Chapter 2
Time and Distance

Abstract An elementary geometric fact, stated as the intercept theorem, makes an observed clock run visibly slower, if it moves away in the line of sight and to run visibly faster by the inverse factor, if it approaches the observer with the same velocity. This Doppler effect of light in the vacuum is particularly simple, because, different from the Doppler effect of sound, it depends only on the relative velocity of the light source and its observer. We employ a referee to determine whether moving clocks are equal and how the times between pairs of events compare. This time endows spacetime with a geometric structure, the distance, which is similar to but also different from Euclidean distance. From the Doppler effect we determine the addition of velocities, time dilation and length contraction and clarify the related paradoxes.

2.1 Theorem of Minkowski

Consider as in Fig. 2.1 a clock $\mathscr{C}$ and an observer $\mathscr{O}$ with another clock. Both move uniformly along straight worldlines and meet in the event O, the origin, where their worldlines intersect. There both clocks are set to zero for simplicity, so that in the following we can speak of times rather than of time differences. When the observer $\mathscr{O}$ looks at the clock $\mathscr{C}$, which moves away from him uniformly in the line of sight, then he reads off the time $t_{\mathscr{C}}$ which passed on $\mathscr{C}$ until the emission of the light. At the moment of observation his own clock shows a time $t_{\mathscr{O}}$. This time of reception is proportional to the time of emission[1]

$$t_{\mathscr{O}} = \kappa(\mathscr{O}, \mathscr{C})\, t_{\mathscr{C}} \quad \text{for } t_{\mathscr{C}} > 0, \tag{2.1}$$

[1] κ and ν are the Greek letters kappa and nu.

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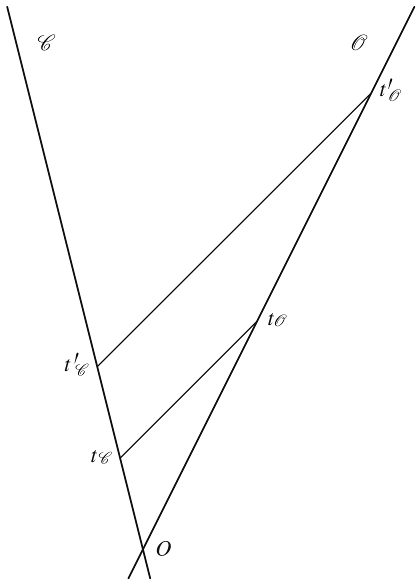


Fig. 2.1 Intercept theorem

with a coefficient $\kappa(\mathscr{O}, \mathscr{C})$, which does not depend on $t_{\mathscr{C}}$ [7]: if the observer later reads off the time $t'_{\mathscr{C}}$, then the triangle $Ot'_{\mathscr{C}}t'_{\mathscr{O}}$ is similar to $Ot_{\mathscr{C}}t_{\mathscr{O}}$ and all distances are enlarged by the same factor. Therefore the ratios $t_{\mathscr{O}}/t_{\mathscr{C}}$ and $t'_{\mathscr{O}}/t'_{\mathscr{C}}$ coincide.

If a quartz is carried along with the clock $\mathscr{C}$ and oscillates n times during the time $t_{\mathscr{C}}$ with a frequency $\nu_{\mathscr{C}} = n/t_{\mathscr{C}}$, then the observer $\mathscr{O}$ sees these oscillations while on his own clock the time $t_{\mathscr{O}}$ elapses. So he observes the frequency

$$\nu_{\mathscr{O}} = \frac{1}{\kappa(\mathscr{O}, \mathscr{C})}\nu_{\mathscr{C}}. \tag{2.2}$$

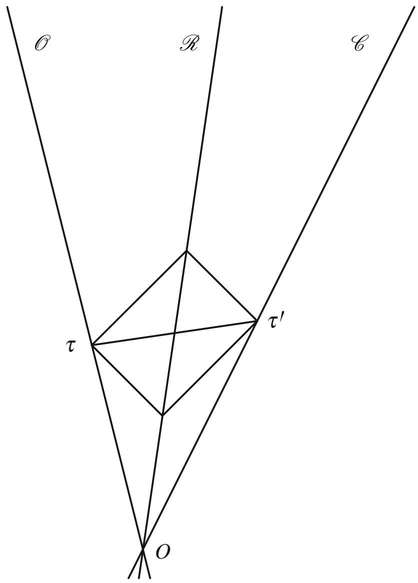
This visible change of frequency of the clock which moves in the line of sight is the longitudinal Doppler effect. It is related to the Doppler effect of sound, which one can hear as whining drop of the pitch of passing police cars or racing cars.

As one cannot distinguish rest from uniform motion, the Doppler factor $\kappa(\mathscr{O}, \mathscr{C})$ only depends on the relative velocity of $\mathscr{C}$ and $\mathscr{O}$ and, contrary to the Doppler effect of sound, not on the velocity with respect to a medium.

Moreover, κ depends on whether both clocks run equally fast. For two clocks at rest this can be easily seen. For the moving clocks $\mathscr{O}$ and $\mathscr{C}$ this is more difficult. One has to correct for the various and changing times which it takes light to run from the clock to the observer who compares both clocks.

However, no correction is necessary for a referee $\mathscr{R}$ as in Fig. 2.2 who is always in the middle of the clocks. Flashes of light which he emits at some time to $\mathscr{O}$ and $\mathscr{C}$ are reflected and return in the same instant. Because the referee is always in the middle, the runtimes of light to and from $\mathscr{O}$ and $\mathscr{C}$ are equal.

Fig. 2.2 Comparison of clocks



Both clocks run the same if they show the referee equal times,

$$\tau' = \tau. \tag{2.3}$$

This is the geometric definition of equal lengths on intersecting straight worldlines of moving observers. Without exception the definition agrees with the physical behavior of real equal clocks.

We continue the worldlines of the light rays, which are received and reflected by $\mathscr{C}$ as it shows the time τ', to the worldline of the observer $\mathscr{O}$ and denote in Fig. 2.3 with t_- and t_+ the times shown by the clock of $\mathscr{O}$ as he emits the light ray to $\mathscr{C}$ and receives it, respectively. Due to (2.1) the clock $\mathscr{O}$ shows the time

$$\tau = \kappa(\mathscr{O}, \mathscr{R})\kappa(\mathscr{R}, \mathscr{O})t_- \tag{2.4}$$

when the light ray emitted at time t_- and reflected by $\mathscr{R}$ arrives. This is because τ is a multiple of the time at which the light ray was reflected by $\mathscr{R}$, and this time is a multiple of the time t_- at which the light ray was emitted by $\mathscr{O}$. By the same reason

$$t_+ = \kappa(\mathscr{O}, \mathscr{R})\kappa(\mathscr{R}, \mathscr{O})\tau. \tag{2.5}$$

Thus, $t_+/\tau = \tau/t_-$ and $\tau^2 = t_+t_- = t^2 - r^2$ (1.6). Moreover, the equal clocks show equal times, $\tau' = \tau$. This proves the

Theorem of Minkowski *Let two observers $\mathscr{O}$ and $\mathscr{C}$ move linearly and uniformly and meet in some event O, when they set their equal clocks to zero. Then the time τ,*

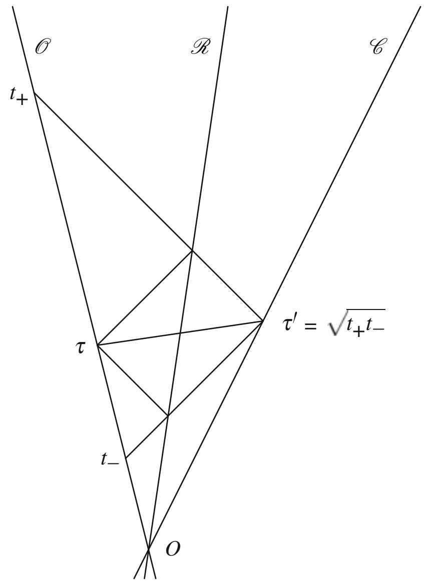


Fig. 2.3 Theorem of Minkowski

which elapses on the clock $\mathscr{C}$ until an event E, is the geometric mean of the time t_- shown by the clock of the observer $\mathscr{O}$ when he emits light to E and the time t_+ which his clock shows when he receives light from E,

$$\tau^2 = t_+ t_- = t^2 - r^2. \tag{2.6}$$

This relation is as important for the geometry of spacetime as the Pythagorean theorem $c^2 = a^2 + b^2$ for Euclidean geometry. According to the Pythagorean theorem in Euclidean geometry all points on a circle are equally far away from the center. The equation $\tau^2 = t^2 - r^2$ implies that in spacetime points of equal temporal distance to the origin O lie on hyperbolas.

Three Equal Clocks

The definition, that equal clocks show their referee equal times differences, is consistent: the clock $\mathscr{O}_3$ equals the clock $\mathscr{O}_1$ if it equals the clock $\mathscr{O}_2$ and if the clock $\mathscr{O}_2$ equals the clock $\mathscr{O}_1$ (Fig. 2.4).

If the clocks move in the same direction and meet in a common event, then the relation

$$t^4 = t_+^2 t_-^2 = t_{++} t_{+-} t_{-+} t_{--} \tag{2.7}$$

holds. As in (2.4) one has

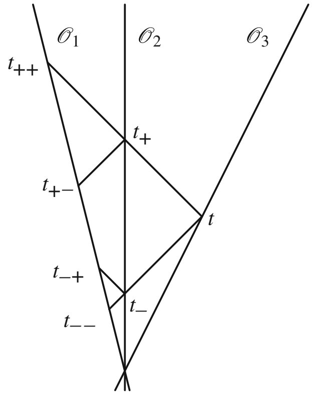


Fig. 2.4 Three equal clocks

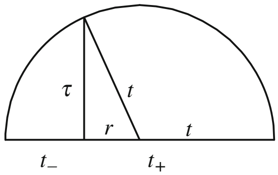


Fig. 2.5 Geometric mean

$$t_{-+} = \kappa(\mathscr{O}_1, \mathscr{O}_2)\kappa(\mathscr{O}_2, \mathscr{O}_1)t_{--} \tag{2.8}$$

and

$$t_{++} = \kappa(\mathscr{O}_1, \mathscr{O}_2)\kappa(\mathscr{O}_2, \mathscr{O}_1)t_{+-}. \tag{2.9}$$

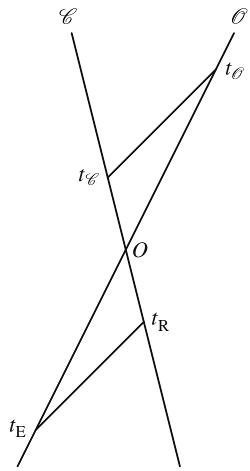
Therefore $t^4 = t_{++}^2 t_{--}^2$ and the clock of $\mathscr{O}_3$ equals the clock of $\mathscr{O}_1$. If two clocks equal a third then they equal each other. This holds also, if the worldlines of the clocks do not lie in a plane or do not intersect because the worldlines can be translated and rotated without changing the clocks.

Construction of the Referee

To construct the worldline of the referee between two observers $\mathscr{O}$ and $\mathscr{C}$ one draws the light rays through a point τ' on one worldline. They intersect the other worldline in t_+ and t_- and determine the geometric mean τ of t_+ and t_-. The worldline of the referee is the straight line through the intersection of the light rays through τ and τ'. This worldline also passes the origin O, because τ is the geometric mean of t_+ and t_- (Fig. 2.5).

The geometric mean $\sqrt{t_+ t_-}$ is constructed in Euclidean geometry by help of a circle with a diameter, which consists of the line segments t_+ and t_-. Its radius is the arithmetic mean $t = (t_+ + t_-)/2$, the line segment t_+ is longer by $r = (t_+ - t_-)/2$,

Fig. 2.6 Towards and away



$t_+ = t + r$, the line segment t_- is shorter $t_- = t - r$. The orthogonal line through the endpoint of t_+ cuts the circle with a segment of length $\tau = \sqrt{t^2 - r^2} = \sqrt{t_+ t_-}$.

2.2 Addition of Velocities

In Fig. 2.3 the Doppler factor $\tau'/t_- = \kappa(\mathscr{C}, \mathscr{O})$ is the ratio of the time of reception to the time of emission (2.1) of light rays sent from the observer $\mathscr{O}$ to $\mathscr{C}$, and $t_+/\tau' = \kappa(\mathscr{O}, \mathscr{C})$ is the ratio for the way back. Both clocks are equal, $\tau = \tau'$. Therefore (2.4) and (2.5) state that the Doppler factor $\kappa(\mathscr{O}, \mathscr{C})$, by which $\mathscr{O}$ sees frequencies of $\mathscr{C}$ shifted, equals the Doppler factor $\kappa(\mathscr{C}, \mathscr{O})$, by which $\mathscr{C}$ perceives shifted frequencies of $\mathscr{O}$

$$\kappa(\mathscr{C}, \mathscr{O}) = \kappa(\mathscr{O}, \mathscr{C}). \tag{2.10}$$

On motion in the line of sight the Doppler shift is reciprocal.

From this reciprocity alone, from $t_+ = \kappa\tau$ together with $\tau = \kappa t_-$, one can conclude Minkowski's theorem and the dependence of the Doppler factor on the relative velocity,

$$\kappa^2 = \frac{t_+}{t_-}, \quad \tau^2 = t_+ t_- . \tag{2.11}$$

The relations $t_+ = t + r$ and $t_- = t - r$ (1.6) imply

$$\kappa^2 = \frac{t+r}{t-r} = \frac{1 + r/t}{1 - r/t}, \quad \tau^2 = t^2 - r^2 = \left(1 - \frac{r^2}{t^2}\right) t^2, \tag{2.12}$$

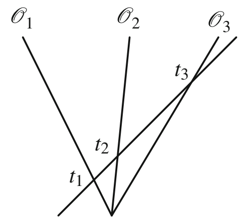


Fig. 2.7 Addition of velocities

and, because $v = r/t$ is the velocity with which the clock $\mathcal{C}$ moves away from the observer $\mathcal{O}$,

$$\kappa(v) = \sqrt{\frac{1+v}{1-v}} = \frac{1+v}{\sqrt{1-v^2}}, \tag{2.13}$$

$$v = \frac{\kappa^2 - 1}{\kappa^2 + 1}, \tag{2.14}$$

$$\tau = \sqrt{1-v^2}\,t. \tag{2.15}$$

If the observer $\mathcal{O}$ emits a pulse of light at a time $t_{\mathrm{E}} < 0$, while the clock moves towards him (recedes with negative velocity) then, as Fig. 2.6 shows, the ratio $\kappa(-v) = t_{\mathrm{R}}/t_{\mathrm{E}}$ of the times of reception and emission is the inverse of the ratios of the times which the clocks show later, when they move away from each other

$$\kappa(-v) = \frac{t_{\mathrm{R}}}{t_{\mathrm{E}}} = \frac{t_{\mathcal{C}}}{t_{\mathcal{O}}} = \frac{1}{\kappa(v)}. \tag{2.16}$$

A clock, which moves away from an observer, appears slower, because it shows him the time $t_{\mathcal{C}} = t_{\mathcal{O}}/\kappa$, when his own and equal clock shows $t_{\mathcal{O}}$ and $\kappa(v)$ is larger than 1 for positive velocity $v > 0$.

On motion in the line of sight an approaching clock appears faster, because during the approach the Doppler factor is inverse to the Doppler factor during recession.

With (2.14) one can determine the velocity v (as is routinely done by traffic authorities) by measuring the Doppler shift κ. It retains its value, if one exchanges observer and observed object. Therefore, two observers who move in the line of sight measure the same relative velocity. We use (2.13) to determine the relative velocities of several observers (Fig. 2.7).

If three observers, $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$, move in the same direction and register the times on their clocks at which a light pulse passes then these times are proportional,

$$t_2 = \kappa_{21} t_1, \quad t_3 = \kappa_{32} t_2, \quad t_3 = \kappa_{31} t_1. \tag{2.17}$$

From $\kappa_{31} t_1 = \kappa_{32}\kappa_{21} t_1$ one immediately concludes

$$\kappa_{31} = \kappa_{32}\kappa_{21}. \tag{2.18}$$

The Doppler factor κ_{31}, by which $\mathcal{O}_3$ sees his clock run faster than the clock of $\mathcal{O}_1$, is the product of the Doppler factor κ_{32}, by which $\mathcal{O}_3$ observes his clock run faster than the clock of $\mathcal{O}_2$ with the Doppler factor κ_{21} for the observer $\mathcal{O}_2$ and the clock of $\mathcal{O}_1$.

In terms of velocities (2.13) and squared this means (in our units with $c = 1$)

$$\frac{1+v_{31}}{1-v_{31}} = \frac{1+v_{32}}{1-v_{32}}\,\frac{1+v_{21}}{1-v_{21}} \tag{2.19}$$

or, solved for v_{31},

$$v_{31} = \frac{v_{32}+v_{21}}{1+v_{32}v_{21}}. \tag{2.20}$$

The velocity v_{31}, with which $\mathcal{O}_3$ sees the observer $\mathcal{O}_1$ recede, is not the sum $v_{32} + v_{21}$ of the velocity v_{32}, with which $\mathcal{O}_3$ observes $\mathcal{O}_2$ recede, and the velocity v_{21}, with which $\mathcal{O}_2$ perceives the recession of $\mathcal{O}_1$. The naive addition of velocities is only approximately correct as long as in ordinary life v_{32} and v_{21} are small compared to the speed of light, $c = 1$.

Up to the sign in the denominator velocities add like inclinations. If the bed of a tipper lorry is inclined by an angle α, then on even ground it has the slope $m_1 = \tan\alpha$. If the truck drives a street with slope $m_2 = \tan\beta$ then its bed has an overall angle $\alpha + \beta$ to the horizontal and the overall inclination

$$m_3 = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\cos\alpha\sin\beta + \sin\alpha\cos\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{m_1+m_2}{1-m_1m_2}. \tag{2.21}$$

We define the rapidity σ as the logarithm of the Doppler factor κ,

$$\sigma = \ln\kappa = \frac{1}{2}\ln\frac{1+v}{1-v}, \quad v = \frac{e^{\sigma} - e^{-\sigma}}{e^{\sigma} + e^{-\sigma}} = \tanh\sigma. \tag{2.22}$$

To the addition of rapidities there corresponds the multiplication of the Doppler factors, $\kappa = e^{\sigma}$. These rapidities, not the velocities, add on motion of several observers in the same direction.

2.3 Time Dilation

If the time t elapses on a clock between two events O and E, then on a second equal clock, which moves relative to the first one with a velocity v, the shorter time (2.15)

$$\tau = \sqrt{1-v^2}\,t \tag{2.23}$$

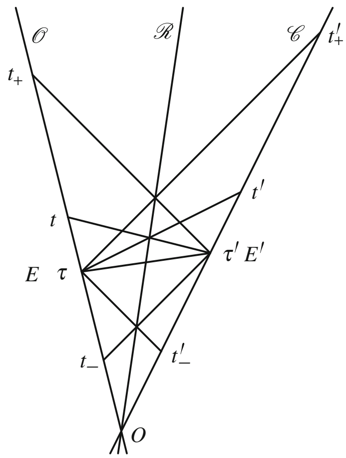


Fig. 2.8 Reciprocal dilation of time

goes by between the corresponding events which are simultaneous to O and E for the first clock.

Time dilation is reciprocal. This can be deduced from Fig. 2.8 where we have continued the light rays of Fig. 2.3 to the worldlines of the observers $\mathscr{O}$ and $\mathscr{C}$.

The events are denoted by the times on the clocks which the observers carry along.

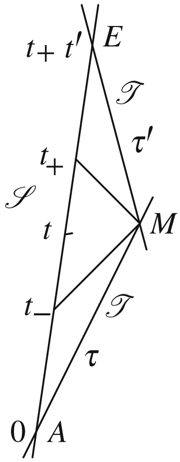
All the time the referee $\mathscr{R}$ is in the middle of $\mathscr{O}$ and $\mathscr{C}$ and sees both clocks show equal times. Therefore the times t_- and t'_- coincide as do τ and τ' and also t_+ and t'_+, because light from each pair of events in which the clocks show these times reaches the referee in the same instant.

For the observer $\mathscr{O}$ the event E', in which the moving clock $\mathscr{C}$ shows the time $\tau' = \tau$, is simultaneous to the event, in which his own clock shows the arithmetic mean $t = (t_+ + t_-)/2$ of the time t_-, which it shows, when light to E' starts and the time t_+, at which the reflected light returns. So for $\mathscr{O}$ the event E' occurs at time t, but the moving clock shows less time, $\tau = \sqrt{t_+ t_-} = \sqrt{1 - v^2}\,t$ (2.15), which is smaller than the arithmetic mean t (given that the velocity v does not vanish).

For $\mathscr{C}$ the event in which his clock shows the time $t' = (t'_+ + t'_-)/2 = t$ is simultaneous to the event E, when the clock of $\mathscr{O}$, which moves with respect to $\mathscr{C}$, shows the time $\tau = \sqrt{t'_+ t'_-} = \sqrt{1 - v^2}\,t$. So for $\mathscr{C}$ the clock of $\mathscr{O}$ runs slower just as well.

Time dilation is reciprocal because the observers do not agree on which events are simultaneous. In Euclidean geometry the corresponding fact is commonplace: if you look in horizontal direction from a lighthouse at sea level to a second lighthouse of identical construction also at sea level some miles away then the other lighthouse does not reach the height of the first one because the surface of the earth is curved. Height depends on which direction is horizontal and the horizontal directions of both lighthouses do not coincide.

Fig. 2.9 Twins



Twin Paradox

Reciprocal time dilation appears to be contradictory, if for example one considers twins. The first twin, the traveller $\mathscr{T}$, departs in an event A with a velocity v for Mars and turns back with velocity v' after the arrival in event M. The other twin, the stay-at-home $\mathscr{S}$, waits calmly the time $t + t'$ until the return of his brother. Which twin, if any, is younger in the end E? For each twin, the other has moved. Does this imply the contradiction, that each twin has aged less than the other?

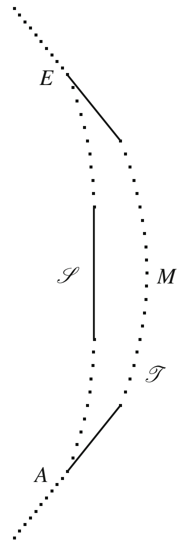
It is often tacitly insinuated that the observations of both twins agree up to the short acceleration at Mars and that from their observations one cannot distinguish the traveller from the stay-at-home. This is wrong.

Each twin sees the other redshifted during the travel to Mars and blueshifted on the way back. In the first period each twin sees the clock of his brother run slower, in the second faster, than his own clock by a Doppler factor which agrees with the Doppler factor of his brother. But the stay-at-home sees the traveller longer redshifted (and age slower) and shorter blueshifted (and age faster) than the traveller sees the stay-at-home. Both twins see the stay-at-home age more than the traveller.

On arrival at Mars M the traveller $\mathscr{T}$ sees his clock show the travel time τ and a redshifted light ray from the stay-at-home $\mathscr{S}$ show the time t_-, which has elapsed on the clock of $\mathscr{S}$ since the start A. The travel time is larger by a Doppler factor κ (2.1), $\tau = \kappa\, t_-$.

During the return trip the traveller observes on his clock the time τ' go by while blueshifted light shows him that the time $t + t' - t_-$ passes on the clock of the stay-at-home until the end E, $\tau' = \kappa'(t + t' - t_-)$. Altogether, he sees the stay-at-home age by

Fig. 2.10 Equal phases of acceleration



$$t + t' = \frac{\tau}{\kappa} + \frac{\tau'}{\kappa'} \tag{2.24}$$

while he has grown older by $\tau + \tau'$.

The stay-at-home sees the traveller redshifted during the time $t_+ = \kappa\tau$, which is longer than τ, and blueshifted during the rest of the waiting time $t + t' - t_+ = \kappa'\tau'$, which is shorter than τ'. Altogether, $\mathscr{S}$ ages by

$$t + t' = \kappa\tau + \kappa'\tau' \tag{2.25}$$

while he observes the traveller grow older by $\tau + \tau'$. This agrees with (2.24) as one confirms with (2.13, 2.15) and with the relation $vt + v't' = 0$ that the traveller returns. Both equations imply

$$t + t' = \frac{\tau}{2}\left(\frac{1}{\kappa} + \kappa\right) + \frac{\tau'}{2}\left(\frac{1}{\kappa'} + \kappa'\right) = \frac{\tau}{\sqrt{1 - v^2}} + \frac{\tau'}{\sqrt{1 - v'^2}}. \tag{2.26}$$

The waiting time $t + t'$ is longer than the travel time $\tau + \tau'$.

"Who rests, rusts" and "Travelling keeps young" correctly states the relativistic effect.

The worldline of the traveller differs from the one of the stay-at-home by the acceleration on arrival at Mars M. Such an acceleration is necessary if the second worldline through the two events A and E is to differ from the straight worldline of the first twin because in flat spacetime there is only one straight line through two points.

But the traveller does not become younger during the acceleration. Even if both twins undergo identical phases of acceleration they can age differently, as Fig. 2.10 shows. There both twins travel together until event A where the stay-at-home $\mathscr{S}$ brakes. The traveller $\mathscr{T}$ brakes later to reach M, there he accelerates to return.

After a fitting waiting time the stay-at-home accelerates in exactly the same way and joins the traveller in E from where both continue their joint flight. Between A and E the twins age differently though their acceleration consisted of equal phases. During these phases they age the same, but the remaining pieces of their worldlines constitute the sides of the triangle AME in Fig. 2.9 and there $\mathscr{S}$ ages more.

Time is what clocks show. The clocks of the twins show different times on return. Therefore, time between two events does not only depend on these events but also on the worldline which the clock passes in between; just as in Euclidean geometry the path length between two points of a curve depends on the path which connects both points. Clocks are like mileage counters.

In a spacetime diagram the different aging of the twins is as paradoxical as in Euclidean geometry the statement that in a triangle each side is shorter than the sum of the other two sides. In order to understand triangles one does not need differential geometry of curved spaces, even if one deals with circles and corners, i.e. with curved trajectories. Similarly, the general theory of relativity is not needed for the solution of the twin paradox. It can be used, but gives the same explanation and the same answer as the special theory of relativity: between every two sufficiently adjacent events on the worldline of every free-falling observer there elapses more time than on all other timelike worldlines connecting these two events.

If the two events are not sufficiently adjacent, then gravity can cause the complication that different world lines of free-falling observers connect these events and that on these worldlines different times go by, even though none of the observers has experienced a sensible acceleration. For instance, a space station may orbit the earth in free fall and a second station launched vertically from the earth may fly past the first in free fall during the motion upwards. If the apogee of the second space station is suitably chosen, it can meet the first space station again on the way downwards after the first station has orbited the earth. During the vertical fall more time has elapsed between the two encounters than in the space station orbiting the earth.

The different aging of the twins can be measured with atomic clocks flying around the earth [14] such that for one twin his velocity adds to the revolution of the earth and subtracts for the other twin. In addition, the gravity on ground and during the flight differs and influences the clocks, just as gravity and motion influence the clocks of GPS satellites. There these relativistic effects are routinely accounted for.

Clocks at sea level, which are carried along with the rotating earth, run equally fast. The rotation does not only lead to different velocities, which depend on the longitude, but also to a flattening of the globe, such that the clocks which move faster are further away from the center. Taken together, the different gravity and the different velocity compensate their effects on the clocks at sea level exactly.

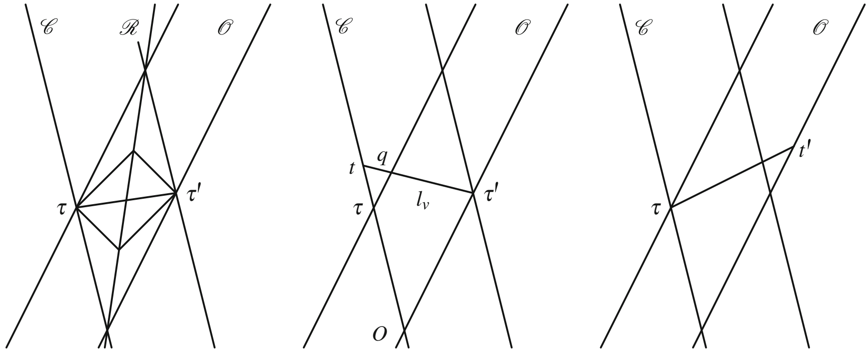


Fig. 2.11 Contraction of moving rods

2.4 Length Contraction

Two moving measuring rods have the same length, if they are equally long for a referee $\mathscr{R}$, who, just as in the left Fig. 2.11 is always in their middle [23]. The beginning of each rod traverses the worldline of the corresponding observer $\mathscr{C}$ and $\mathscr{O}$, the end traverses a parallel worldline. As the referee $\mathscr{R}$ confirms, the rods of $\mathscr{C}$ and $\mathscr{O}$ have the same length, because in the events τ and τ', which are simultaneous for him and equally far away, both ends of both rods coincide.

A moving rod is shorter than an equal rod at rest by the same factor $\sqrt{1-v^2}$ by which a moving clock runs slower than an equal clock at rest. This can be deduced from the middle of Fig. 2.11. There we have omitted all auxiliary lines and shown the segment from t to τ' which consists of events which occur simultaneously for the observer $\mathscr{C}$. At this moment, his measuring rod extends from t to τ' and the right ends of both rods coincide. The moving rod is shorter, its left end intersects the line segment from t to τ' in the event q.

The triangles $t\ O\ \tau'$ and $t\ \tau\ q$ are similar, therefore the length l_v of the segment $\tau' q$ relates to the length l of the segment $\tau' t$ as the length of $O\tau$ to the length of Ot. But $\tau = \sqrt{1-v^2}t$ is the length of $O\tau$ and t the length of Ot. Therefore, a measuring rod which moves uniformly with a velocity v has the shorter length

$$l_v = \sqrt{1-v^2}l, \tag{2.27}$$

if compared to an equal measuring rod of length l at rest.

As the right in Fig. 2.11 shows, length contraction is reciprocal. For the observer $\mathscr{O}$ the events τ and t' occur simultaneous and the measuring rod of $\mathscr{C}$ is shorter.

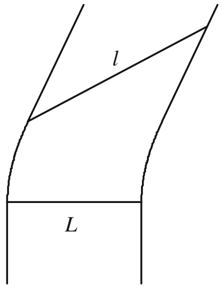


Fig. 2.12 Accelerated Rockets

Equally Accelerated Rockets

We consider two rockets which we idealize as points. Initially they rest in a distance L, later they are accelerated in an equal way such that, as in Fig. 2.12, their worldlines are related by a translation by L. For an observer at rest both rockets have a distance L at all times. After the acceleration the rockets follow straight worldlines with velocity v. If the crews of the rockets then measure the mutual distance with measuring rods which they carry along, they obtain some value l. For the observer at rest, this rod is moving and contracted and has length $L = \sqrt{1 - v^2}l$, because it reaches from one rocket to the other. So l is larger than L.

A rope as considered in [5, Chap. 9], initially spanned between the rockets and stretched to rupture, snaps immediately, if the rockets and the rope are accelerated equally.

This is also what the crews of both rockets observe. For them the rocket in front reaches the final velocity earlier and veers away from the rear rocket.

If one wants to accelerate the constituents of the rope, which rest initially until a time $t = 0$, to a velocity v, such that their distances, as seen by the constituents, remain unchanged, then one has to accelerate the pieces in the rear more but for a shorter time than the pieces in front such that all points at r, $0 \leq r \leq L$ traverse worldlines $x(t) = \sqrt{(r + R)^2 + t^2} - R$ during the times $0 \leq t \leq v(r + R)/\sqrt{1 - v^2}$ and move straight and uniformly afterwards. Here $1/R$ is the acceleration of the last point in the rear.

Length Paradox

Just as time dilation leads to the twin paradox, length contraction seemingly leads to a contradiction, if one considers whether a car with high speed fits into a garage of equal length. For the owner of the garage it is at rest and the moving car is shorter, therefore the car fits into the garage. Seen from the driver, however, the garage is shorter and does not fit the car.

The situation is depicted in the spacetime Fig. 2.11, where $\mathscr{C}$ and the parallel worldline represent the owner at the gate and the rear wall of the garage while $\mathscr{O}$ and the parallel worldline correspond to the front and rear fender of the car.

Consider a red flash of light, emitted by a photo sensor in the event τ', when the front fender hits the garage wall, and a green flash of light, which is emitted in the event τ, when the rear fender passes the gate. The referee sees both flashes in the same instant and, because he is in the middle of the gate and the wall and the runtimes of light from τ and τ' to him are equal, confirms that the garage and the car have equal length.

The owner of the garage $\mathscr{C}$ observes the red flash from τ' after the green one. If he accounts for the runtime of light, he concludes that the front bumper had hit the garage wall in the event τ' at the time t after the event τ, in which the rear fender passed the gate. For him, the car had fitted into the garage at time t, the car war shorter.

The driver $\mathscr{O}$ sees the green flash of light from the rear of his car τ later than the red flash. If he account for the runtime of light he concludes that the green flash τ had been emitted at the time t' after the red flash. For him, the front fender had hit the wall before the rear fender had passed the gate. So he concludes that the garage is shorter than the car.

This in not a contradiction and not a paradox. Observers, who move relative to each other, do not have to agree on the order of events which are not cause and effect as in the case under consideration. The passage of the rear fender through the gate does not cause the crash of the front fender on the wall and vice versa.

Both observers agree that a fast, slim car can pass a slim garage of equal length if the car in addition has some transverse velocity, just as one can thread long yarn through the narrow eye of a needle.

With some transverse velocity of the car the worldlines of the front and rear fender no longer lie in the plane of the Fig. 2.11. They can intersect the plane in the events q and τ. For the garage owner these events are simultaneous, before them the car was on one side and afterwards it is on the other: the car has passed the garage, which is longer than the car. Also the driver observes his car pass the garage, though it is shorter than his car. He first drives around the wall with his front fender and later passes the gate with his rear.

Whether a fast car fits through a garage does not only depend on the length but also on the temporal sequence of the events just as it depends on the direction of a long ladder whether it fits through a low door.

2.5 Doppler Effect

If a clock $\mathscr{C}$ moves with a velocity v in direction $\mathbf{e}$ at an angle θ to the line of sight, then its distance to an observer $\mathscr{O}$ changes by $\mathrm{d}r = v\cos\theta\,\mathrm{d}t$ during the short time $\mathrm{d}t$. The changed distance cause a changed runtime of light and light rays $\underline{l}$ and $\overline{l}$ from two events on the clock which started with a time difference $\mathrm{d}t$ reach the observer

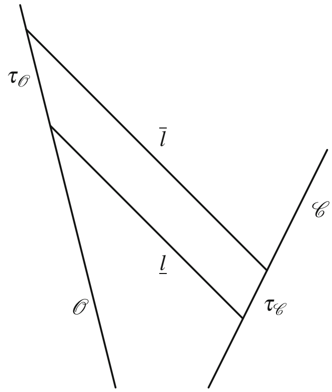


Fig. 2.13 Doppler effect

(in units with $c = 1$) with a time difference

$$\tau_{\mathscr{O}} = \mathrm{d}t + v\cos\theta\,\mathrm{d}t. \tag{2.28}$$

In this consideration we use times, velocities and angles as determined by the observer $\mathscr{O}$.

On the moving clock $\mathscr{C}$ the time $\tau_{\mathscr{C}} = \sqrt{1-v^2}\mathrm{d}t$ elapses between the emission of the two flashes of light. This follows from (2.13), because spacetime is homogeneous and time flows between the origin $(0, 0, 0, 0)$ and (t, x, y, z) the same as between (t_0, x_0, y_0, z_0) and $(t_0 + \mathrm{d}t, x_0 + v_x\mathrm{d}t, y_0 + v_y\mathrm{d}t, z_0 + v_z\mathrm{d}t)$

Consequently the observer $\mathscr{O}$ sees the time

$$\tau_{\mathscr{C}} = \frac{\sqrt{1-v^2}}{1+v\cos\theta}\tau_{\mathscr{O}} \tag{2.29}$$

pass by on the moving clock while on his own, equal clock the time $\tau_{\mathscr{O}}$ passes. Equation (2.13), $\tau_{\mathscr{O}} = \kappa\tau_{\mathscr{C}}$, is the special case in which the clock recedes in the line of sight with $\cos\theta = 1$.[2]

If an oscillator is carried along with the clock and oscillates n-times with a frequency $\nu_{\mathscr{C}} = n/\tau_{\mathscr{C}}$, the the observer sees these n oscillations while the time $\tau_{\mathscr{O}}$ passes on his own clock. He observes the frequency $\nu_{\mathscr{O}} = n/\tau_{\mathscr{O}}$,

$$\nu_{\mathscr{O}} = \frac{\sqrt{1-v^2}}{1+v\cos\theta}\,\nu_{\mathscr{C}}. \tag{2.30}$$

[2] Figure 2.13 depicts the worldlines of the observer $\mathscr{O}$ and the clock $\mathscr{C}$ in a plane. However, we consider the general case in which the worldline of the observer is parallel to the plane and does not intersect the worldline of the clock. Note that in spacetime diagrams the frequency of light is not a property of a light ray but pertains to the distance of two events on parallel light rays.

If $v\cos\theta > \sqrt{1-v^2}-1$, then the clock is seen slower and the frequency of light from the clock is shifted to smaller values of red light, it is redshifted.

Otherwise, if $v\cos\theta < \sqrt{1-v^2}-1$ and the clock moves towards the observer, then it appears faster and its light is blueshifted. This change of the perceived frequency is the Doppler effect. It is commonly used to measure velocities.

On motion crosswise to the line of sight, $\cos\theta = 0$, the transversal Doppler effect $\tau_{\mathscr{C}} = \sqrt{1-v^2}\tau_{\mathscr{O}}$ directly shows the slowdown of moving clocks, because the distance between source and observer just does not change.

The Doppler shift is usually time dependent (because the direction changes) and reciprocal only for motion in the line of sight. If the observer $\mathscr{O}$ sends two flashes of light with a delay of $\mathrm{d}t = \hat{\tau}_{\mathscr{O}}$ to the clock then the second flash reaches the clock later by $\mathrm{d}t' = \mathrm{d}t + v\cos\theta\mathrm{d}t'$ that is $\mathrm{d}t' = \hat{\tau}_{\mathscr{O}}/(1-v\cos\theta)$. During this interval the time $\hat{\tau}_{\mathscr{C}} = \sqrt{1-v^2}\mathrm{d}t'$ elapses on the moving clock. Seen from the clock, frequencies from $\mathscr{O}$ are shifted to

$$\hat{\nu}_{\mathscr{C}} = \frac{1-v\cos\theta}{\sqrt{1-v^2}}\hat{\nu}_{\mathscr{O}}. \tag{2.31}$$

This agrees with $\hat{\nu}_{\mathscr{C}} = \sqrt{1-v^2}/(1+v\cos\theta')\hat{\nu}_{\mathscr{O}}$ (2.30) because θ' is the angle to the line of sight, changed by aberration (3.19), with which $\mathscr{C}$ sees $\mathscr{O}$ move.

Apparent Superluminal Velocity

A jet of gas streams out of the quasar 3C273 with a measurable angular velocity [17, Chap. 11]. If one multiplies the observed angular velocity with the known distance one obtains a velocity of seven times the speed of light for the crosswise motion. The quasar seems to emit particles with superluminal velocity.

This conclusion is wrong, the product of the distance with the observed angular velocity is not the velocity transverse to the line of sight.

The clock $\mathscr{C}$ in Fig. 2.13 moves by $v\sin\theta\,\mathrm{d}t = r\,\mathrm{d}\theta$ within the short time $\mathrm{d}t$ transverse to the line of sight, where r denotes its present distance. The flashes of light $\underline{l}$ and $\overline{l}$ reach the observer with a difference angle $\mathrm{d}\theta$ and a time difference $\tau_{\mathscr{O}} = \mathrm{d}t + v\cos\theta\,\mathrm{d}t$ because $\overline{l}$ starts later by $\mathrm{d}t$ and has to pass a distance which is larger by $\mathrm{d}r = v\cos\theta\,\mathrm{d}t$. So the observed angular velocity $\omega_{\mathscr{O}} = \mathrm{d}\theta/\tau_{\mathscr{O}}$ and the apparent transverse velocity $u = r\,\omega_{\mathscr{O}}$ are

$$\omega_{\mathscr{O}} = \frac{v\sin\theta}{r(1+v\cos\theta)}, \quad u = \frac{v\sin\theta}{1+v\cos\theta}. \tag{2.32}$$

This velocity u becomes maximal for the angle $\cos\theta = -v$ between the direction of motion and the line of sight and in this case has the value $v/\sqrt{1-v^2}$. This value can be arbitrary large though $|v|$ is smaller than $c = 1$, the speed of light.

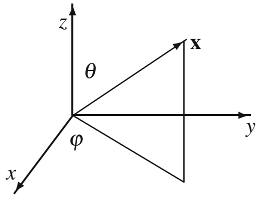


Fig. 2.14 Spherical Coordinates

2.6 Spacetime Coordinates

We can denote the events E in spacetime simply by the values[3] $(t_+, t_-, \theta, \varphi)$, the light coordinates of E, which an observer $\mathcal{O}$ determines as he sends light to E and receives it from E. He reads the times of emission, t_-, and reception, t_+, from his clock and determines the direction of the outgoing light ray by means of, say, two angles θ and φ.

Primarily coordinates only have to denote the events uniquely, at least in some range of their values. Other coordinates, which are invertible functions of the light coordinates, are equally conceivable. In particular, light coordinates are related in a simple way to inertial coordinates (t, x, y, z), in which particles, which move uniformly on straight lines, traverse straight coordinate lines (Fig. 2.14).

The time t and the distance $r = \sqrt{x^2 + y^2 + z^2}$, at which the event E occurs, are the arithmetic mean and half the difference of the light coordinates t_+ and t_- (1.4, 1.5),

$$t = \frac{t_+ + t_-}{2}, \quad r = \frac{t_+ - t_-}{2}. \tag{2.33}$$

The direction of the outgoing light ray from $\mathcal{O}$ to the event E is opposite to the incident direction of the light ray from E because the observer does not rotate but uses reference directions that do not change in time.

The angles θ and φ of the light ray to E and the distance r are the spherical coordinates and define the cartesian spatial coordinates of the event by

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r\mathbf{e}_{\theta,\varphi} = r\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}. \tag{2.34}$$

For events on the worldline of the observer $\mathcal{O}$ one has $t_+ = t_-$, hence $\mathbf{x} = 0$. In particular, the origin O has coordinates $(0, 0, 0, 0)$.

If $\mathcal{O}$ emits a light ray at time t_0 in the direction $\mathbf{e}_{\theta,\varphi}$, then the light ray passes events for which t_-, θ and φ are constant

$$t = \frac{t_+ + t_0}{2}, \quad \mathbf{x}(t) = \frac{t_+ - t_0}{2}\mathbf{e}_{\theta,\varphi}, \tag{2.35}$$

[3] θ and φ are the Greek letters theta and phi.

or, if we express the variable t_+ in terms of t, then the light ray is given by the map

$$\Gamma : t \mapsto \big(t, \mathbf{x}(t)\big) = \big(t, \mathbf{e}_{\theta,\varphi}\cdot(t - t_0)\big). \tag{2.36}$$

This is a worldline parameterized by t, which at the time t_0 intersects the worldline of the observer. In the coordinates (t, x, y, z) it is a straight worldline which is traversed with the speed of light $c = 1$, since the speed $\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$ is a unit vector. The equations also holds for $t < t_0$ for a light pulse incident from the opposite direction $-\mathbf{e}_{\theta,\varphi}$. For such a light ray $t_+ = t_0$ is constant and $t = (t_0 + t_-)/2$ and $\mathbf{x} = -\mathbf{e}_{\theta,\varphi}(t_0 - t)$. While light coordinates $(t_+, t_-, \theta, \varphi)$ of a passing light ray are discontinuous in the event, in which it intersects the worldline of the observer, inertial coordinates are continuous.

Displacing the light ray by $\mathbf{x}_0 + \mathbf{e}_{\theta,\varphi}t_0$ yields more generally the light ray which passes $\mathbf{x}_0$ at the time $t = 0$,

$$\Gamma : t \mapsto \big(t, \mathbf{x}(t)\big) = \big(t, \mathbf{e}_{\theta,\varphi}\cdot t + \mathbf{x}_0\big). \tag{2.37}$$

If the worldline of a linearly and uniformly moving particle passes the origin O at time $t = 0$, the observer $\mathcal{O}$ sees afterwards all events on this worldline from the same direction. The angles θ and φ are constant, except at $t = 0$. The particle departs into the opposite of the direction from which it approached and the angles change discontinuously from θ to $= \pi - \theta$ and from φ to $\varphi + \pi$ at $t = 0$.

According to (2.11) one has $t_+ = \kappa^2 t_-$ for events on the straight worldline of the particle. For its coordinates this means

$$t = (\kappa^2 + 1)\frac{t_-}{2}, \quad \mathbf{x} = (\kappa^2 - 1)\frac{t_-}{2}\mathbf{e}_{\theta,\varphi}\,, \tag{2.38}$$

or, if we express t_- in terms of t and use (2.14), the worldline is given by

$$\Gamma : t \mapsto \big(t, \mathbf{x}(t)\big) = \big(t, \mathbf{v}\,t\big) \quad \text{with} \quad \mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\kappa^2 - 1}{\kappa^2 + 1}\mathbf{e}_{\theta,\varphi}. \tag{2.39}$$

Translating the worldline by $\mathbf{x}_0$ one obtains more generally the worldline of an uniformly moving particle which passes the point $\mathbf{x}_0$ at time $t = 0$,

$$\Gamma : t \mapsto \big(t, \mathbf{x}(t)\big) = \big(t, \mathbf{v}\cdot t + \mathbf{x}_0\big). \tag{2.40}$$

So the coordinates (t, x, y, z) which we have constructed from the light coordinates t_+, t_-, θ and φ are inertial coordinates in which particles, which move straight and uniformly, traverse straight coordinate lines.

2.7 Scalar Product and Length Squared

Together, the time and the spatial coordinates of each event constitute an ordered set (t, x, y, z) of four real numbers, and each such four-tuple corresponds to one and only one event. In Special Relativity spacetime, the set of all events, is $\mathbb{R}^4$. We denote the components of the four-tuple which corresponds to a particular event E either by t_E, x_E, y_E and z_E or we use a name like u for the four-tuple $u = (u^0, u^1, u^2, u^3)$ and enumerate the components with a superscript. It depends on the context whether the superscript denotes an exponent (rarely) or enumerates a footnote or a component.

The homogeneity of spacetime makes it a vector space. If one shifts all events $u = (u^0, u^1, u^2, u^3)$, which participate in some physical process, by $s = (s^0, s^1, s^2, s^3)$ in space and time, then the events

$$u + s = (u^0 + s^0, u^1 + s^1, u^2 + s^2, u^3 + s^3) \tag{2.41}$$

can participate in an equally possible process.

The scaled versions of spacetime diagrams consist of events

$$au = (au^0, au^1, au^2, au^3) \tag{2.42}$$

scaled by a common factor a. However, elementary physical processes are not scale invariant. While scaled diagrams of physical processes with free pointlike particles correspond to equally possible physical processes, this is not true for interacting particles, for example, one has never observed an enlarged hydrogen atom (electron and proton bound by electromagnetic interactions).

The set $\mathbb{R}^4$, equipped with the operations of addition and multiplication by a scale factor, is a four-dimensional vector space. Its elements are called four-vectors.[4]

On an uniformly moving clock, which passes the two events (t_0, x_0, y_0, z_0) and $(t_0 + t, x_0 + x, y_0 + y, z_0 + z)$, there elapses the time

$$\tau^2 = t^2 - x^2 - y^2 - z^2. \tag{2.43}$$

This follows from (2.13), because spacetime is homogeneous and time flows between the origin $(0, 0, 0, 0)$ and the event (t, x, y, z) the same as between (t_0, x_0, y_0, z_0) and $(t_0 + t, x_0 + x, y_0 + y, z_0 + z)$.

The time between two events does not depend on the details of the clock used to measure it. The time is a measure for distance, i.e. a geometric structure, in spacetime.

[4] Without mentioning it explicitly we shall consider different copies of $\mathbb{R}^4$, e.g. spacetime or the set of four-velocities, four-momenta or four-accelerations. Vectors from different spaces cannot be added, because they differ in units. e.g. a velocity $\mathbf{v}$ cannot be added to a position $\mathbf{x}$. What can be added is the image $\mathbf{v}t$ of a velocity $\mathbf{v}$ under the linear map t, which maps it to the space of positions. Though vectors from different four-spaces cannot be added, their directions can be compared, because, as we shall see, the Lorentz group acts on each of these spaces and the x-direction, for example, is the set of vectors which is invariant under rotations around the x-axis and under boosts in y- and z-directions (also compare page 89).

Events with equal temporal distance one does not find in a plane $t = \text{const}$ as in nonrelativistic physics or on a sphere $x^2 + y^2 + z^2 = r^2 = \text{const}$ as in Euclidean geometry, but on an hyperboloid $t^2 - x^2 - y^2 - z^2 = \tau^2 = \text{const}$. The square of the temporal distance between two events is not subject to the Pythagorean theorem but to the theorem of Minkowski.

The clock does not depend on which observer determines coordinates for the events. If another observer measures light coordinates t'_+,t'_-, θ' and φ' and converts them into spacetime coordinates (t'_0, x'_0, y'_0, z'_0) and $(t'_0 + t', x'_0 + x', y'_0 + y', z'_0 + z')$ of the two events, then the sums of squares appearing in (2.43) have to agree

$$t^2 - x^2 - y^2 - z^2 = t'^2 - x'^2 - y'^2 - z'^2. \tag{2.44}$$

The sum of squares plays a central role in relativistic physics. We introduce the related scalar product of four-vectors like $u = (u^0, u^1, u^2, u^3)$ and $w = (w^0, w^1, w^2, w^3)$

$$u{\cdot}w := u^0 w^0 - u^1 w^1 - u^2 w^2 - u^3 w^3. \tag{2.45}$$

As length squared of a four-vector w we define[5]

$$w^2 = w{\cdot}w = (w^0)^2 - (w^1)^2 - (w^2)^2 - (w^3)^2. \tag{2.46}$$

In this notation, the time τ between events u and w is given by

$$\tau^2 = (u - w)^2. \tag{2.47}$$

The scalar product (2.45) maps each pair of four-vectors to a real number and is symmetric and linear in each argument, (a denotes an arbitrary real factor)

$$v{\cdot}w = w{\cdot}v, \tag{2.48}$$
$$u{\cdot}(v + w) = u{\cdot}v + u{\cdot}w, \quad v{\cdot}(a\,w) = a\,(v{\cdot}w), \tag{2.49}$$

but, different from Euclidean geometry, not definite. Lightlike vectors have length squared zero though they do not vanish. The scalar product is nondegenerate, i.e. the scalar product of a vector v vanishes with all other vector if and only if $v = 0$ vanishes.

The scalar product of two vectors u and v can be written as the difference of lengths squared

$$u{\cdot}v = \frac{1}{4}((u + v)^2 - (u - v)^2). \tag{2.50}$$

Since different observers determine different coordinates but the same lengths squared of differences of four-vectors (2.44), scalar products of difference vectors

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[5] The reader has to deduce from the context whether the length squared or the y-component of a vector is meant.

do not depend on the coordinate system of the respective observer either,

$$u\cdot v = u'\cdot v'. \tag{2.51}$$

If the length squared w^2 is positive we call w timelike, if it is negative we call w spacelike, if $w^2 = 0$, $w \neq 0$, w is called lightlike. A timelike or lightlike vector w is future directed, if its component w^0 is positive, otherwise it is past directed.

Two events A and B are mutually spacelike if the corresponding difference vector from B to A

$$w_{AB} = (t_A - t_B, x_A - x_B, y_A - y_B, z_A - z_B) \tag{2.52}$$

is spacelike. Correspondingly we define lightlike or timelike pairs of events.

An event B can cause an effect A only, if w_{AB} is future directed timelike or lightlike.

Events on a light ray are mutually lightlike.

Events on the worldline of an observer are mutually timelike since each observer is slower than light. If his worldline is straight then the length squared of the difference vector of two of his events is the square of the time which passes on his clock between the two events.

Orthogonal

To construct the line $\mathscr{O}_\perp$, which orthogonally intersects the line $\mathscr{O}$ in the point t, one chooses two points on $\mathscr{O}$, t_+ and t_-, which are equally far away from t, and determines a second point E which is equally far away from t_+ and t_-. The orthogonal line $\mathscr{O}_\perp$ is the line through t and E.

This is true in Euclidean geometry and in spacetime. In spacetime, however, the distance is given by τ (2.43). If t_- and t_+ are two events on the worldline of the observer $\mathscr{O}$ and if t is in their middle then the intersections E and E' of the light rays through t_- and t_+ lie on the orthogonal line through t, because E and E' are equally far away from t_- and from t_+, to wit the distance vanishes because the separations are lightlike (Fig. 2.15).

The worldline $\mathscr{O}$ consists of events which are equilocal for the observer. The events on $\mathscr{O}_\perp$ are equitemporal for him (Fig. 1.6). The lines of equilocal events are orthogonal to the lines of equitemporal events.

Using the vector v from t_- to t and from t to t_+ and the vector w from t to E, the light ray from t_- to E is $v + w$. The vector $v - w$ is the light ray back from E to t_+. The length squared of the lightlike vectors $v + w$ and $v - w$ vanishes,

$$\begin{aligned} 0 =&(v + w)^2 = v^2 + 2v\cdot w + w^2,\\ 0 =&(v - w)^2 = v^2 - 2v\cdot w + w^2. \end{aligned} \tag{2.53}$$

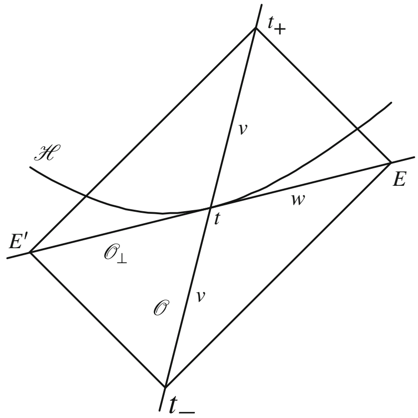

Fig. 2.15 Orthogonal vectors with hyperbola

Therefore $v^2 = -w^2$ and the scalar product of the orthogonal vectors vanishes,

$$v \cdot w = 0. \tag{2.54}$$

The length squared v^2 is the square of the time between the events t_- and t on the worldline of the observer $\mathscr{O}$. This is the runtime of light and therefore the distance from $\mathscr{O}$ to the event E. Because $v^2 = -w^2$ the square of the distance of two simultaneous events, which are separated by the spacelike vector w, is $-w^2$.

Orthogonal: *The vector w from an event t on the worldline of a uniformly moving observer to an event E, which occurs simultaneously for him, is orthogonal in terms of the scalar product* (2.45) *to his worldline. The negative length squared $-w^2$ is the square of the distance between E and the observer.*

The hyperbola $\mathscr{H}$ through t around t_- is defined to consist of points which are obtained from t_- by equally long translations $u(s)$

$$u(s) = \sqrt{1+s^2}\,v + sw, \quad u(s)^2 = v^2, \tag{2.55}$$

where s varies in the real numbers. In particular, the point t on the hyperbola corresponds to $s = 0$. In terms of the length squared of spacetime, all points of $\mathscr{H}$ are equally far away from t_-.

Each vector from t_- to a point A on the orthogonal line $\mathscr{O}_\perp$ is of the form $x(s) = v + sw$, where s is some real number. Because of $v \cdot w = 0$ it is as long as the vector $-v + sw$ from t_+ to A.

Because $\sqrt{1+s^2} > 1$ for $s \neq 0$, all points of $\mathscr{H}$ apart from t lie on the side of $\mathscr{O}_\perp$ which is opposite to t_-, one has $u(s) = x(s) + a(s)v$ with a positive $a(s)$. In addition, t belongs to both $\mathscr{O}_\perp$ and $\mathscr{H}$. Therefore both curve touch each other at t and the straight line $\mathscr{O}_\perp$ is tangent to the hyperbola $\mathscr{H}$ in the point t. The tangent at t is orthogonal to the vector from t_- to t.

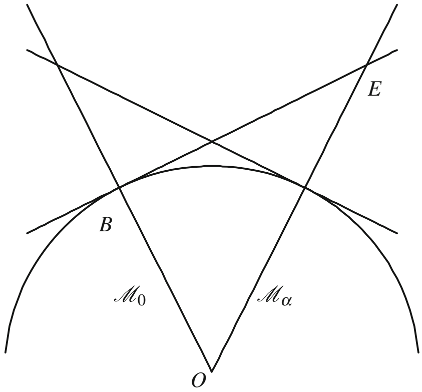

Fig. 2.16 Rotated measuring rods

The same conclusion is obtained by differentiating $u(s)^2$ with respect to s . The tangential vector $t(s) = \frac{\mathrm{d}u}{\mathrm{d}s}(s)$ is orthogonal to the position vector $u(s)$,

$$u(s)\cdot u(s) = \text{constant} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}s}\cdot u = 0. \tag{2.56}$$

2.8 Perspectives

If one takes bearing in horizontal direction from a lighthouse at sea level to a second lighthouse of identical construction also at sea level some miles away then the other lighthouse does not reach the same height because the surface of the earth is curved (see Fig. 2.16). Height is a perspective quantity. It depends on which direction is horizontal and the horizontal directions of both lighthouses do not coincide.

Perspective shortening is physically relevant, Because one can change the height of a ladder by rotation, it may pass a low door though the ladder is longer than the height of the door and though rotations leave the sizes of the door and the ladder unchanged.

The Fig. 2.16 depicts the perspective height of two measuring rods $\mathscr{M}_0$ and $\mathscr{M}_\alpha$ in Euclidean geometry which are rotated with respect to each other. The circle consists of points of equal distance to the center; each tangent vector is orthogonal to the position vector.

The measuring rods intersect in the point O. For an observer, who measures height with $\mathscr{M}_0$, all point on the straight line through B, which is orthogonal to $\mathscr{M}_0$, are equally high. In particular the point E is as high as B and higher as the end point of $\mathscr{M}_\alpha$. Rotated measuring rods reach less high.

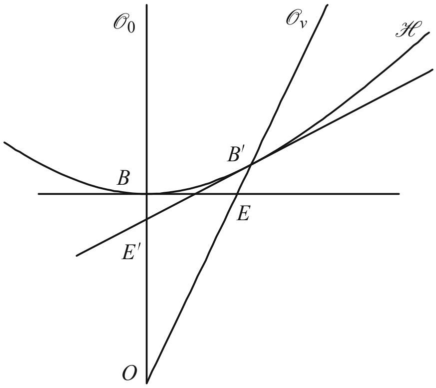

Fig. 2.17 Time dilation

The perspective shortening of height is reciprocal. Judged from $\mathscr{M}_\alpha$ the rotated rod $\mathscr{M}_0$ is lower.

If one replaces in Fig. 2.16 the circle by a hyperbola, one obtains the geometric relations in spacetime.

In Fig. 2.17 the hyperbola $\mathscr{H}$ consists of events with equal temporal distance τ to the origin O (2.43). Equal uniformly moving clocks of observers $\mathscr{O}_0$ and $\mathscr{O}_v$, who pass the origin, show the same laps of time, τ, when their worldlines intersect the hyperbola.

The tangents in B and B' are orthogonal to the worldlines of the observers $\mathscr{O}_0$ and $\mathscr{O}_v$ respectively (2.56). Therefore they consist of events which occur simultaneously for $\mathscr{O}_0$ or $\mathscr{O}_v$. The tangents intersect the worldline of the other observer before the time τ has elapsed on it.

If the time τ elapses on a clock between two events, then the shorter time $\tau_{EO} = \tau_{E'O} = \sqrt{1 - v^2}\tau$ (2.13) passes between the simultaneous events on a moving clock, just as two points on a vertical ladder have a shorter distance than equally high points on a tilted ladder. The perspective relations in spacetime are reciprocal as in Euclidean geometry.

The abbreviated summary "moving clocks run slower" suppresses the specifications of the segments OE and OB or OE' and OB' the duration of which is to be compared. The abbreviation is the reason for misunderstandings, because "running slower" is an order relation and the clock of $\mathscr{O}_0$ cannot run slower and also faster than the clock of $\mathscr{O}_v$. In fact, both clocks are equal as confirmed by the referee in Fig. 2.2.

Contraction of moving measuring rods can be read of Fig. 2.18 which is the mirrored version of Fig. 2.17. The beginning and the end of uniformly moving measuring rods of observers $\mathscr{O}_0$ and $\mathscr{O}_v$ traverse pairs of parallel straight worldlines. Both rods have the length l, because the left ends coincides in O and the right ends B and B' lie on the auxiliary hyperbola, which consists of points P which satisfy $-{w_{PO}}^2 = l^2$.

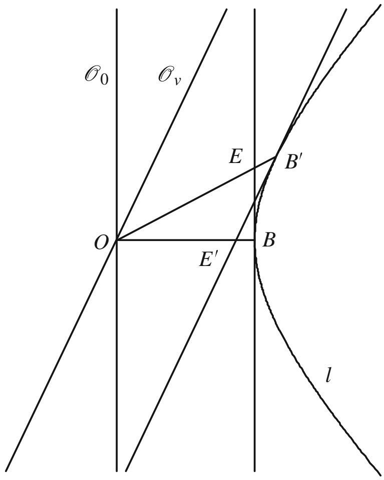

Fig. 2.18 Contraction of moving rods

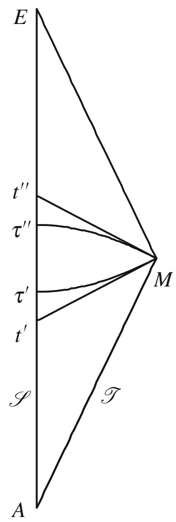

Fig. 2.19 Twin paradox

The segments OB and OB' are orthogonal to the worldlines of the observers and $\mathscr{O}_v$, because they are position vectors and parallel to tangent vectors of the hyperbola (page 43).

Therefore the events O, E' and B are simultaneous for $\mathscr{O}_0$. At this moment, the left ends coincide, but the right end of the moving rod only reaches to E', so the moving rod is shorter than the own rod which reaches until B.

For $\mathscr{O}_v$ the events O, E and B' are simultaneous. Then the left ends coincide in O and the rod, which moves relative to $\mathscr{O}_v$ only reaches to E and is shorter than the own rod which reaches until B'.

In Fig. 2.19 the waiting time and the travel times of the twin paradox are not determined from the Doppler factors as in Fig. 2.9 but compared by the auxiliary hyperbola from M to τ' with origin at A and the hyperbola from M to τ'' with origin at E.

The line segments from A to τ' and from τ'' to E on the worldline of the stay-at-home $\mathscr{S}$ last as long as for the traveller $\mathscr{T}$ the travel to and from Mars. The stay-at-home has aged in addition during the time which passed between τ' and τ''. On the straight worldline of the stay-at-home more time has passed than on the worldline of the traveller with a kink.

The tangents Mt' and Mt'' of the hyperbolas consist of events, which are simultaneous to the arrival at Mars for an uniformly moving observer $\mathscr{O}_{\text{to}}$, who also flies to Mars respectively for an uniformly moving observer $\mathscr{O}_{\text{from}}$, who flies back. The tangents intersect the worldline of $\mathscr{S}$ in t' and t'' confirming that for observers who fly to or from Mars the clock of the stay-at-home shows less time than simultaneously on their own clocks.

But the events t' and t'' do not coincide, they are simultaneous to the arrival at Mars for different observers. Between t' and t'' the stay-at-home ages so much that in the end he has grown older than the traveller.