures in Theorem 2.1, the stable region for fractional-order system  $G_3(s)$  with  $\beta = 0.5$  is shown in Fig. 2.5.

*Case 4*  $w(\alpha) = \sum_{k=1}^n b_k \delta(\alpha - k\beta)$ ,  $(0 < n\beta < 1)$ .

In this case, the DODE (2.1) converts to the so-called LTI commensurate fractional-order system

**Fig. 2.5** The stable boundary of fractional-order system (2.5)  $G_3(s)$



$$\sum_{k=1}^n b_{k0} D_t^{k\beta} x(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t). \quad (2.7)$$

Let  $\hat{x}(t) = [x(t) \ D^\beta x(t) \ D^{2\beta} x(t) \ \cdots \ D^{(n-1)\beta} x(t)]^T$ , (2.7) can be converted to the following equivalent form

$${}_0D t^\beta \hat{x}(t) = \hat{A} \hat{x}(t) + \hat{B} u(t) \quad (2.8)$$

where  $\hat{A} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ \frac{A}{b_n} - \frac{b_1}{b_n} I & -\frac{b_2}{b_n} I & \cdots & -\frac{b_{n-1}}{b_n} I \end{bmatrix}$ ,  $\hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{B}{b_n} \end{bmatrix}$ .

Now we have changed Case 4 to Case 3, which can be similarly analyzed.

The following can be obtained under the similar analysis procedure in the previous cases,



$$\begin{aligned}
x(t) &= L^{-1} \left[ \left( \left( \int_0^1 \sum_{n=1}^N b_n \delta(\alpha - \beta_n) s^\alpha d\alpha \right) I - A \right)^{-1} B \right] \\
&= L^{-1} \left[ \left( \sum_{n=1}^N b_n s^{\beta_n} I - A \right)^{-1} B \right].
\end{aligned}$$

In the following, Case 4 will not be considered.

### 2.3 Time-Domain Analysis: Impulse Responses

*Case 1*  $w(\alpha) = 1$

As the transfer function of distributed-order system for Case 1 with the assumption that  $D = 0$  is  $G_1(s) = C \ln s ((s - 1)I - A \ln s)^{-1} B$ , using the similar method of impulse response for distributed-order integrator/differentiator in Li et al. (2010), the inverse Laplace transform of  $G_1(s)$  can be derived as follows.

$$\begin{aligned}
y_1(t) &= L^{-1} \{G_1(s)\} \\
&= C \left( \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{-st} \ln s (sI - (I + \ln s A))^{-1} ds \right) B \\
&= C \left( \int_0^\infty e^{-xt} (x + 1) A_1^{-1} dx \right) B
\end{aligned} \tag{2.9}$$

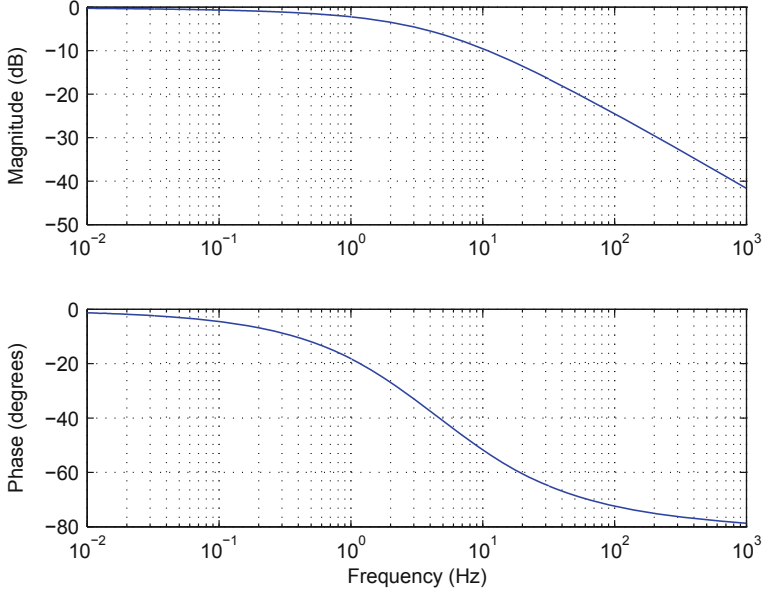
where  $A_1 := ((x + 1)I + A \ln x)^2 + (A\pi)^2$ .

*Case 2*  $w(\alpha) = \alpha$

Following the same procedures, the transfer function of distributed-order system for Case 2 with  $D = 0$  is  $G_2(s) = C \ln^2 s ((s - 1 - \ln s)I - A \ln^2 s)^{-1} B$ , using the similar method of impulse response for distributed-order integrator/differentiator in Li et al. (2010), the inverse Laplace transform of  $G_2(s)$  can be derived as follows.

$$\begin{aligned}
y_2(t) &= L^{-1} \{G_2(s)\} \\
&= C \left( \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st} \ln^2 s \left( (1 - s + s \ln s)I - \ln^2 s A \right)^{-1} ds \right) B \\
&= C \left( \int_0^\infty e^{-xt} \left( \frac{((1 + x - x \ln x)I + (\ln^2 x - \pi^2)A)^2}{+\pi^2(xI + 2 \ln x A)^2} \right) A_2^{-1} dx \right) B
\end{aligned} \tag{2.10}$$

where  $A_2 := ((1 + x - x \ln x)I + (\ln^2 x - \pi^2)A)^2 + \pi^2(xI + 2 \ln x A)^2$ .



**Fig. 2.6** The Bode plot of distributed-order system  $g_2(s) = \frac{\ln^2 s}{1-s+s \ln s + \ln^2 s}$

*Case 3*  $w(\alpha) = \delta(\alpha - \beta)$ ,  $(0 < \beta < 1)$

The transfer function of fractional-order system for Case 3 with the assumption that  $D = 0$  is  $G_3(s) = C(s^\beta I - A)^{-1}B$ , the inverse Laplace transform of  $G_3(s)$  with null initial condition is

$$y_3(t) = C \left( t^{\beta-1} E_{\beta,\beta}(At^\beta) \right) B \quad (2.11)$$

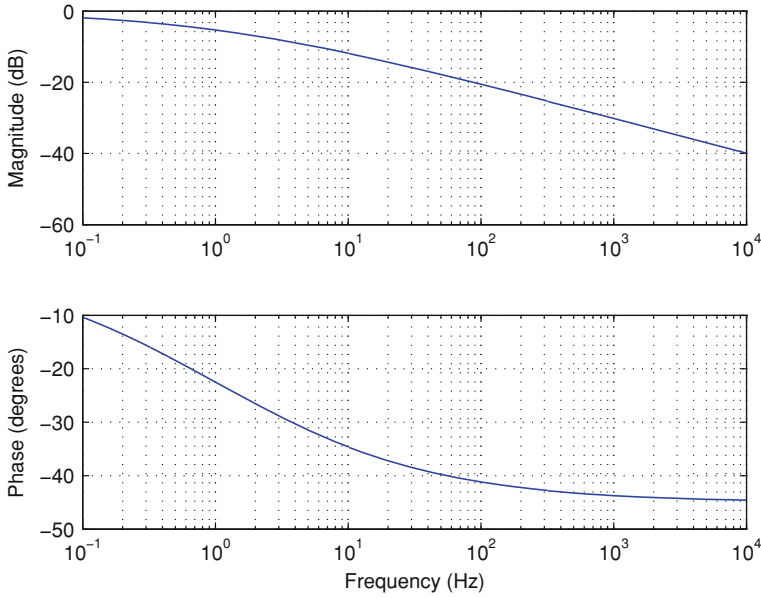
where  $E_{\alpha,\beta}(\cdot)$  is the Mittag-Leffler function in two parameters defined as in Podlubny (1999)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\Re(\alpha, \beta) > 0).$$

*Remark 2.5* Computing (2.9), (2.10) and (2.11) can be easily realized in MATLAB numerically.

## 2.4 Frequency-Domain Response: Bode Plots

Generally, the frequency domain response has to be obtained by the direct evaluation of the irrational transfer function of distributed-order system along the imaginary axis for  $s = j\omega$ ,  $\omega \in (0, \infty)$ . For simplicity, Bode plots of some scalar transfer functions for Case 1 to Case 3 are shown as follows.



**Fig. 2.7** The Bode plot of fractional-order system  $g_3(s) = \frac{1}{s^{0.5}+1}$

For  $w(\alpha) = 1$ , the frequency-domain response of  $g_1(s) = \frac{\ln s}{s-1+\ln s}$  is shown in Fig. 2.6.

For  $w(\alpha) = \alpha$ , the frequency-domain response of  $g_2(s) = \frac{\ln^2 s}{1-s+s \ln s+\ln^2 s}$  is shown in Fig. 2.7.

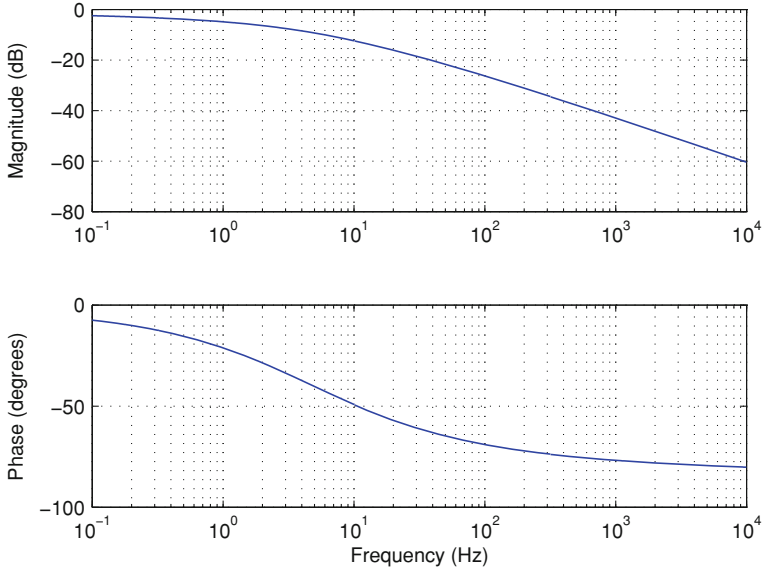
For  $w(\alpha) = \delta(\alpha - \beta)$ , ( $0 < \beta < 1$ ), the frequency-domain response of  $g_3(s) = \frac{1}{s^{0.5}+1}$  is shown in Fig. 2.8.

## 2.5 Numerical Examples

In this section, numerical examples are shown to demonstrate the effectiveness of the proposed results.

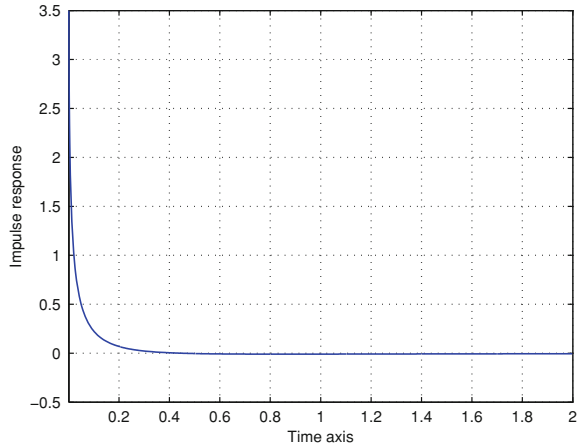
*Example 1* Consider a distributed-order system with Case 1 described with parameters given as  $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$  and  $D = 0$ .

The eigenvalues of  $A$  are  $\lambda_1 = 1 + 2j$  and  $\lambda_2 = 1 - 2j$ , so it can be known from Theorem 2.1 that this distributed-order system is bounded-input bounded-output stable. Using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.9 and 2.10, respectively.



**Fig. 2.8** The Bode plot of distributed-order system  $g_1(s) = \frac{\ln s}{s-1+\ln s}$

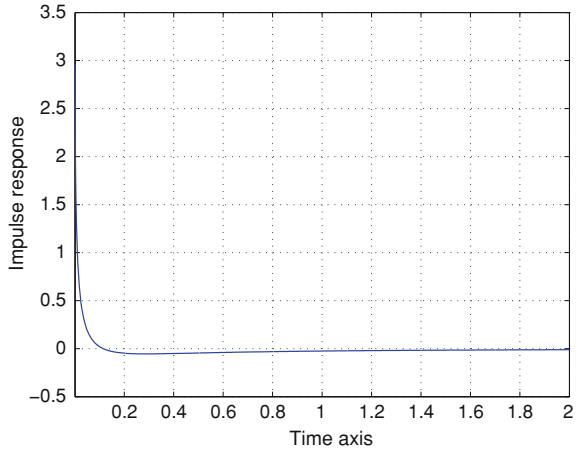
**Fig. 2.9** The state  $x_1$  of stable distributed-order system (2.1) for Case 1



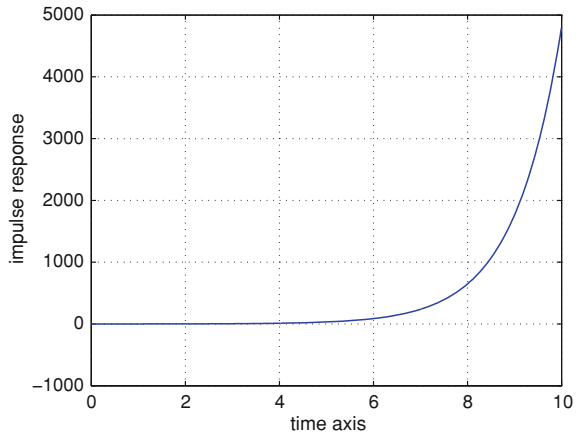
**Example 2** Consider a distributed-order system with Case 1 described with parameters given as  $A = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $C = [2 \ 1]$  and  $D = 0$ .

The eigenvalues of  $A$  are  $\lambda_1 = 2 + 2j$  and  $\lambda_2 = 2 - 2j$ , and it can be known from Theorem 2.1 that this distributed-order system is not bounded-input bounded-output stable. Using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.11 and 2.12, respectively.

**Fig. 2.10** The state  $x_2$  of stable distributed-order system (2.1) for Case 1



**Fig. 2.11** The state  $x_1$  of unstable distributed-order system (2.1) for Case 1



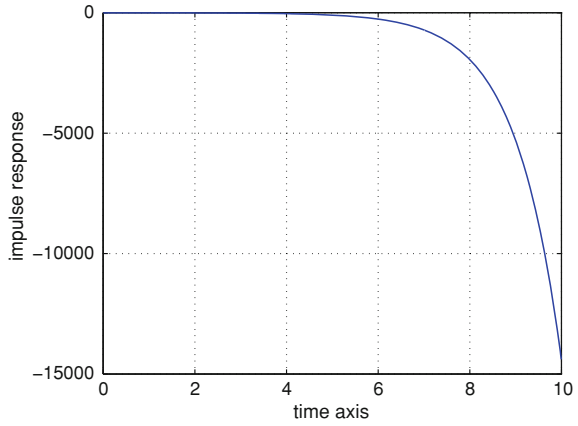
*Example 3* Consider a distributed-order system with Case 2 described with parameters given as  $A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $C = [2 \ 1]$  and  $D = 0$ .

The eigenvalues of  $A$  are  $\lambda_1 = 1 + 3j$  and  $\lambda_2 = 1 - 3j$ , so it can be known from Theorem 2.2 that this distributed-order system is bounded-input bounded-output stable, and the states of impulse response with null initiations are shown in Figs. 2.13 and 2.14, respectively.

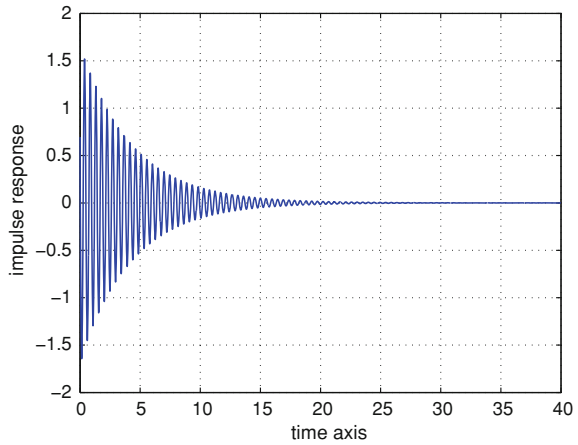
*Example 4* Consider a distributed-order system with Case 2 described with parameters given as  $A = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $C = [2 \ 1]$  and  $D = 0$ .

The eigenvalues of  $A$  are  $\lambda_1 = 2 + 2j$  and  $\lambda_2 = 2 - 2j$ , it can be known from Theorem 2.2 that this distributed-order system is not bounded-input bounded-output

**Fig. 2.12** The state  $x_2$  of unstable distributed-order system (2.1) for Case 1



**Fig. 2.13** The state  $x_1$  of stable distributed-order system (2.1) for Case 2



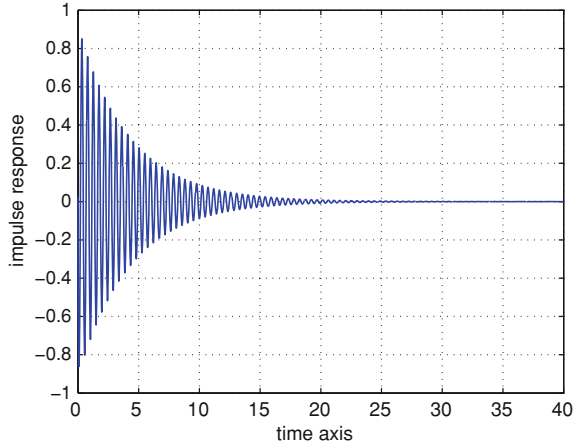
stable, and by using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.15 and 2.16, respectively.

*Example 5* Consider a fractional-order system for Case 3 described with parameters given as  $\alpha = 0.5$ ,  $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $C = [2 \ 1]$  and  $D = 0$ .

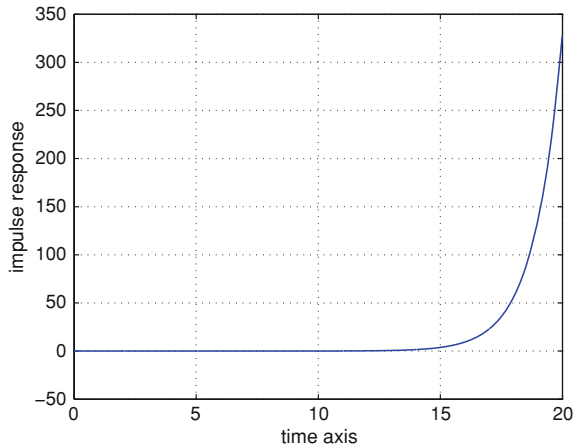
The eigenvalues of  $A$  are  $\lambda_1 = 2j$  and  $\lambda_2 = -2j$ , it can be known from Theorem 2.3 that this fractional-order system is bounded-input bounded-output stable. Using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.17 and 2.18, respectively.

*Example 6* Consider a fractional-order system for Case 3 described with parameters given as  $\alpha = 2/3$ ,  $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $C = [2 \ 1]$  and  $D = 0$ .

**Fig. 2.14** The state  $x_2$  of stable distributed-order system (2.1) for Case 2



**Fig. 2.15** The state  $x_1$  of unstable distributed-order system (2.1) for Case 2

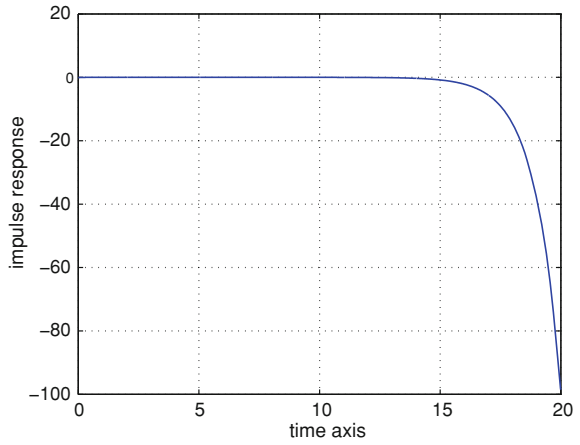


Since the eigenvalues of  $A$  are  $\lambda_1 = 1 + j$  and  $\lambda_2 = 1 - j$ , it can be known from Theorem 2.3 that this fractional-order system is bounded-input bounded-output stable. Using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.19 and 2.20, respectively.

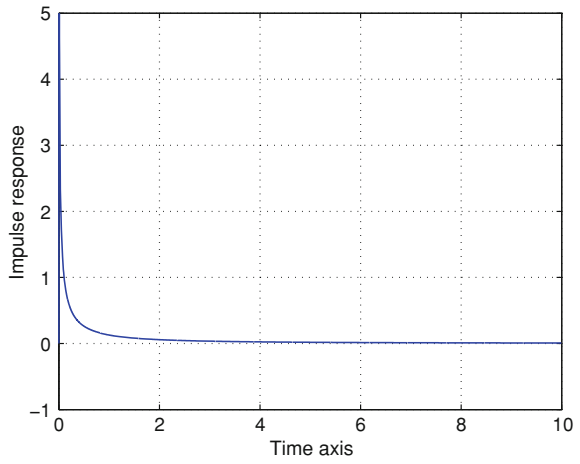
## 2.6 Chapter Summary

In this chapter, the bounded-input bounded-output stability conditions for four kinds of linear time-invariant distributed-order system whose integral interval being  $(0, 1)$  have been derived for the first time. Based on the final value property of Laplace transform, sufficient and necessary conditions of stability for distributed-order sys-

**Fig. 2.16** The state  $x_2$  of unstable distributed-order system (2.1) for Case 2



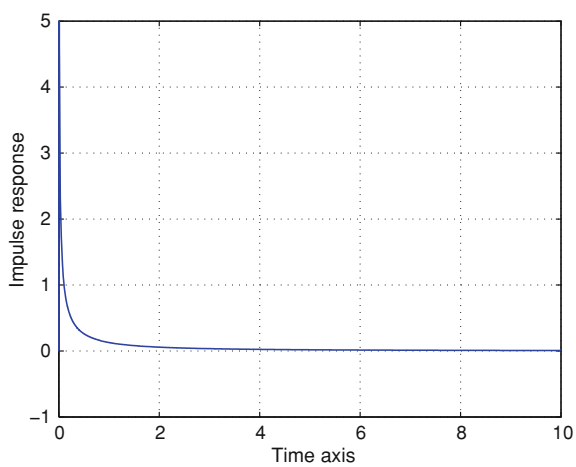
**Fig. 2.17** The state  $x_1$  of stable fractional-order system (2.5) for Case 3



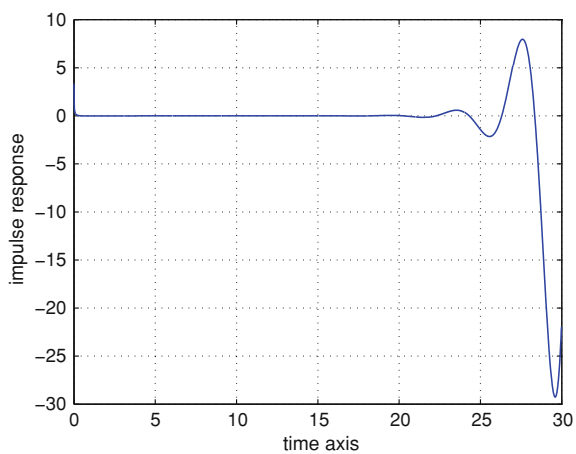
tems are presented. In addition, time-domain and frequency-domain responses are presented with six illustrative numerical examples. Detailed MATLAB codes are shown in Appendix A.



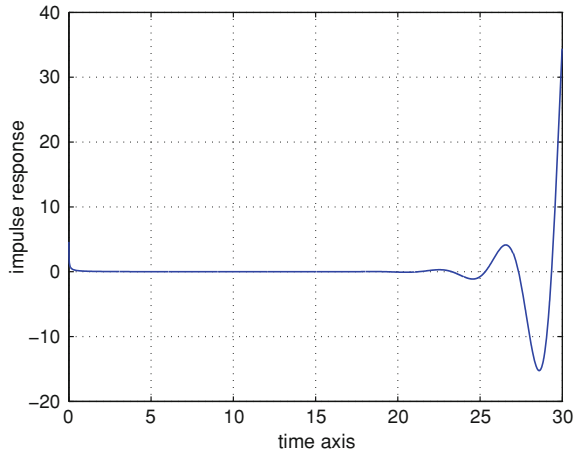
**Fig. 2.18** The state  $x_2$  of stable fractional-order system (2.5) for Case 3



**Fig. 2.19** The state  $x_1$  of unstable fractional-order system (2.5) for Case 3



**Fig. 2.20** The state  $x_2$  of unstable fractional-order system (2.5) for Case 3



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