Chapter 5

Change of Variables Formula, Improper Multiple Integrals

In this chapter we discuss the change of variables formula for multiple integrals, improper multiple integrals, functions defined by integrals, Weierstrass' approximation theorem, and the Fourier transform.

5.1 Change of variables formula

The "change of variable" formula, also called the "u-substitution rule" in one variable is known from elementary calculus and tells us that; *if $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $\varphi : [a, b] \to \mathbb{R}$ is a one-to-one function of class C^1 with $\varphi'(x) > 0$ for $x \in (a, b)$, then with $u = \varphi(x)$,*

$$\int_a^b f(\varphi(x))\varphi'(x)dx = \int_{\varphi(a)}^{\varphi(b)} f(u)du. \tag{5.1}$$

To prove this, first we find a differentiable function F such that $F' = f$, that is, an antiderivative F of f (this is possible by the Fundamental Theorem of Calculus). Then

$$\int_{\varphi(a)}^{\varphi(b)} f(u)du = F(\varphi(b)) - F(\varphi(a)).$$

On the other hand, by the Chain Rule $(F \circ \varphi)'(x) = f(\varphi(x))\varphi'(x)$. Hence, again by the Fundamental theorem,

$$\int_a^b f(\varphi(x))\varphi'(x)dx = (F \circ \varphi)(b) - (F \circ \varphi)(a) = F(\varphi(b)) - F(\varphi(a)).$$

However, one has to be careful here, for (5.1) is fine as it stands when φ is increasing. If φ were decreasing, (i.e., $\varphi'(x) < 0$) the right side of (5.1) is

$$\int_{\varphi(b)}^{\varphi(a)} f(u)du = -\int_{\varphi(a)}^{\varphi(b)} f(u)du.$$

This is corrected by rewriting (5.1) as

$$\int_a^b f(\varphi(x))|\varphi'(x)|dx = \int_{\varphi(a)}^{\varphi(b)} f(u)du.$$

Setting $\Omega = [a, b]$, so that $\varphi(\Omega)$ is the interval with endpoints $\varphi(a)$ and $\varphi(b)$ we write

$$\int_{\varphi(\Omega)} f(u)du = \int_{\Omega} f(\varphi(x))|\varphi'(x)|dx. \tag{5.2}$$

We remark that this result is also valid if f is not continuous, but merely integrable, because sets of measure zero contribute nothing to either integral.

We shall now generalize (5.2) for multiple integrals. First we define what is meant by a "change of variables" in an n-dimensional integral.

Definition 5.1.1. Let U be an open set in $\mathbb{R}^n$. Let $\varphi : U \to \mathbb{R}^n$ be a one-to-one function of class $C^1(U)$, such that the Jacobian $J_\varphi(x) = \det D_\varphi(x) \neq 0$ for all $x \in U$. Then φ is called a *change of variables* in $\mathbb{R}^n$.

Note our hypothesis is slightly redundant, for if $\varphi : U \to \varphi(U)$ is a one-to-one function such that both φ and φ^{-1} are of class C^1, that is, if φ is a diffeomorphism, then the Chain Rule implies that D_φ is nonsingular, so that $\det D_\varphi(x) \neq 0$ and hence φ is a change of variables.

On the other hand, if φ is a change of variables in $\mathbb{R}^n$, then the Inverse Function theorem (Theorem 3.8.2) and Corollary 3.8.3 tell us $\varphi(U)$ is open in $\mathbb{R}^n$ and the function $\varphi^{-1} : \varphi(U) \to U$ is of class C^1. In other words, a change of variables in $\mathbb{R}^n$ is just a *diffeomorphism* in $\mathbb{R}^n$.

The basic issue for changing variables in multiple integrals is as follows: *Let U be open in $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^n$ a diffeomorphism. Suppose Ω is a bounded simple set (i.e., its boundary has n-dimensional volume zero) with $\overline{\Omega} \subset U$. Given an integrable function $f : \varphi(\Omega) \to \mathbb{R}$ we want to change the integral $\int_{\varphi(\Omega)} f(y)dy$ into an appropriate integral over Ω (which we hope to be easier to compute).* In fact, we shall prove the following n-dimensional analogue of (5.2)

$$\int_{\varphi(\Omega)} f(y)dy = \int_{\Omega} f(\varphi(x))|J_{\varphi}(x)|dx. \tag{5.3}$$

Since the sets over which we integrate are bounded simple sets it is natural to ask, whether a diffeomorphism maps (bounded) simple sets to (bounded) simple sets.

Lemma 5.1.2. *Let U be open in $\mathbb{R}^n$, $\varphi : U \to \mathbb{R}^n$ a diffeomorphism whose resrtiction on Ω° is a diffeomorphism and Ω a bounded set with $\overline{\Omega} \subset U$. Then*

$$\partial(\varphi(\Omega)) = \varphi(\partial(\Omega)).$$

Proof. Since φ is continuous and $\overline{\Omega}$ is compact (by the Heine-Borel theorem), it follows that $\varphi(\overline{\Omega})$ is also compact and so $\varphi(\overline{\Omega})$ is closed and bounded. Hence, $\varphi(\overline{\Omega}) = \overline{\varphi(\Omega)}$. At the same time, since $\varphi \in C^1(\Omega)$ and $J_{\varphi}(x) \neq 0$ for all $x \in \Omega$, by Corollary 3.8.3, φ maps open sets onto open sets. In particular, $\varphi(\Omega^{\circ})$ is open. Thus, points of $\partial(\varphi(\Omega))$ can not be images of points of $\Omega^{\circ} = \Omega \backslash \partial\Omega$, that is, $\partial(\varphi(\Omega)) \subseteq \varphi(\partial(\Omega))$. On the other hand, let $x \in \partial(\Omega)$. Then there are sequences $\{x_k\}$ in Ω and $\{y_k\}$ in $U \backslash \Omega$ such that $x_k \to x$ and $y_k \to x$. The continuity of φ implies $\varphi(x_k) \to \varphi(x)$ and $\varphi(y_k) \to \varphi(x)$. Since φ is one-to-one on U, it follows that $\varphi(y_k) \notin \varphi(\Omega)$ and hence $\varphi(x) \in \partial(\varphi(\Omega))$. Therefore $\varphi(\partial(\Omega)) \subseteq \partial(\varphi(\Omega))$. Thus, $\partial(\varphi(\Omega)) = \varphi(\partial(\Omega))$. $\square$

Remark 5.1.3. In developing the theory of integration one uses n-dimensional rectangles in a number of places. However one could use n-dimensional cubes instead. An n-dimensional open *cube* centered at $a \in \mathbb{R}^n$ of side length $2r$ is the set

$$C_r(a) = \{x \in \mathbb{R}^n : ||a - x||_\infty < r\},$$

where the norm is the"max-norm" (or "box-norm")

$$||x||_\infty = \max\{|x_1|, ..., |x_n|\}.$$

As we have proved in Theorem 1.7.2 all norms in $\mathbb{R}^n$ are equivalent, in particular $||x||_\infty \leq ||x|| \leq \sqrt{n}||x||_\infty$. So that a cube C in $\mathbb{R}^n$ contains a ball B in $\mathbb{R}^n$ and vice versa. Furthermore, knowing beforehand that a function f is integrable over a bounded simple set Ω contained in a cube C, by using a partition $\mathcal{P} = \{C_1, ..., C_k\}$ of C involving subcubes of *equal side length* $2r$, (so that $||\mathcal{P}|| \to 0$ if and only if $r \to 0$ if and only if $k \to \infty$), one obtains approximating Riemann sums of f that have substantial computational advantages. In the sequel we shall use n-dimensional cubes as our basic sets rather than rectangles in $\mathbb{R}^n$.

Proposition 5.1.4. *Let $S \subset \mathbb{R}^n$ be a set of measure zero and suppose that $\varphi : S \to \mathbb{R}^n$ satisfies a Lipschitz condition. Then $\varphi(S)$ has measure zero.*

Proof. Since φ satisfies Lipschitz condition, there is $M > 0$ such that,

$$||\varphi(x) - \varphi(x')|| \leq M||x - x'||,$$

for all $x, x' \in S$. Because S has measure zero, given $\epsilon > 0$, there is a sequence of closed cubes C_j such that $S \subset \bigcup_{j=1}^{\infty} C_j$ and $\sum_{j=1}^{\infty} \nu(C_j) < \epsilon$. Let the side length of C_j be r_j and $d_j = \sqrt{n}r_j$ its diameter. Then for $x, x' \in S \cap C_j$ we have

$$||\varphi(x) - \varphi(x')||_\infty \leq ||\varphi(x) - \varphi(x')|| \leq M||x - x'|| \leq M\sqrt{n}||x - x'||_\infty.$$

Therefore $||\varphi(x) - \varphi(x')||_\infty \leq M\sqrt{n}d_j$, and so $\varphi(S \cap C_j)$ is contained in the cube K_j with side length $2M\sqrt{n}d_j$. The cubes K_j cover $\varphi(S)$ and

$$\sum_{j=1}^{\infty} \nu(K_j) = (Mn)^n \sum_{j=1}^{\infty} \nu(C_j) \leq (Mn)^n \epsilon.$$

Since $(Mn)^n$ is constant and $\epsilon > 0$ is arbitrary, $\varphi(S)$ has measure zero. □

Lemma 5.1.5. *Let $U \subseteq \mathbb{R}^n$ be open and $\varphi : U \to \mathbb{R}^n$ be $C^1(U)$. Suppose S is a compact set in U. If S has volume zero, then $\varphi(S)$ also has volume zero.*

Proof. Since U is open, for each $x \in S$ we can choose an open ball $B_x = B_{\delta_x}(x) \subseteq U$. The collection $\{B_x : x \in S\}$ is an open cover for S. Since S is compact a finite number of these balls cover S. That is, $S \subseteq \bigcup_{j=1}^N B_{x_j} \subseteq U$.

Let $S_j = S \cap B_{x_j}$. Since $S_j \subset S$ and $\nu(S) = 0$ we have $\nu(S_j) \leq \nu(S) = 0$. Since $\varphi \in C^1(U)$, Corollary 3.4.8 tells us that φ satisfies a Lipschitz condition on B_{x_j} ans so on S_j. Proposition 5.1.4 then implies that $\nu(\varphi(S_j) = 0$ for all $j = 1, ..., N$ (here because S is compact the notions of volume zero and measure zero coincide).

Now $S = \bigcup_{j=1}^N [S \cap B_{x_j}] = \bigcup_{j=1}^N S_j$ and so $\varphi(S) = \bigcup_{j=1}^N \varphi(S_j)$. Therefore

$$\nu(\varphi(S)) = \nu\left(\bigcup_{j=1}^N \varphi(S_j)\right) \leq \sum_{j=1}^N \nu(\varphi(S_j)) = 0.$$

□

Proposition 5.1.6. *Let U be open in $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^n$ a diffeomorphism. If Ω is a bounded simple set with $\overline{\Omega} \subset U$, then $\varphi(\Omega)$ is also bounded and simple.*

Proof. First note that since $\varphi(\Omega) \subseteq \varphi(\overline{\Omega})$ and $\varphi(\overline{\Omega})$ is compact (i.e., closed and bounded), the set $\varphi(\Omega)$ is also bounded. At the same time, since Ω is simple, $\nu(\partial(\Omega)) = 0$. It follows from Lemma 5.1.5 that $\nu(\varphi(\partial(\Omega))) = 0$. Now since φ is a diffeomorphism, Lemma 5.1.2 tells us $\partial(\varphi(\Omega)) = \varphi(\partial(\Omega))$. Hence $\nu(\partial(\varphi(\Omega))) = \nu(\varphi(\partial(\Omega))) = 0$, and $\varphi(\Omega)$ is simple. □

5.1.1 Change of variables; linear case

In this subsection we prove the linear change of variables formula. Here $\varphi = T$, where T is a linear map on $\mathbb{R}^n$, in which case $J_T = \det T$ is a

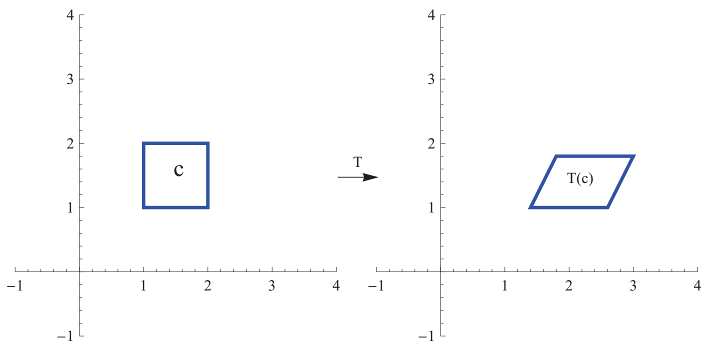

Figure 5.1: Linear Transformation

constant. The proof does not use elementary transformations, rather it uses the polar decomposition.

Proposition 5.1.7. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and $\Omega \subset \mathbb{R}^n$ a bounded simple set. Then*

$$\nu(T(\Omega)) = |\det T|\nu(\Omega). \tag{5.4}$$

Proof. We identify T with its standard matrix representation. If T is singular (i.e., $\det T = 0$), then $T(\mathbb{R}^n)$ is a *proper* subspace of $\mathbb{R}^n$ of dimension $k < n$. By Corollary 4.1.25, $\nu(T(\Omega)) = 0$, and so the statement holds trivially.

Now suppose T is invertible (i.e., $\det T \neq 0$). Then T is an isomorphism on $\mathbb{R}^n$. Proposition 5.1.6 tells us that $T(\Omega)$ is a bounded simple set. By the Polar Decomposition theorem (Theorem 2.4.11), T can be written as $T = OP$, where O is an orthogonal matrix and P a positive definite symmetric matrix. Therefore P is orthogonally conjugate to a diagonal matrix D with positive entries. That is, $P = O_1 D O_1^{-1}$ where O_1 and O_1^{-1} are orthogonal matrices. Since $|\det(AB)| = |\det(A)||\det(B)|$ and if A is orthogonal $|\det A| = 1$, (see, the discussion following Definition 2.4.9)) we see that $|\det T| = |\det((OO_1)DO_1^{-1})| = \det D$.

First we shall prove (5.4) for a cube $C = [-s, s]^n$ in $\mathbb{R}^n$. Since any orthogonal transformation O is an isometry (Theorem 2.4.10), it leaves distances fixed. Hence shapes are fixed and so the geometric effect of the application of O leaves volumes unchanged. Thus it is enough to see how D affects the volume of C. Since D is diagonal with positive diagonal entries $\lambda_1, ..., \lambda_n$ we have

$$D(C) = [-\lambda_1 s, \lambda_1 s] \times ... \times [-\lambda_n s, \lambda_n s].$$

Therefore

$$\nu(D(C)) = (2\lambda_1 s) \cdots (2\lambda_n s) = \lambda_1 \cdots \lambda_n (2s)^n = (\det D)\nu(C).$$

Hence

$$\nu(T(C)) = |\det T|\nu(C).$$

Now, let C be a cube containing Ω and partition C into subcubes of equal side length $2r$. Since Ω is simple its characteristic function χ_Ω is integrable over C, that is, $\nu_*(\Omega) = \nu(\Omega) = \nu^*(\Omega)$ (see Remark 4.1.41). Let C_i be the subcubes for which $C_i \subset \Omega$ with $i = 1, ...p$ and C_j be those subcubes for which $C_j \cap \Omega \neq \emptyset$ with $j = 1, ..., q$. Then $\bigcup_{i=1}^p C_i \subset \Omega \subset \bigcup_{j=1}^q C_j$. Therefore $\bigcup_{i=1}^p T(C_i) \subset T(\Omega) \subset \bigcup_{j=1}^q T(C_j)$. Since the result holds for cubes, it follows that

$$|\det T| \sum_{i=1}^{p} \nu(C_i) \leq \nu(T(\Omega)) \leq |\det T| \sum_{j=1}^{q} \nu(C_j).$$

Letting $r \to 0$ we get

$$|\det T|\nu(\Omega) \leq \nu(T(\Omega)) \leq |\det T|\nu(\Omega).$$

Thus again, $\nu(T(\Omega)) = |\det T|\nu(\Omega)$. □

Corollary 5.1.8. *(The volume of a ball in $\mathbb{R}^n$). Let $B^n(r) = \{x \in \mathbb{R}^n : ||x|| \leq r\}$ be the ball of radius $r > 0$ in $\mathbb{R}^n$. Then*

$$\nu(B^n(r)) = c_n r^n,$$

where $c_n = \nu(B^n)$ is the volume of the n-dimensional unit ball B^n.[1]

[1]Explicit expressions for c_n are given in Examples 5.1.33 and 5.3.27.

Proof. Since $||rx|| = r||x||$, the ball $B^n(r)$ is the image of B^n under the linear map $T(x) = rx$ on $\mathbb{R}^n$. Furthermore, since $\det T = r^n$, Proposition 5.1.7 gives $\nu(B^n(r)) = \nu(T(B^n)) = r^n c_n$. For example, for $n = 3$, Example 4.3.19 tells us $c_3 = \frac{4}{3}\pi$, and of course $c_2 = \pi$. □

Theorem 5.1.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded simple set and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a nonsingular linear map. If $f : T(\Omega) \to \mathbb{R}$ is an integrable function, then $(f \circ T)|\det T|$ is integrable over Ω and*

$$\int_{T(\Omega)} f(y)dy = \int_{\Omega} f(T(x))|\det T|dx. \tag{5.5}$$

Proof. For $x \in \Omega$ let $F(x) = f(T(x))$ and set $y = T(x)$. Let C be a cube containing Ω and partition C into subcubes $\mathcal{P} = \{C_j : j = 1, ..., m\}$ of equal side length $2r$. Then for each j, the sets $\{T(C_j)\}$ form a partition of $T(\Omega)$. From Proposition 5.1.7, $\nu(T(C_j)) = |\det T|\nu(C_j)$.

Since f is integrable over $T(\Omega)$, by the definition of the integral over a set, $f\chi_{T(\Omega)}$ is integrable over (any) cube C' containing $T(\Omega)$. Moreover, since $\chi_\Omega = \chi_{T(\Omega)} \circ T$, for any $y_j = T(x_j) \in T(C_j)$ we can write

$$\sum_{j=1}^{m} f\chi_{T(\Omega)}(y_j)\nu(T(C_j)) = \sum_{j=1}^{m} F\chi_\Omega(x_j)|\det T|\nu(C_j) = |\det T|S_{\mathcal{P}}(F\chi_\Omega). \tag{5.6}$$

Let $d = \max_{j=1,...,m}\{d(T(C_j)\}$. The continuity of T implies that as $r \to 0$ then also $d \to 0$.

Now passing to the limit in (5.6) as $r \to 0$ the first sum converges to $\int_{T(\Omega)} f(y)dy$, which automatically implies the existence of the limit of the second sum (i.e., the integrability of $F = f \circ T$ on Ω) and the equality

$$\int_{T(\Omega)} f(y)dy = |\det T| \int_{\Omega} f(T(x))dx = \int_{\Omega} f(T(x))|\det T|dx.$$

□

The meaning of the determinant

Definition 5.1.10. Let $\{v_1, ..., v_k\}$ be linearly independent vectors in $\mathbb{R}^n$ ($k \leq n$). We define the *k-dimensional parallelepiped* $\Pi = \Pi(v_1, ..., v_k)$ with adjacents sides the vectors v_1,...,v_k (also called the *edges* of Π) to be the set of all $x \in \mathbb{R}^n$ such that

$$x = c_1 v_1 + ... + c_k v_k,$$

where c_j are scalars with $0 \leq c_j \leq 1$, for $j = 1, ..., k$.

A 2-dimensional parallelepiped is a *parallelogram*, and a higher dimensional analogue is called a *parallelepiped*. The 3-dimensional version of the next result was obtained in Proposition 1.3.35.

Proposition 5.1.11. *Let $v_1, ..., v_n$ be n linearly independent vectors in $\mathbb{R}^n$ and let Π be the parallelepiped $\Pi(v_1, ..., v_n)$. If $A = [v_1 ... v_n]$ is the $n \times n$ matrix with columns the vectors $v_1, ..., v_n$, then*

$$\nu(\Pi) = |\det A|.$$

Proof. Let T be the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = Ax$. Then $T(e_i) = v_i$, where $\{e_i : i = 1, ..n\}$ are the standard (unit) basic vectors in $\mathbb{R}^n$. Therefore, since T is linear it maps the unit cube $C^n = [0, 1]^n$ onto the parallelepiped $\Pi = \Pi(v_1, ..., v_n)$. Proposition 5.1.7 implies

$$\nu(\Pi) = |\det T|\nu(C^n) = |\det T|.$$

□

5.1.2 Change of variables; the general case

Here we shall extend Theorem 5.1.9 for nonlinear change of variables. The principal idea is based on the local approximation of a C^1 mapping $\varphi : U \subset \mathbb{R}^n \to \mathbb{R}^n$ at a point $a \in U$ by its differential $D_\varphi(a)$. *The local replacement of a nonlinear relation by a linear one is a basic idea of mathematics.* Recall from the Linear Approximation theorem (Theorem 3.1.3) that $D_\varphi(a) : \mathbb{R}^n \to \mathbb{R}^n$ is such that

$$D_\varphi(a)(h) \approx \varphi(a + h) - \varphi(a)$$

in a neighborhood of the point $a \in U$, and from Theorem 3.2.7, that the standard matrix of $D_\varphi(a)$ is the $n \times n$ matrix $D_\varphi(a) = (\frac{\partial \varphi_j}{\partial x_i}(a))$, where φ_j are the component functions of φ and x_i the components of x for $i, j = 1, ..., n$. Using the Linear Approximation theorem, we will show that for a sufficiently small cube C centered at $a \in U$

$$\nu(\varphi(C)) \approx |\det D_\varphi(a)| \nu(C).$$

Lemma 5.1.12. *Let $U \subset \mathbb{R}^n$ be open, and let $\varphi : U \to \varphi(U)$ be a diffeomorphism in $\mathbb{R}^n$ and $a \in U$ be fixed. Let $C \subset U$ be a cube centered at a. Then*

$$\lim_{C \downarrow a} \frac{\nu(\varphi(C))}{\nu(C)} = |\det D_\varphi(a)|,$$

where $C \downarrow a$ means that C shrinks to its center $a \in C$.

Here the C can be either a cube or a ball in U centered at a. It is sufficient to consider only balls $B_r(a)$, for replacing the norm $|| \cdot ||$ by the norm $|| \cdot ||_\infty$ the same type of argument works also for cubes. Thus what we are trying to prove is

$$\lim_{r \to 0} \frac{\nu(\varphi(B_r(a)))}{\nu(B_r(a))} = |\det D_\varphi(a)|.$$

Proof. We begin with the Linear Approximation theorem:

$$\varphi(x) = \varphi(a) + D_\varphi(a)(x - a) + \epsilon(x)||x - a||,$$

where $\epsilon(x)$ tends to 0 as $x \to a$. Since φ is a C^1 *change of variable*, $\det D_\varphi(a) \neq 0$ and so $D_\varphi(a)^{-1}$ exists. Applying this linear transformation to the equation above yields,

$$D_\varphi(a)^{-1}\varphi(x) = D_\varphi(a)^{-1}\varphi(a) + (x - a) + ||x - a|| D_\varphi(a)^{-1}\epsilon(x),$$

Hence by the triangle inequality[2]

$$||D_\varphi(a)^{-1}\phi(x) - D_\phi(a)^{-1}\varphi(a)|| \leq ||x - a|| + ||x - a|| \, ||D_\varphi(a)^{-1}\epsilon(x)||.$$

[2]For the case of the cube, here use the triangle inequality for the box-norm $|| \cdot ||_\infty$ and the estimate $||x - a|| \leq \sqrt{n}||x - a||_\infty$.

Finally, estimating the effect of the linear transformation $D_\varphi(a)^{-1}$ there is a positive constant β so that

$$||D_\varphi(a)^{-1}\epsilon(x)|| \le \beta||\epsilon(x)||.$$

Thus,

$$||D_\varphi(a)^{-1}\varphi(x) - D_\varphi(a)^{-1}\varphi(a)|| \le ||x-a|| + \beta||\epsilon(x)||\,||x-a|| < (1+\beta||\epsilon(x)||)r.$$

Since $||x-a|| < r$, $||\epsilon(x)|| \to 0$ as $r \to 0$. Now let $0 < t < 1$ and choose $r > 0$ small enough so that $1 + \beta||\epsilon(x)||$ lies strictly between $1-t$ and $1+t$. This means $D_\varphi(a)^{-1}(\varphi(B_r(a)))$ is contained in a ball centered at $D_\varphi(a)^{-1}\varphi(a)$ of radius $(1+t)r$. If for some x, $D_\varphi(a)^{-1}\varphi(x)$ lies on the boundary of this ball, then

$$||D_\varphi(a)^{-1}\varphi(x) - D_\varphi(a)^{-1}\varphi(a)|| = (1+t)r > (1-t)r,$$

so that $D_\varphi(a)^{-1}\varphi(x)$ lies outside this smaller ball of radius $(1-t)r$. Since, $D_\varphi(a)^{-1}$ is a *homeomorphism* in a neighborhood of a, $D_\varphi(a)^{-1}(\varphi(B_r(a)))$ also contains this smaller ball. In particular, by Corollary 5.1.8 we have

$$c_n(1-t)^n r^n \le \nu(D_\varphi(a)^{-1}(\varphi(B_r(a))) \le c_n(1+t)^n r^n.$$

That is,

$$(1-t)^n \le \frac{\nu(D_\varphi(a)^{-1}(\varphi(B_r(a)))}{\nu(B_r(a))} \le (1+t)^n.$$

But since $D_\phi(a)^{-1}$ is linear, Proposition 5.1.7 tells us

$$\nu(D_\varphi(a)^{-1}(\varphi(B_r(a))) = |\det D_\varphi(a)^{-1}|\nu(\varphi(B_r(a))).$$

Thus,

$$(1-t)^n \le \frac{\nu(\varphi(B_r(a)))}{|\det D_\varphi(a)|\nu(B_r(a))} \le (1+t)^n. \tag{5.7}$$

Letting $t \to 0$ (so that $r \to 0$) yields the conclusion. □

Lemma 5.1.13. *Let $U \subset \mathbb{R}^n$ be open and let Ω be a bounded simple set with $\overline{\Omega} \subset U$. Then there exist a compact set S of the form $S = \bigcup_{i=1}^N C_i$, where $\{C_1, ..., C_N\}$ are essential disjoint closed cubes, such that $\Omega \subset S \subset U$.*

Proof. Since $\overline{\Omega}$ is compact and U is open we may cover $\overline{\Omega}$ by open cubes lying completely inside U. By the compactness of $\overline{\Omega}$ a finite number of these cubes $\left\{C_{r_j}(x_j) : x_j \in \overline{\Omega},\ j = 1, ..., p\right\}$ of sides $2r_j$ respectively, also cover $\overline{\Omega}$. Let $\delta = \min\{r_1, ..., r_p\}$. Then the open cubes $\left\{C_\delta(x) : x \in \overline{\Omega}\right\}$ cover $\overline{\Omega}$ and by compactness so does a finite number of them $\left\{C_\delta(x_k) : x_k \in \overline{\Omega},\ k = 1, ..., q\right\}$ and each closed cube $\overline{C_\delta(x_k)} \subset U$. In general these closed cubes overlap. Partitioning each of these to smaller closed subcubes and counting the overlaping subcubes once, we obtained the required finite collection of essential disjoint closed (sub)cubes $\{C_i : i = 1, ..., N\}$ whose union S is contained in U. ☐

Next we prove the principal result[3] of this section.

Theorem 5.1.14. *(Change of variables formula). Let U be an open set in $\mathbb{R}^n$ and let $\varphi : U \to \mathbb{R}^n$ be a diffeomorphism. Let Ω be a bounded simple set with $\overline{\Omega} \subset U$. If $f : \varphi(\Omega) \to \mathbb{R}$ is integrable on $\varphi(\Omega)$, then $(f \circ \varphi)|\det D_\varphi|$ is integrable over Ω and*

$$\int_{\varphi(\Omega)} f(y)dy = \int_\Omega f(\varphi(x))|\det D_\varphi(x)|dx. \tag{5.8}$$

Proof. Set $J_\varphi = \det D_\varphi$. We show that $(f \circ \varphi)|J_\varphi|$ is integrable over Ω. We shall use Theorem 4.1.42 which characterizes integrable functions in terms of their points of discontinuity. Let D be the set of discontinuities of f in $\varphi(\Omega)$. Then $E = \varphi^{-1}(D)$ is the set of discontinuities of $f \circ \varphi$ in Ω. Since f is integrable on $\varphi(\Omega)$ the set D has measure zero. As $D \subset \varphi(\Omega) \subset \varphi(\overline{\Omega}) \subset \varphi(U)$ and $\varphi(\overline{\Omega})$ is compact contained in the open set $\varphi(U)$, we can find a finite number of closed cubes $\{C_i : i = 1, ..., N_1\}$ such that $D \subset \cup_{j=1}^{N_1} C_i \subset \varphi(U)$ (as in Lemma 5.1.13). Set $K = \cup_{i=1}^{N_1} C_i$, then K is compact and $D \subset K \subset \varphi(U)$. As

[3]The Change of Variables formula was first proposed by Euler when he studied double integrals in 1769, and it was generalized to triple integrals by Lagrange in 1773. Although it was used by Legendre, Laplace, Gauss, and first generalized to n variables by Mikhail Ostrogradski in 1836, it resisted a fully rigorous proof for a surprisingly long time. The theorem was first completely proved 125 years later, by Elie Cartan in a series of papers beginning in the mid-1890s. A popular proof adapted by many authors is the one given by J. Schwartz (1954) in [29]. In the proof given here effort has been made in avoiding as many technicalities as possible. A quite different approach to the problem can be found in P. Lax [20] and [21].

in the proof of Theorem 4.1.30 we may write $D = \bigcup_{k=1}^{\infty} D_{\frac{1}{k}}$ with each $D_{\frac{1}{k}}$ compact. Since $\varphi^{-1} \in C^1(\varphi(U))$ and each $D_{\frac{1}{k}}$ has measure zero, Lemma 5.1.5 and Proposition 4.1.21 tell us that $E = \varphi^{-1}(D)$ has also measure zero. Hence $f \circ \varphi$ is integrable. Since $|J_\varphi|$ is continuous on Ω, the set of points of discontinuity of $(f \circ \varphi)|J_\varphi|$ is the set E and so $(f \circ \varphi)|J_\varphi|$ is integrable on Ω.

We first prove (5.8) for a closed cube $C \subset U$. Let $\mathcal{P} = \{C_j : j = 1, ..., m\}$ be a partition of C into subcubes of equal side length $2r$ centered at x_j. Then $\varphi(C_j)$ is a bounded simple set for each j. In addition $\{\varphi(C_j)\}$ are mutually essentially disjoint and $\varphi(C) = \bigcup_{j=1}^{m} \varphi(C_j)$. From (5.7) of Lemma 5.1.12 we have

$$(1-t)^n |J_\varphi(x_j)| \nu(C_j) \leq \nu(\varphi(C_j)) \leq |J_\varphi(x_j)| \nu(C_j)(1+t)^n. \tag{5.9}$$

Let $y_j = \varphi(x_j)$. We set $F = f \circ \varphi$ and *suppose* $f \geq 0$. From (5.9) we have

$$(1-t)^n \sum_{j=1}^{m} F(x_j)|J_\varphi(x_j)|\nu(C_j) \leq \sum_{j=1}^{m} f(y_j)\nu(\varphi(C_j)) \leq$$

$$\leq (1+t)^n \sum_{j=1}^{m} F(x_j)|J_\varphi(x_j)|\nu(C_j).$$

Letting $t \to 0$ yields

$$\sum_{j=1}^{m} F(x_j)|J_\varphi(x_j)|\nu(C_j) \leq \sum_{j=1}^{m} f(y_j)\nu(\varphi(C_j)) \leq \sum_{j=1}^{m} F(x_j)|J_\varphi(x_j)|\nu(C_j). \tag{5.10}$$

The far left and far right are each a Riemann sum $S_{\mathcal{P}}(F|J_\varphi|)$ for the function $F|J_\varphi|$. Let $d = \max_{j=1,...,m}\{d(\varphi(C_j))\}$. Since φ is (uniformly) continuous on C, if $r \to 0$ then also $d \to 0$. Now since the functions f and $F|J_\varphi|$ are integrable on $\varphi(C)$ and C respectively, letting $r \to 0$ in (5.10) we get

$$\int_{\varphi(C)} f(y)dy = \int_C f(\varphi(x))|J_\varphi(x)|dx. \tag{5.11}$$

To remove the assumption that $f \geq 0$, we write $f = (f + c) - c$ where the constant $c \geq 0$ is sufficiently large that $f + c \geq 0$ on Ω, for example $c = \sup |f(x)|$. The argument just given applies to $f + c$ and to the constant function c. By the linearity of the integral (Theorem 4.2.1) subtracting the results we get (5.11).

Next we prove (5.8) for a finite union $S = \bigcup_{i=1}^{N} C_i$ of essential disjoint closed cubes with $\Omega \subset S \subset U$. That such a set S exists is guaranteed from Lemma 5.1.13. Now the finite additivity of the integral (Corollary 4.2.5) yields

$$\int_{\varphi(S)} f(y)dy = \sum_{i=1}^{N} \int_{\varphi(C_i)} f(y)dy = \sum_{i=1}^{N} \int_{C_i} (f(\varphi(x))|J_\varphi(x)|dx$$
$$= \int_S (f(\varphi(x))|J_\varphi(x)|dx.$$

Finally, we obtain (5.8) for Ω. Let $y = \varphi(x)$ with $x \in S$, where $\Omega \subset S \subset U$. Since $\chi_\Omega = \chi_{\varphi(\Omega)} \circ \varphi$, where χ_Ω is the characteristic function of Ω, the definition of the integral over a set gives

$$\int_{\varphi(\Omega)} f(y)dy = \int_{\varphi(S)} (f\chi_{\varphi(\Omega)})(y)dy = \int_S [(f\chi_{\varphi(\Omega)}) \circ \varphi |J_\varphi|](x)dx$$

$$= \int_S [(f\circ\varphi)|J_\varphi|\chi_\Omega](x)dx = \int_\Omega ((f\circ\varphi)|J_\varphi|)(x)dx = \int_\Omega f(\varphi(x))|J_\varphi(x)|dx.$$

□

Taking $f \equiv 1$ which is integrable everywhere we get

Corollary 5.1.15. *Let U be open in $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^n$ a diffeomorphism. Let Ω be a bounded simple set with $\overline{\Omega} \subset U$. Then*

$$\nu(\varphi(\Omega)) = \int_\Omega |\det D_\varphi(x)|dx.$$

In certain circumstances a direct application of the Change of Variables formula (5.8)) is not valid, mainly because φ fails to be one-to-one on the boundary of Ω. The following slight improvement of Theorem 5.1.14 is then useful.

Corollary 5.1.16. *Let U be open in $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^n$ be a C^1 map. Let Ω be a bounded simple set with $\overline{\Omega} \subset U$. Suppose $f : \varphi(\Omega) \to \mathbb{R}$ is integrable. Then the change of variables formula (5.8) remains valid if φ is only asssumed to be a diffeomorphism on the interior of Ω.*

Proof. Let C be a cube containing Ω and partition C into subcubes of equal side length $2r$. Let $C_1, ..., C_p$ be the subcubes contained in Ω, and $A = \Omega \backslash K$, where $K = \bigcup_{i=1}^{p} C_i$. Then $K \subset \Omega$ and so $\nu(K) \leq \nu(\Omega)$. Since Ω is simple, the difference $\nu(\Omega) - \nu(K)$ can be made arbitrarily small by taking r sufficiently small. That is, $\nu(A) = 0$ ans so also $\nu(\overline{A}) = 0$. Therefore by Lemma 5.1.5 (with $S = \overline{A}$) it follows that $\nu(\varphi(A)) \leq \nu(\varphi(\overline{A})) = 0$. Theorem 5.1.14 and the finite additvity of the integral yield

$$\int_{\varphi(K)} f(y)dy = \sum_{i=1}^{p} \int_{\varphi(C_i)} f(y)dy$$

$$= \sum_{i=1}^{p} \int_{C_i} (f(\varphi(x))|J_\varphi(x)|dx = \int_K (f(\varphi(x))|J_\varphi(x)|dx.$$

Finally, as $\Omega = K \cup A$, $\varphi(\Omega) = \varphi(K) \cup \varphi(A)$ and the integral over sets of volume zero is zero, the additivity of the integral gives $\int_{\varphi(\Omega)} f(y)dy = \int_\Omega (f(\varphi(x))|J_\varphi(x)|dx$. ☐

More generally a further extension of the Change of Variables formula can be made allowing singularities of φ by using a special case of *Sard's theorem*: Let $\varphi : U \to \mathbb{R}^n$ be of class C^1 on an open set $U \subset \mathbb{R}^n$ and

$$A = \{x \in U : \det D_\varphi(x) = 0\}.$$

Then $\varphi(A)$ has *measure zero*, (see Theorem 4.5.1). Hence the places where $|\det D_\varphi(x)| \equiv 0$ on Ω contribute nothing to the integrals in (5.8).

Next we give an extension of the Change of Variables formula[4] for continuous real valued functions f on $\mathbb{R}^n$ with *compact support* (see, Section 4.4).

[4] Although the integral of f is taken over $\mathbb{R}^n$ and such integrals are studied in the next section, since the integrand f has compact support it is an integral over any cube containing the support of f.

Theorem 5.1.17. *Let U be an open set in $\mathbb{R}^n$ and φ be a diffeomorphism of U with its image $\varphi(U)$. Then*

$$\int_{\mathbb{R}^n} f(y)dy = \int_{\mathbb{R}^n} f(\varphi(x))|\det D_\varphi(x)|dx, \tag{5.12}$$

for any continuous function f with compact support whose support lies in $\varphi(U)$.

Proof. Let $y \in \text{supp}(f)$ and V_y be a neighborhood of y contained in the open set $\varphi(U)$. Since the $\{V_y\}$ cover $\text{supp}(f)$ we can find a finite number of them $\{V_{y_i} : i = 1, ..., k\}$ which also cover. Let ψ_i be a partition of unity subordinate to this cover. By Corollary 4.4.4 $f = \sum_{i=1}^{k} \psi_i f$, where $\psi_i f$ is a continuous function whose support lies in V_{y_i}. By the Change of Variables formula (5.8) equation (5.12) holds for each $\psi_i f$, $i = 1, ..., k$ and hence by linearity of the integral also for their sum. ☐

We now apply the Change of Variables formula to a number of examples.

5.1.3 Applications, polar and spherical coordinates

Let $\Omega \subset \mathbb{R}^n$ be a bounded simple set and $f : \Omega \to \mathbb{R}$ an integrable function. One of the purposes of the Change of Variables formula is to simplify the computation of the multiple integral $\int_\Omega f$ on which either the integrand f or the region Ω is complicated and for which direct computation is difficult. Therefore, a C^1 change of variables $\varphi : \Omega^* \to \Omega$ is chosen, with $\Omega = \varphi(\Omega^*)$, so that the integral is easier to compute with the new integrand $(f \circ \varphi)|J_\varphi|$, or with the new region $\Omega^* = \varphi^{-1}(\Omega)$. In this notation the Change of Variables formula becomes

$$\int_\Omega f = \int_{\varphi(\Omega^*)} f = \int_{\Omega^*} (f \circ \varphi)|J_\varphi|. \tag{5.13}$$

The Change of Variables formula (5.13) for *double integrals* is

$$\int\int_\Omega f(x, y)dxdy = \int\int_{\Omega^*} f(x(u, v), y(u, v))|J_\varphi(u, v)|dudv,$$

where $\varphi(u,v)=(x(u,v),y(u,v))$ for $(u,v)\in\Omega^*=\varphi^{-1}(\Omega)$ and

$$|J_\varphi(u,v)|=|\det D_\varphi(u,v)|=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u}\end{pmatrix}\right|.$$

The Change of Variables formula (5.13) for *triple integrals* is

$$\int\int\int_\Omega f(x,y,z)dxdydz$$

$$=\int\int\int_{\Omega^*} f(x(u,v,w),y(u,v,w),z(u,v,w))|J_\varphi(u,v,w)|dudvdw,$$

where $\varphi(u,v,w)=(x(u,v,w),y(u,v,w),z(u,v,w))$ for $(u,v,w)\in\Omega^*=\varphi^{-1}(\Omega)$ and

$$|J_\varphi(u,v,w)|=|\det D_\varphi(u,v,w)|=\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|=\left|\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{pmatrix}\right|.$$

Example 5.1.18. Let $f(x,y)=xy$ and let Ω be the parallelogram bounded by the lines $x-y=0$, $x-y=1$, $x+2y=0$ and $x+2y=6$. Using the Change of Variables formula compute $\int\int_\Omega f(x,y)dxdy$.

Solution. The equations of the bounding lines of Ω suggest the linear change of variables $u=x-y$, $v=x+2y$. That is, the linear mapping $L:\mathbb{R}^2\to\mathbb{R}^2$ given by $L(x,y)=(x-y,x+2y)$. The standard matrix of L is $L=\begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix}$, and $\det L=3$. Hence L is an isomorphism on $\mathbb{R}^2$.

The vertices of the parallelogram Ω are $(0,0)$, $(2,2)$, $(\frac{8}{3},\frac{5}{3})$, and $(\frac{2}{3},-\frac{1}{3})$ and these are mapped by L to $(0,0)$, $(0,6)$, $(1,6)$, and $(1,0)$, respectively. Since L is linear, it follows that the image of the parallelogram Ω under L is the rectangle

$$\Omega^*=\{(u,v):0\le u\le 1, 0\le v\le 6\}.$$

In fact, we are actually interested in the inverse mapping $T=L^{-1}$ from the uv-plane to the xy-plane defined by

$$x=\tfrac{1}{3}(2u+v),\quad y=\tfrac{1}{3}(v-u).$$

That is, the linear transformation $T(u,v) = \frac{1}{3}(2u+v, v-u)$ with Jacobian $J_T(u,v) = \det T = \frac{1}{3}$ and $T(\Omega^*) = \Omega$.

Hence the Change of Variables formula yields

$$\int\int_{\Omega} xydxdy = \frac{1}{9}\int\int_{\Omega^*}(2u+v)(v-u)|\frac{1}{3}|dudv$$

$$= \frac{1}{27}\int_0^1\int_0^6 (2u+v)(v-u)dudv = \int_0^6 (-\frac{2}{3}+\frac{1}{2}v+v^2)dv = \frac{77}{27}.$$

Example 5.1.19. Compute the integral

$$\int\int_{\Omega} e^{\frac{y-x}{y+x}}dxdy,$$

where $\Omega = \{(x,y) : x \geq 0,\ \ y \geq 0,\ \ x+y \leq 1\}$.

Solution. Here the difficulty in the computation of the integral comes from the integrand. Note that the function $f(x,y) = e^{\frac{y-x}{y+x}}$ is not defined at $(0,0)$ and, in fact, does not even have a limit there. However, no matter how one defines f at $(0,0)$ the function is integrable by Theorem 4.1.42. The set Ω is a simple region in $\mathbb{R}^2$. Actually, Ω is the triangle bounded by the line $x+y=1$ and the coordinate axes. The integrand suggests the use of the new variables

$$u = y-x,\ v = y+x,$$

i.e., the linear transformation $L(x,y) = (y-x, y+x)$. L is an isomorphism on $\mathbb{R}^2$ and its inverse $T = L^{-1}$ is the linear transformation

$$T(u,v) = (\frac{v-u}{2}, \frac{v+u}{2}).$$

The standard matrix representation is $T = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, and its Jacobian $J_T(u,v) = \det T = -\frac{1}{2}$. To find the image Ω^* of Ω under L in the uv-plane, note that the lines $x=0$, $y=0$ and $x+y=1$ are mapped onto the lines $u=v$, $u=-v$ and $v=2$, respectively. The points inside Ω satisfy $0 < x+y < 1$ and these are mapped into points of Ω^* satisfying $0 < v < 1$. Hence Ω^* is the triangle in the uv-plane bounded by the

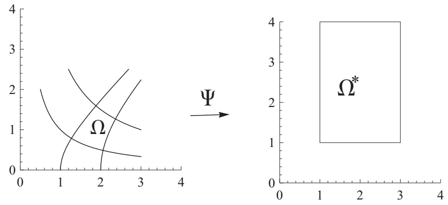

Figure 5.2: $\psi(x,y) = (xy, x^2 - y^2)$

lines $v = u$, $v = -u$ and $v = 2$. Now $T(\Omega^*) = \Omega$ and the Change of Variables formula yields

$$\int\int_{\Omega} e^{\frac{y-x}{y+x}} dxdy = \frac{1}{2}\int\int_{\Omega^*} e^{\frac{u}{v}} dudv$$

$$= \frac{1}{2}\int_0^1 \left(\int_{-v}^{v} e^{\frac{u}{v}} du\right) dv = \frac{1}{2}\int_0^1 (e - \frac{1}{e})vdv = \frac{1}{4}(e - e^{-1}) = \frac{1}{2}\sinh(1).$$

Example 5.1.20. Let $f(x,y) = x^2 + y^2$. Evaluate $\int\int_{\Omega} f(x,y)dxdy$, where Ω is the region in the first quadrant of the xy-plane bounded by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 4$.

Solution. To simplify the region Ω we make the substitution $u = xy$, $v = x^2 - y^2$. This transformation $\psi : \mathbb{R}^2 \to \mathbb{R}^2$

$$\psi(x,y) = (xy, x^2 - y^2)$$

is one-to-one on Ω and maps the region Ω in the xy-plane onto the rectangle Ω^* in the uv-plane bounded by the lines $u = 1$, $u = 3$, $v = 1$, and $v = 4$, ie $\psi(\Omega) = \Omega^* = [1,3] \times [1,4]$. Its Jacobian is

$$J_\psi(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix} = -2(x^2 + y^2) \neq 0$$

for all $(x,y) \in \Omega$. As before, we are interested in its inverse $\varphi = \psi^{-1}$ given by

$$\varphi(u,v) = \left(\left[\frac{(4u^2 + v^2)^{\frac{1}{2}} + v}{2}\right]^{\frac{1}{2}}, \left[\frac{(4u^2 + v^2)^{\frac{1}{2}} - v}{2}\right]^{\frac{1}{2}}\right).$$

The Jacobian of φ is

$$J_\varphi(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{J_\psi(x,y)} = -\frac{1}{2(x^2+y^2)} = -\frac{1}{2(4u^2+v^2)^{\frac{1}{2}}}.$$

Now $\Omega = \varphi(\Omega^*)$ and the Change of Variables formula gives

$$\int\int_\Omega f(x,y)dxdy = \int\int_{\Omega^*} f(\varphi(u,v))|J_\varphi(u,v)|dudv$$

$$= \int_1^3\int_1^4 (4u^2+v^2)^{\frac{1}{2}} \cdot \frac{1}{2(4u^2+v^2)^{\frac{1}{2}}} dudv = \frac{1}{2}\int_1^3\int_1^4 dudv = 3.$$

Example 5.1.21. (*The solid torus*). Let $0 < a < b$ be fixed. The solid torus in $\mathbb{R}^3$ is the region Ω obtained by revolving the disk $(y-b)^2 + z^2 \leq a^2$ in the yz-plane, about the z-axis. Calculate the volume of Ω. (See, Figure 4.5)

Solution. Note that the mapping $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the equations

$$x = (b + r\cos\phi)\cos\theta$$
$$y = (b + r\cos\phi)\sin\theta$$
$$z = r\sin\phi,$$

maps the rectangle $R = \{(\theta,\phi,r) : \theta,\phi \in [0,2\pi],\ r \in [0,a]\}$ onto the torus, ie $\varphi(R) = \Omega$. In particular, φ is one-to-one in the interior of R. Its Jacobian is

$$J_\varphi(\theta,\phi,r) = \frac{\partial(x,y,z)}{\partial(\theta,\phi,r)} = r(b + r\cos\phi),$$

which is positive in the interior of R. By Corollary 5.1.16, the Change of Variables formula applies, and Corollary 5.1.15 gives

$$\nu(\Omega) = \nu(\varphi(R)) = \int\int\int_R r(b + r\cos\phi)d\theta d\phi dr = 2\pi^2 a^2 b.$$

Polar coordinates

The map which changes from polar coordinates to rectangular coordinates is

$$\varphi(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y).$$

Its Jacobian is

$$J_\phi(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r,$$

which is zero for $r=0$. φ maps the set $[0,+\infty)\times[0,2\pi)$ in the $r\theta$-plane onto $\mathbb{R}^2$. Note that on this set φ is not one-to-one, since for $0\leq\theta<2\pi$ it sends all points $(0,\theta)$ into $(0,0)$. However φ restricted to $S=(0,+\infty)\times(0,2\pi)$ (the interior of $[0,+\infty)\times[0,2\pi)$) is one-to-one and $J_\phi(r,\theta)\neq 0$ on S. Although $\varphi(S)$ excludes the non-negative x-axis, this *set has* 2*-dimensional volume zero and therefore contributes nothing to the value of an integral*. Note in particular, that φ maps the rectangle $R_\alpha=[0,\alpha]\times[0,2\pi]$ onto the disk $B_\alpha=\{(x,y): x^2+y^2\leq\alpha^2\}$. Thus by restricting φ in the interior of R_α we can apply the Change of Variables theorem (Corollary 5.1.16) to convert integration over B_α into integration over R_α.

Example 5.1.22. Compute

$$\int\int_{B_\alpha} e^{-(x^2+y^2)}dxdy,$$

where $B_\alpha=\{(x,y): x^2+y^2\leq\alpha^2\}$.

Solution. Changing to polar coordinates and using (5.13) with $\Omega=B_\alpha$ and $\Omega^*=[0,\alpha]\times[0,2\pi]$ we get

$$\int\int_{B_\alpha} e^{-(x^2+y^2)}dxdy = \int_0^\alpha\int_0^{2\pi} e^{-r^2}rdrd\theta$$

$$= \int_0^\alpha\left(e^{-r^2}r\int_0^{2\pi}d\theta\right)dr = \pi\int_{-\alpha^2}^0 e^u du = \pi[1-e^{-\alpha^2}].$$

In particular, letting $\alpha \to \infty$ we find[5]

$$\int\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dxdy = \pi.$$

Example 5.1.23. Compute

$$\int\int_{\Omega} \sin\sqrt{x^2+y^2}\,dxdy,$$

where $\Omega = \left\{(x,y) : \pi^2 < x^2+y^2 \leq 4\pi^2\right\}$.

Solution. Changing to polar coordinates we see that $\Omega = \varphi(\Omega^*)$, where $\Omega^* = (\pi, 2\pi] \times [0, 2\pi]$ and we get

$$\int\int_{\Omega} \sin\sqrt{x^2+y^2}\,dxdy = \int_{\pi}^{2\pi}\int_{0}^{2\pi} \sin r \cdot r dr d\theta$$

$$= 2\pi \int_{\pi}^{2\pi} r \sin r dr = 2\pi \left[-r\cos r\Big|_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos r dr \right]$$

$$= -6\pi^2 + 2\pi \sin r\Big|_{\pi}^{2\pi} = -6\pi^2.$$

Example 5.1.24. Compute

$$\int\int_{\Omega} \sqrt{\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}}\,dxdy,$$

where $\Omega = \left\{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1,\ \ x > 0,\ \ y > 0\right\}$.

Solution. Changing to polar coordinates using the transformation

$$\varphi(r,\theta) = (ar\cos\theta, br\sin\theta) = (x,y),$$

we see that $\Omega^* = (0,1] \times [0, \frac{\pi}{2})$ and $J_\varphi(r,\theta) = abr$. Now the integral becomes

[5]The integral $\int\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dxdy = \pi$ is improper. See Section 5.2, for a detailed discussion on improper multiple integrals.

$$ab\int_0^1\int_0^{\frac{\pi}{2}}\sqrt{\frac{1-r^2}{1+r^2}}rdrd\theta = ab\frac{\pi}{2}\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}rdr$$
$$= ab\frac{\pi}{2}\int_0^{\sqrt{2}}\sqrt{2-\rho^2}d\rho,$$

where $\rho^2 = 1 + r^2$. Finally, setting $\rho = \sqrt{2}\sin t$, this latter integral yields $\frac{\pi}{2}(\frac{\pi}{4} - \frac{1}{2})ab$.

Example 5.1.25. Find the area of the region S in $\mathbb{R}^2$ bounded by the circles $r = 1$ and $r = \frac{2}{\sqrt{3}}\cos\theta$ (outside of the circle $r = 1$).

Solution. Solving $1 = \frac{2}{\sqrt{3}}\cos\theta$ we find the (polar) coordinates of the points of intersections of the given circles to be $(1, \frac{\pi}{6})$ and $(1, \frac{11\pi}{6})$ (the reader is recommended to make a drawing; in cartesian coordinates the circle $r = \frac{2}{\sqrt{3}}\cos\theta$ is $(x - \frac{1}{\sqrt{3}})^2 + y^2 = \frac{1}{3}$ and the circle $r = 1$ is, of course, $x^2 + y^2 = 1$). By the symmetry of the region the area is

$$A(S) = \int\int_S rdrd\theta = 2\int_0^{\frac{\pi}{6}}\left(\int_1^{\frac{2}{\sqrt{3}}}\cos\theta rdr\right)d\theta$$
$$= \int_0^{\frac{\pi}{6}}(\frac{4}{3}\cos^2\theta - 1)d\theta = \frac{3\sqrt{3}-\pi}{18}.$$

Cylindrical coordinates

The *cylidrical coordinates* are just the polar coordinates in the xy-plane with the z-coordinate added on, (Figure 5.3),

$$\varphi(r, \theta, z) = (r\cos\theta, r\sin\theta, z).$$

To get a one-to-one mapping we must keep $r > 0$ and restrict θ, say, in $0 < \theta \leq 2\pi$. The Jacobian of φ is easily seen to be $J_\varphi(r, \theta, z) = r$, and the Change of Variables formula (5.13) becomes

$$\int\int\int_\Omega f(x, y, z)dxdydz = \int\int\int_{\Omega^*} f(r\cos\theta, r\sin\theta, z)rdrd\theta dz.$$

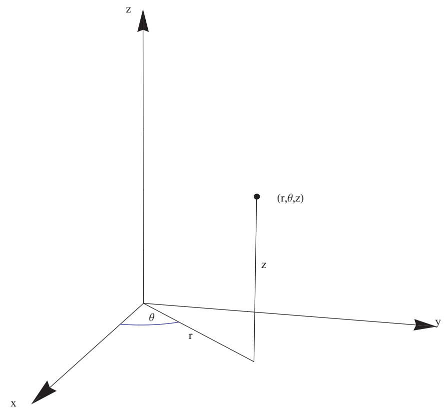

Figure 5.3: Cylindrical coordinates

Example 5.1.26. Find the volume of the region Ω bounded by the surfaces $(\frac{x^2+y^2}{a^2})^2 + \frac{z}{b} = 1$, $z = 0$, where $a, b > 0$.

Solution. As we know from Definition 4.1.39

$$\nu(\Omega) = \int_{\Omega} 1 = \int\int\int_{\Omega} dxdydz.$$

Changing to cylidrical coordinates we get

$$\nu(\Omega) = \int\int\int_{\Omega^*} rdrd\theta dz = \int_0^a \int_0^{2\pi} \left(\int_0^{b(1-\frac{r^4}{a^4})} dz \right) drd\theta$$

$$= 2\pi b \int_0^a r\left(1 - \frac{r^4}{a^4}\right) dr = \frac{2}{3}\pi a^2 b.$$

Example 5.1.27. Find the moment of inertia I_L of a cylinder $x^2+y^2 = a^2$ of height h, if its density at each point is propotional to the distance of this point from the axis of the cylinder, with respect to a line L parallel to the axis of the cylinder and at distance b from it.

Solution. Let Ω be the cylinder. The moment of inertia of Ω about the line L is

$$I_L = \int\int\int_{\Omega} [d(x,y,z)]^2 \rho(x,y,z) dxdydz,$$

where $d(x,y,z)$ is the distance from (x,y,z) to the line L, and $\rho(x,y,z)$ is the density. Placing the base of the cylinder on the xy-plane with its center at the origin, the line L is described by $x=b$, $y=0$. The density function is $\rho(x,y,z) = k\sqrt{x^2+y^2}$ and $[d(x,y,z)]^2 = (x-b)^2+y^2$. Passing to cylindrical coordinates we get

$$\begin{aligned} I_L &= \int_0^h \int_0^{2\pi} \int_0^a [(r\cos\theta - b)^2 + (r\sin\theta)^2](kr) r dr d\theta dz \\ &= \int_0^h \int_0^{2\pi} \int_0^a kr^2[r^2+b^2-2r\cos\theta] dr d\theta dz \\ &= k\int_0^h \int_0^{2\pi} \int_0^a r^2[r^2+b^2] dr d\theta dz + 0 \\ &= 2\pi k a^3 h \left(\frac{a^2}{5} + \frac{b^2}{3}\right). \end{aligned}$$

Spherical coordinates

Given $(x,y,z)\in\mathbb{R}^3$, the *spherical coordinates* (r,θ,ϕ) are defined by

$$\Phi(r,\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) = (x,y,z).$$

Here $r=\sqrt{x^2+y^2+z^2}$, θ is the longitude (the angle from the x-axis to the vector $(x,y,0)$), and ϕ is the co-latitude (the angle from the positive z-axis to the vector (x,y,z)). (see, Figure 5.4)

It is readily seen that Φ maps the rectangle $R=[0,\alpha]\times[0,2\pi]\times[0,\pi]$ in the $r\theta\phi$-space into the ball $B_\alpha = \{(x,y,z) : x^2+y^2+z^2 \leq \alpha^2\}$.

Again Φ is not globally invertible; we have $\Phi(r,\theta,\phi) = \Phi(r,\theta + 2k\pi, \phi + 2m\pi) = \Phi(r,\theta+(2k+1)\pi, \phi+(2m+1)\pi)$ for each $k,m\in\mathbb{Z}$. The map Φ is one-to-one on $\{(r,\theta,\phi) : r>0, 0\leq\theta<2\pi, 0<\phi<\pi\}$ and maps this set onto $\mathbb{R}^3 - \{(0,0,z) : z\in\mathbb{R}\}$. Since $\{(0,0,z) : z\in\mathbb{R}\}$, has volume zero does not affect the value of an integral. Now

$$J_\Phi(r,\theta,\phi) = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$$

$$= \det\begin{pmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{pmatrix} = -r^2\sin\phi.$$

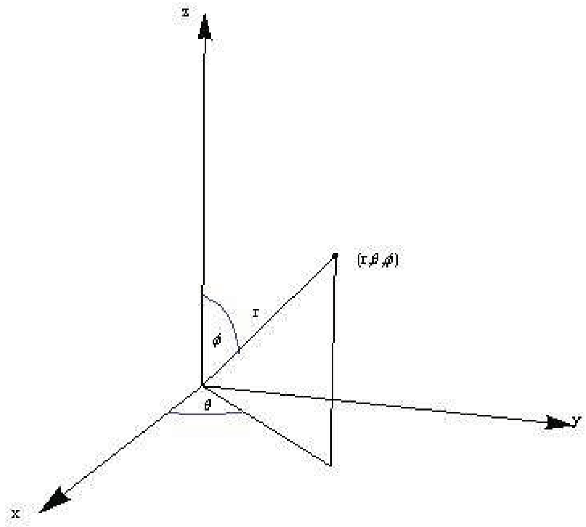

Figure 5.4: Spherical coordinates (r, θ, ϕ)

Hence the formula for integration in spherical coordinates is

$$\int\int\int_{\Omega} f(x,y,z)dxdydz$$

$$= \int\int\int_{\Omega^*} f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)r^2\sin\phi drd\theta d\phi.$$

Example 5.1.28. Compute the integral $\int\int\int_{\Omega} e^{(x^2+y^2+z^2)^{\frac{3}{2}}}dxdydz$, where Ω is the unit ball in $\mathbb{R}^3$.

Solution. Changing to spherical coordinates $\Omega = \Phi(\Omega^*)$, with $\Omega^* = [0,1]\times[0,2\pi]\times[0,\pi]$. Now the formula of integration in spherical coordinates gives

$$\int\int\int_{\Omega} e^{(x^2+y^2+z^2)^{\frac{3}{2}}}dxdydz = \int\int\int_{\Omega^*} e^{r^3}r^2\sin\phi drd\theta d\phi$$

$$= \int_0^1\int_0^{\pi}\left(r^2e^{r^3}\sin\phi\int_0^{2\pi}d\theta\right)d\phi = 2\pi\int_0^1\left(r^2e^{r^3}\int_0^{\pi}\sin\phi d\phi\right)dr$$

$$= 4\pi \int_0^1 r^2 e^{r^3} dr = \frac{4}{3}\pi \int_0^1 e^t dt = \frac{4\pi(e-1)}{3},$$

where we set $t = r^3$.

Example 5.1.29. Compute $\int\int\int_\Omega \left(\frac{1}{\sqrt{x^2+y^2}} + \frac{1}{z}\right) dxdydz$, where Ω is the region in $\mathbb{R}^3$ bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2 = 1$, and $z > 0$.

Solution. The region of integration Ω represents an "ice cream cone". Note that Ω is an elementary region[6] in $\mathbb{R}^3$. However, both the region Ω and the integrand suggest a change to spherical coordinates. Then $\Omega = \Phi(\Omega^*)$, where

$$\Omega^* = \left\{(r,\theta,\phi) : 0 < r < 1,\ 0 \le \theta < 2\pi,\ 0 < \phi < \frac{\pi}{4}\right\}.$$

Now the Change of Variables formula yields

$$\int\int\int_\Omega \left(\frac{1}{\sqrt{x^2+y^2}} + \frac{1}{z}\right) dxdydz$$

$$= \int\int\int_{\Omega^*} \left(\frac{1}{r\sin\phi} + \frac{1}{r\cos\phi}\right) r^2 \sin\phi drd\theta d\phi$$

$$= \int_0^{\frac{\pi}{4}} \left((1+\tan\phi)\int_0^1 r\left(\int_0^{2\pi} d\theta\right) dr\right) d\phi$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \left((1+\tan\phi)\int_0^1 rdr\right) d\phi$$

$$= \pi \int_0^{\frac{\pi}{4}} (1+\tan\phi)d\phi = \pi[(\phi - \log(\cos\phi))|_0^{\frac{\pi}{4}}] = \pi\left[\frac{\pi}{4} - \log\left(\frac{\sqrt{2}}{2}\right)\right].$$

[6] $\Omega = \{(x,y,z) : -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}, -\sqrt{\frac{1}{2}-x^2} \le y \le \sqrt{\frac{1}{2}-x^2}, \sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}\}$. Note that working with rectangular coordinates the integral becomes quite tedious.

Example 5.1.30. Find the volume of the region Ω in $\mathbb{R}^3$ above the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = z$.

Solution. Here $\Omega = \Phi(\Omega^*)$, where

$$\Omega^* = \left\{(r, \theta, \phi) : 0 \le r < \cos\phi,\ 0 \le \theta < 2\pi,\ 0 < \phi < \frac{\pi}{4}\right\}.$$

Hence the volume $\nu(\Omega)$ is equal to

$$\int\int\int_{\Omega} 1 dxdydz = \int_0^{\frac{\pi}{4}} \left(\int_0^{\cos\phi} \left(\int_0^{2\pi} d\theta\right) r^2 \sin\phi dr\right) d\phi$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \left(\sin\phi \int_0^{\cos\phi} r^2 dr\right) d\phi = \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \cos^3\phi \sin\phi d\phi$$

$$= \frac{2\pi}{3} \int_{\frac{\sqrt{2}}{2}}^{1} t^3 dt = \frac{2\pi}{3} \cdot \frac{3}{16} = \frac{\pi}{8},$$

where we set $t = \cos\phi$.

Example 5.1.31. Find the centroid of a ball of radius a. Find the moment of inertia of a ball of constant density with respect to a diameter.

Solution. Place the ball B with its center at the origin, so that B is described by $x^2 + y^2 + z^2 \le a^2$. From Example 4.3.19 the volume of the ball is $\nu(B) = \frac{4}{3}\pi a^3$. Using spherical coordinates

$$\overline{x} = \frac{1}{\nu(B)} \int\int\int_B x dxdydz = \frac{1}{\nu(B)} \int_0^a \int_0^{2\pi} \int_0^{\pi} (r\cos\phi) r^2 \sin\phi d\phi d\theta dr = 0.$$

By symmetry $\overline{x} = \overline{y} = \overline{z}$. Hence its centroid is its center $(0, 0, 0)$.

Let the density of the ball be $\rho(x, y, z) = k$. By symmetry $I_x = I_y = I_z$. Since $I_x + I_y + I_z = 2I_0$, where I_0 is the moment of inertia about the origin, $I_0 = \int\int\int_B (x^2 + y^2 + z^2) k dxdydz$, we have

$$I_x = I_y = I_z = \frac{2}{3} I_0 = \frac{2}{3} \int_0^a \int_0^{\pi} \int_0^{2\pi} k r^4 \sin\phi d\theta d\phi dr = \frac{8\pi k a^5}{15}.$$

Since the mass of the ball is $m = \frac{4\pi k a^3}{3}$, the answer can be expressed $\frac{2}{5} a^2 m$.

Exercise 5.1.32. Show that the centroid of the portion of the ball of radius a in the first octant is the point $\frac{3a}{2}(1,1,1)$.

Spherical coordinates in $\mathbb{R}^n$

This is the higher dimensional analogue of the spherical coordinates in $\mathbb{R}^3$. Given $(x_1,...,x_n) \in \mathbb{R}^n$ we let $r = ||x|| = \sqrt{x_1^2 + ... + x_n^2}$, and $\Theta = (\theta, \phi_1, ..., \phi_{n-2})$, a generic point on the unit sphere, S^{n-1}.
For $n \geq 2$ the n-dimensional spherical mapping $\Phi_n : \mathbb{R}^n \to \mathbb{R}^n$ written as

$$\Phi_n(r,\Theta) = \Phi_n(r,\theta,\phi_1,...,\phi_{n-2}) = (x_1,...,x_n)$$

is defined by

$$x_1 = r\cos\phi_1,$$

$$x_2 = r\sin\phi_1\cos\phi_2,$$

$$x_3 = r\sin\phi_1\sin\phi_2\cos\phi_3,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$x_{n-1} = r\sin\phi_1\cdots\sin\phi_{n-2}\cos\theta,$$

$$x_n = r\sin\phi_1\cdots\sin\phi_{n-2}\sin\theta.$$

Φ_n maps the rectangle

$$R_\alpha = [0,\alpha]\times[0,2\pi]\times[0,\pi]\times\cdots\times[0,\pi]$$

$$= \{(r,\theta,\phi_1,...,\phi_{n-2}) \in \mathbb{R}^n : r \in [0,\alpha],\ \theta \in [0,2\pi],\ \phi_i \in [0,\pi], i=1,...,n-2\}$$

onto the ball $B_\alpha = \{x \in \mathbb{R}^n : ||x|| \leq \alpha\}$.

Thus for $n = 2$

$$\Phi_2(r,\Theta) = \Phi_2(r,\theta) = (r\cos\theta, r\sin\theta),$$

which are just the polar coordinates in $\mathbb{R}^2$, while for $n = 3$

$$\Phi_3(r,\Theta) = \Phi_3(r,\theta,\phi_1) = (r\cos\phi_1, r\sin\phi_1\cos\theta, r\sin\phi_1\sin\theta),$$

which are the spherical coordinates in $\mathbb{R}^3$ (viewing the x_1-axis as the z-axis and ϕ_1 as ϕ).

The Jacobian of Φ_n can be calculated by induction and is

$$J_{\Phi_n}(r,\Theta) = J_{\Phi_n}(r,\theta,\phi_1,...,\phi_{n-2}) = \frac{\partial(x_1,x_2,x_3,...,x_n)}{\partial(r,\theta,\phi_1,...,\phi_{n-2})}$$

$$= r^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\cdots\sin^2\phi_{n-3}\sin\phi_{n-2} = r^{n-1}\prod_{k=1}^{n-2}\sin^{n-k-1}\phi_k.$$

Clearly, Φ_n is not one-to-one on R_α, however Φ_n restricted to the interior of R_α it is one-to-one with $J_{\Phi_n}(r,\Theta) > 0$ and the Change of Variables formula applies.

Example 5.1.33. (*The volume of the unit ball in* $\mathbb{R}^n$). Find the volume of the n-dimensional unit ball,

$$B^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}.$$

Solution. In Corollary 5.1.8 we denoted the volume $\nu(B^n) = c_n$. Now

$$c_n = \int_{B^n} 1 = \int_{R_1} J_{\Phi_n}(r,\Theta)$$

$$= \frac{2\pi}{n}\left[\int_0^\pi\cdots\int_0^\pi \sin^{n-2}\phi_1\sin^{n-3}\phi_2\cdots\sin\phi_{n-2}d\phi_1\cdots d\phi_{n-2}\right]$$

$$= \frac{2\pi}{n}\prod_{k=1}^{n-2}\left[\int_0^\pi \sin^k\phi d\phi\right] = \frac{2\pi}{n}\prod_{k=1}^{n-2} I_k = \frac{2\pi}{n}\left[I_1 I_2\cdots I_{n-2}\right],$$

where $I_k = \int_0^\pi \sin^k\phi d\phi$. From elementary calculus we know that for $k = 1, 2, ...$

$$I_{2k} = \int_0^\pi \sin^{2k}\phi d\phi = \frac{(2k)!\pi}{2^{2k}(k!)^2},$$

$$I_{2k-1} = \int_0^\pi \sin^{2k-1}\phi d\phi = \frac{2^{2k-1}((k-1)!)^2}{(2k-1)!}.$$

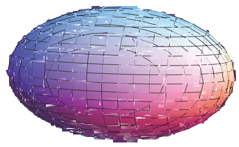

Figure 5.5: Ellipsoid: $x^2 + 2y^2 + 3z^2 = 1$

Now, since $I_{k-1}I_k = \frac{2\pi}{k}$, when $n = 2m$ $(m = 1, 2, ...)$, we have

$$c_{2m} = \frac{2\pi}{2m}\left[\frac{\pi^{m-1}}{(m-1)!}\right] = \frac{\pi^m}{m!},$$

while when $n = 2m + 1$ $(m = 0, 1, ...)$,

$$c_{2m+1} = \frac{2\pi}{2m+1}\left[\frac{\pi^{m-1}}{(m-1)!} \cdot I_{2m-1}\right] = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m+1)}.$$

Thus,

$$c_{2m} = \frac{\pi^m}{m!}, \quad c_{2m+1} = \frac{2^{m+1}\pi^m}{1 \cdot 3 \cdot 5 \cdots (2m+1)}. \tag{5.14}$$

Hence $c_1 = 2$, $c_2 = \pi$, $c_3 = \frac{4\pi}{3}$, $c_4 = \frac{\pi^2}{2}$, $c_5 = \frac{8\pi^2}{15}$, $c_6 = \frac{\pi^3}{6}$, $c_7 = \frac{16\pi^3}{105}$, and so on.

Example 5.1.34. Find the volume of the n-dimensional solid ellipsoid

$$E^n = \left\{x \in \mathbb{R}^n : \frac{x_1^2}{a_1^2} + ... + \frac{x_n^2}{a_n^2} \leq 1\right\}.$$

Solution. Consider the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$T(x_1, ..., x_n) = (a_1x_1, ..., a_nx_n).$$

The solid ellipsoid E^n is the image of the unit ball B^n under T. By Proposition 5.1.7

$$\nu(E^n) = \nu(T(B^n)) = |\det T|\nu(B^b) = a_1a_2 \cdots a_n\nu(B^n).$$

In particular, the volume of the 3-dimensional ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1$ is $\frac{4}{3}\pi abc$.

Example 5.1.35. (*Invariant integrals on groups*)(*).
We now give an application of the Change of Variables formula to calculating *Invariant integrals on groups* . Rather than deal with generalities, we shall do this with an example which has enough complexity to illustrate the general situation well.

Consider the set G of all matrices of the following form.

$$G=\left\{x=\begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} : u\neq 0, v\in\mathbb{R}\right\}.$$

which evidently can be identified with the (open) left and right half planes. We write $x=(u,v)$. As the reader can easily check, G is a group under matrix multiplication (in particular G is closed under multiplication). G is called the *affine group of the real line* $\mathbb{R}$. We define left and right translation (L_g and R_g respectively) in G as follows: $L_g(x)=gx$, $R_g(x)=xg$. The reader can easily check that each L_g and R_g is a diffeomorphism of G. Now as such they also operate on functions. Namely if $f\in C_c(G)$, the continuous functions on G with compact support, we write $L_g(f)(x)=f(gx)$ and $R_g(f)(x)=f(xg)$. Notice that $L_{gh}=L_gL_h$ and $R_{gh}=R_gR_h$ for each g and $h\in G$.

We say an integral over G is *left invariant* if

$$\int_G L_g(f)=\int_G f$$

for every $g\in G$ and $f\in C_c(G)$. Similarly, an integral over G is called *right invariant* if

$$\int_G R_g(f)=\int_G f$$

for every $g\in G$ and $f\in C_c(G)$. We now calculate such left or right invariant integrals with the help of the Change of Variables formula. To do this we first find the derivatives and Jacobians of the L_g's and R_g's. A direct calculation, which we leave to the reader, shows that if $g=(u,v)$, then $D(L_g)=uI$ and hence $|\det D(L_g)|=u^2$. Similarly $|\det D(R_g)|=|u|$. The first observation we make is that these Jacobians are independant of x (as well as v) and only depend on g. They are also nowhere zero. (This latter point is obvious in our case, but is true in general since L_g and R_g are diffeomorphisms).

Now consider $f \in C_c(G)$. Then $\frac{f}{|\det D(L_g)|}$ is also $\in C_c(G)$ and in fact for each g, as f varies over $C_c(G)$ so does $\frac{f}{|\det D(L_g)|}$. Similarly for each g, as f varies over $C_c(G)$ so does $\frac{f}{|\det D(R_g)|}$. Now let $\phi \in C_c(G)$ and L_g be a change of variables. The Change of Variables formula (5.8) tells us

$$\int_G \phi(gx)|\det D(L_g)|dudv = \int_G \phi(x)dudv.$$

Since this holds for all ϕ and we can take $\phi = \frac{f}{|\det D(L_g)|}$, we get

$$\int_G \frac{f(gx)}{|\det D(L_g)(gx)|}|\det D(L_g)|dudv = \int_G \frac{f(x)}{|\det D(L_g)(x)|}dudv.$$

After taking into account that $L_{gh} = L_g L_h$, the Chain Rule and the fact that $|\det(AB)| = |\det(A)||\det(B)|$ we get

$$\int_G f(gx)\frac{dudv}{|\det D(L_g)|} = \int_G f(x)\frac{dudv}{|\det D(L_g)|},$$

for every $f \in C_c(G)$. Thus

$$\frac{dudv}{|\det D(L_g)|} = \frac{dudv}{u^2}$$

is a left invariant integral for G. Similaly, $\frac{dudv}{|u|}$ is a right invariant integral for G. Notice that here left invariant and right invariant integrals are not the same.

EXERCISES

1. Evaluate the integral

$$\int\int_\Omega \sqrt{x+y}dxdy,$$

where Ω is the parallelogram bounded by the lines $x + y = 0$, $x + y = 1$, $2x - 3y = 0$ and $2x - 3y = 4$.

2. Let $n > 0$. Compute the integral

$$\int\int_{\Omega}(x+y)^n dxdy,$$

where $\Omega = \{(x,y) : x \geq 0,\ \ y \geq 0,\ \ x+y \leq 1\}$.

3. Evaluate the integral

$$\int_0^1 \int_0^x \sqrt{x^2+y^2}dydx,$$

using the transformation $x = u$, $y = uv$.

4. Evaluate the integral

$$\int\int_{\Omega} \frac{3y}{\sqrt{1+(x+y)^3}}dxdy,$$

where $\Omega = \{(x,y) : x+y < a, x > 0, y > 0\}$.
Hint. Set $u = x+y$, $v = x-y$.

5. Evaluate

$$\int\int_{\Omega} \sin(\frac{x-y}{x+y})dxdy,$$

where $\Omega = \{(x,y) : x \geq 0,\ \ y \geq 0,\ \ x+y \leq 1\}$.

6. Evaluate

$$\int\int_{\Omega} \sin x \sin y \sin(x+y)dxdy,$$

where $\Omega = \left\{(x,y) : x \geq 0,\ \ y \geq 0,\ \ x+y \leq \frac{\pi}{2}\right\}$.

7. Evaluate

$$\int\int_{\Omega} \sqrt{x^2+y^2}dxdy,$$

where Ω is the region by the circles $x^2+y^2 = 4$, $x^2+y^2 = 9$.

8. Evaluate

$$\int\int_{\Omega} \tan^{-1}\left(\frac{y}{x}\right) dxdy,$$

where $\Omega = \left\{(x,y) : x^2+y^2 \geq 1,\ x^2+y^2 \leq 9,\ y \geq \frac{x}{\sqrt{3}}, y \leq x\sqrt{3}\right\}$.
Hint. Change to polar coordinates.

9. Evaluate
$$\int\int_{\Omega} (x-y)^4 e^{x+y} dxdy,$$
where Ω is the square with vertices $(1,0)$, $(2,1)$, $(1,2)$, and $(0,1)$.

10. Evaluate
$$\int_0^1 \int_0^{1-x} y \log(1-x-y) dydx,$$
using the transformation $x = u - uv$, $y = uv$.

11. Show that if $\Omega = \{(x,y) : y \geq 0,\ x^2 + y^2 \leq 1\}$,
$$\int\int_{\Omega} \frac{(x+y)^2}{\sqrt{1+x^2+y^2}} dxdy = \frac{2-\sqrt{2}}{3}\pi.$$

12. Find the volume of the region in $\mathbb{R}^3$ which is above the xy-plane, under the paraboloid $z = x^2 + y^2$, and inside the elliptic cylinder $\frac{x^2}{9} + \frac{y^2}{4} = 1$. *Hint.* Use elliptical coordinates $x = 3r\cos\theta$, $y = 2r\sin\theta$.

13. Use the transformation in $\mathbb{R}^2$ given by $x = r\cos^3\theta$, $y = r\sin^3\theta$ to prove that the volume of the set $\Omega = \left\{(x,y,z) : x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} \leq 1\right\}$ is $\frac{4\pi}{35}$.

14. Let Ω be the region in the octant with $x, y, z \geq 0$, which is bounded by the plane $x = y + z = 1$. Use the change of variables $x = u(1-v)$, $y = uv(1-w)$ $z = uvw$ to compute the integral
$$\int\int\int_{\Omega} \frac{1}{y+z} dxdydz.$$

15. Evaluate the integral
$$\int\int\int_{\Omega} \frac{z}{1+x^2+y^2} dxdydz,$$
where $\Omega = \left\{(x,y,z) : 1 \leq x^2 + y^2 \leq 3, x \geq 0, x \leq y, 1 \leq z \leq 5\right\}$.
Hint. Use cylindrical coordinates.

16. Integrate the function $f(x,y,z) = z^4$ over the ball in $\mathbb{R}^3$ centered at the origin with radius $a > 0$.

17. Evaluate
$$\int\int\int_{\Omega} \frac{1}{\sqrt{x^2+y^2+(z-2)^2}} dxdydz,$$
where Ω is the unit ball $x^2+y^2+z^2 \le 1$.

18. Evaluate
$$\int\int\int_{\Omega} \frac{yz}{1+x} dxdydz,$$
where Ω is the portion of the closed unit ball in $\mathbb{R}^3$ which lies in the positive octant $x, y, z \ge 0$.

19. Evaluate
$$\int\int\int_{\Omega} \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} dxdydz,$$
where Ω is the solid bounded by the two spheres $x^2+y^2+z^2 = a^2$ and $x^2+y^2+z^2 = b^2$, where $0 < b < a$.

Answers to selected Exercises

1. $\frac{8}{15}$. 2. $\frac{1}{6(n+4)}$. 3. $\frac{1}{6}[\sqrt{2} + \log(1+\sqrt{2})])$.

5. 0. 6. $\frac{\pi}{16}$. 7. $\frac{38\pi}{3}$. 8. $\frac{\pi^2}{6}$. 9. $\frac{1}{5}(e^3 - e)$. 10. $-\frac{11}{36}$.

14. $\frac{1}{2}$. 15. $\frac{3}{2}\pi \log 2$. 16. $\frac{4\pi a^7}{35}$. 17. $\frac{2\pi}{3}$. 18. $\frac{19-24\log 2}{36}$.

19. $4\pi \log(\frac{a}{b})$.

5.2 Improper multiple integrals

The situation we encounter here is as follows: we are given a function f defined on a set $\Omega \subseteq \mathbb{R}^n$ where f may not be integrable on Ω according to Definition 4.1.37, either because Ω is *unbounded* or because f is *unbounded* on Ω. Integrals over unbounded regions or integrals of unbounded functions are refered to as *improper integrals*. It is useful and often necessary to be able to integrate over unbounded regions in $\mathbb{R}^n$ or to integrate unbounded functions. As might be expected, by analogy with improper integrals of functions of a single variable, this involves a process of taking a limit of a (proper) multiple integral. However the process of defining improper integrals in dimension $n > 1$ is trickier than in dimension $n = 1$, (this is due to the *great variety* of ways in which a limit can be formed in $\mathbb{R}^n$). In this connection we have the following.

Definition 5.2.1. Let Ω be a set in $\mathbb{R}^n$ (possibly unbounded). An *exhaustion* of Ω is a sequence of bounded simple sets $\{\Omega_k\}$ such that $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset ... \subset \Omega$ with $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$.

Lemma 5.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded simple set and let ν be n-dimensional volume. If $\{\Omega_k\}$ is an exhaustion of Ω, then*

$$\lim_{k\to\infty} \nu(\Omega_k) = \nu(\Omega).$$

Proof. Clearly $\Omega_k \subset \Omega_{k+1} \subset \Omega$ implies that $\nu(\Omega_k) \subset \nu(\Omega_{k+1}) \subset \nu(\Omega)$ and $\lim_{k\to\infty} \nu(\Omega_k) \leq \nu(\Omega)$. To get equality, let $\epsilon > 0$. Note that since $\nu(\partial(\Omega)) = 0$, we can cover $\partial(\Omega)$ by a finite number of open rectangles $R_1, ..., R_N$ of total volume less than ϵ. Let $E = \bigcup_{j=1}^{N} R_j$. Then the set $\Omega \cup E$ is open in $\mathbb{R}^n$ and by construction it contains $\overline{\Omega}$. At the same time, $\nu(\Omega \cup E) \leq \nu(\Omega) + \nu(E) < \nu(\Omega) + \epsilon$.

For each $k = 1, 2, ...$, applying this construction to each set Ω_k of the exhaustion $\{\Omega_k\}$ with $\epsilon_k = \frac{\epsilon}{2^k}$, we obtain a sequence of open sets $\{\Omega_k \cup E_k\}$ such that $\Omega = \bigcup_{k=1}^{\infty} \Omega_k \subset \bigcup_{k=1}^{\infty}(\Omega_k \cup E_k)$, and $\nu(\Omega_k \cup E_k) \leq \nu(\Omega_k) + \nu(E_k) < \nu(\Omega_k) + \epsilon_k$.

Now the collection of open sets $\{E, \Omega_k \cup E_k : k = 1, 2, ...\}$ is an open cover of the compact set $\overline{\Omega}$. So that there is a finite number of these open sets, say, $E, (\Omega_1 \cup E_1), ..., (\Omega_m \cup E_m)$ covering $\overline{\Omega}$. Since $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_m$, the sets $E, E_1, E_2, ..., E_m, \Omega_m$ also form an open cover for $\overline{\Omega}$. Hence,

$$\nu(\Omega) \leq \nu(\overline{\Omega}) \leq \nu(\Omega_m) + \nu(E) + \sum_{i=1}^{m} \nu(E_i) < \nu(\Omega_m) + 2\epsilon.$$

Therefore, $\nu(\Omega) \leq \lim_{k\to\infty} \nu(\Omega_k)$. ∎

We begin by defining the improper integral of a *non-negative* function. Let Ω be a (possibly unbounded) set in $\mathbb{R}^n$ and $\{\Omega_k\}$ an exhaustion of Ω. Suppose $f : \Omega \to \mathbb{R}$ is a non-negative (possibly unbounded) function such that f is integrable over the sets $\Omega_k \in \{\Omega_k\}$. Then the integrals $s_k = \int_{\Omega_k} f$ exist for all $k \in \mathbb{N}$ and by the monotonicity of the integral they form an increasing sequence of real numbers $\{s_k\}$. If this sequence is bounded by Theorem 1.1.20 we get a finite limit

$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \int_{\Omega_k} f < \infty,$$

otherwise it diverges to ∞. Hence the following definition is a natural consequence.

Definition 5.2.3. Let $\{\Omega_k\}$ be an exhaustion of the set $\Omega \subseteq \mathbb{R}^n$ and suppose $f : \Omega \to [0, \infty)$ is integrable over each Ω_k. If the limit

$$\int_{\Omega} f = \lim_{k\to\infty} \int_{\Omega_k} f \tag{5.15}$$

exists and has a value independent of the choice of the sets in the exhaustion of Ω, this limt is called the *improper integral of f over* Ω. When this limit is *finite*, we say that the integral *converges* and f is *integrable* over Ω. If there is no common limit for all exhaustions of Ω or its value is $+\infty$, we say the integral *diverges* and f is *not integrable* over Ω.

Definition 5.2.3 extends the concept of multiple integral to the case of an unbounded region of integration or an unbounded integrand. This calls for the following remark.

Remark 5.2.4. If Ω is a bounded simple set in $\mathbb{R}^n$ and f is integrable over Ω, then the integral over Ω in the sense of Definition 5.2.3 converges and has the same value as the (proper) integral $\int_{\Omega} f$ of Definition 4.1.37.

Indeed, that $f|_{\Omega_k}$ is integrable over Ω_k follows from the Lebesgue criterion (as in Theorem 4.2.1 (6)). Since f is integrable over Ω, it is bounded. Hence $|f(x)| \leq M$ for all $x \in \Omega$, for some $M > 0$. From the additivity of the integral we have

$$\left| \int_{\Omega} f - \int_{\Omega_k} f \right| = \left| \int_{\Omega \backslash \Omega_k} f \right| \leq M\nu(\Omega \backslash \Omega_k). \tag{5.16}$$

Now, since $\nu(\Omega \backslash \Omega_k) = \nu(\Omega) - \nu(\Omega_k)$, letting $k \to \infty$ in (5.16) and using Lemma 5.2.2 we get

$$\int_{\Omega} f = \lim_{k\to\infty} \int_{\Omega_k} f,$$

and the two definitions agree.

The verification that an improper integral converges is simplified by the following proposition. We prove that for a non-negative function the existence of the limit (5.15) is independent of the choice of the exhaustion $\{\Omega_k\}$ of Ω.

Proposition 5.2.5. *Let Ω be a set in $\mathbb{R}^n$ and $f : \Omega \to [0, \infty)$. Let $\{\Omega_k\}$ and $\{\Omega'_m\}$ be exhaustions of Ω. Suppose f is integrable over each Ω_k and each Ω'_m. Then*

$$\lim_{k\to\infty} \int_{\Omega_k} f = \lim_{m\to\infty} \int_{\Omega'_m} f,$$

where the limit may be finite or $+\infty$.

Proof. Let $s = \lim_{k\to\infty} \int_{\Omega_k} f$ be finite. For each fixed $m = 1, 2, \ldots$, the sets $G_{km} = \Omega_k \cap \Omega'_m$, $k = 1, 2, \ldots$ form an exhaustion of the set Ω'_m. Since each Ω'_m is a bounded simple set and f is integrable over Ω'_m it follows from Remark 5.2.4 that

$$\int_{\Omega'_m} f = \lim_{k\to\infty} \int_{G_{km}} f \leq \lim_{k\to\infty} \int_{\Omega_k} f = s.$$

Since $f \geq 0$ and $\Omega'_m \subset \Omega'_{m+1} \subset \Omega$, it follows that $\lim_{m\to\infty} \int_{\Omega'_m} f$ exists and

$$\lim_{m\to\infty} \int_{\Omega'_m} f = s' \leq s.$$

Reversing the roles of the exhaustions shows that $s \leq s'$ also. Thus $s = s'$. If $s = +\infty$, then arguing as in the above paragraph, we have $s \leq s'$ and so $s' = +\infty$ also. □

Example 5.2.6. 1. If $\Omega = \mathbb{R}^n$ and $f : \mathbb{R}^n \to [0, \infty)$ is continuous (or essentially continuous), we can take for the exhaustion to be

$$\Omega_k = B_k = \{x \in \mathbb{R}^n : ||x|| \leq k\},$$

the ball of radius k centered at the origin. Another choice can be

$$\Omega_k = C_k = [-k, k] \times [-k, k] \times \cdots \times [-k, k],$$

the cube of side length $2k$ centered at the origin.

2. If $\Omega = B^n$ the unit ball in $\mathbb{R}^n$ and $f : B^n \to [0, \infty)$ is continuous (or essentially continuous) but $f(x) \to \infty$ as $x \to 0$, we can take Ω_k to be the spherical shells

$$\Omega_k = \left\{ x \in B^n : \frac{1}{k} \leq ||x|| \leq 1 \right\}.$$

Note here that the union of the Ω_k's is $B^n \setminus \{0\}$, but this is immaterial, since, as we know, omission of a single point or even a set of volume zero from a region of integration does not effect the value of an integral.

Our first example of improper double integrals is a classic calculation that leads to the following important integral.

Example 5.2.7. (*Euler-Poisson integral*).

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \tag{5.17}$$

Solution. Exhausting the plane $\mathbb{R}^2$ by a sequence of disks

$$B_k = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 < k \right\}$$

and recalling Example 5.1.22 we have

$$\int\int_{\mathbb{R}^2} e^{-(x^2+y^2)} dxdy = \pi.$$

If we now consider the exhaustion of the plane by the squares $C_k = [-k, k] \times [-k, k]$, then by Fubini's theorem and the fact that $e^{a+b} = e^a e^b$ we have

$$\int\int_{C_k} e^{-(x^2+y^2)} dxdy = \int_{-k}^{k} \left(e^{-x^2} \int_{-k}^{k} e^{-y^2} dy \right) dx$$

$$= \left(\int_{-k}^{k} e^{-y^2} dy\right) \left(\int_{-k}^{k} e^{-x^2} dx\right) = \left(\int_{-k}^{k} e^{-x^2} dx\right)^2.$$

Letting $k \to \infty$, Proposition 5.2.5 tells us that

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \pi.$$

Since any exponential is positive, taking the positive square root of both sides yields the desired integral.

In particular,

$$\int \cdots \int_{\mathbb{R}^n} e^{-(x_1^2+\ldots+x_n^2)} dx_1 \cdots dx_n = \int_{\mathbb{R}^n} e^{-||x||^2} dx = (\pi)^{\frac{n}{2}}.$$

The Euler-Poisson integral is inaccessible by one variable calculus (the antiderivative of e^{-x^2} is not an elementary function). The function e^{-x^2} is known as the *Gaussian*[7] function and comes up in many contexts. Its graph is the "bell-shaped curve". Rescaling it by the factor $\frac{1}{\sqrt{\pi}}$, so that the total area under its graph is 1, gives the *normal distribution* in probability and statistics.

Corollary 5.2.8. *(Comparison test). Let f and g be functions defined on the set $\Omega \subseteq \mathbb{R}^n$ and integrable over exactly the same bounded simple subsets of Ω. Suppose $0 \leq f(x) \leq g(x)$ for all $x \in \Omega$. If the improper integral $\int_\Omega g$ converges, then the integral $\int_\Omega f$ also converges.*

Proof. Let $\{\Omega_k\}$ be an exhaustion of Ω on whose elements both f and g are integrable. By the monotonicity of the integral (Theorem 4.2.1 (2)) we have $\int_{\Omega_k} f \leq \int_{\Omega_k} g$ for all $k = 1, 2, \ldots$. Since the functions are non-negative, letting $k \to \infty$

$$\int_\Omega f \leq \int_\Omega g < \infty.$$

□

[7] C. Gauss (1777-1855). One of the great mathematicians with numerous pioneering contributions in several fields in mathematics, physics and astronomy.

The basic properties of multiple integrals stated in Theorem 4.2.1 are readily extended to improper integrals of non-negative functions using Definition 5.2.3. Thus far we worked with non-negative functions. We shall now define the improper integral for a function $f : \Omega \subseteq \mathbb{R}^n \to (-\infty, \infty)$. The essential point is that the precceding theory can be applied to $|f|$, so that it makes sense to say $\int_\Omega |f|$ converges. Before we go on we need to define the so-called positive and negative parts of f.

Definition 5.2.9. (The functions f^+, f^-).
Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. For $x \in \Omega$ we define the *positive part* f^+ of f by

$$f^+(x) = \max\{f(x), 0\} = \frac{|f(x)| + f(x)}{2}$$

and the *negative part* f^- of f by

$$f^-(x) = \max\{-f(x), 0\} = \frac{|f(x)| - f(x)}{2}.$$

Lemma 5.2.10. *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. Then $f^+ \geq 0$, $f^- \geq 0$, $f^- = (-f)^+$ and*

$$f = f^+ - f^-.$$

Furthermore, $|f| = \max\{f, -f\}$ *and*

$$|f| = f^+ + f^-.$$

The proof of the lemma is left to the reader as an exercise.

Definition 5.2.11. Let Ω be a set in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$. We say f is *absolutely integrable* if the integral $\int_\Omega |f|$ converges.

Proposition 5.2.12. *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$. If $|f|$ is integrable over Ω, then f is integrable.*

Proof. Since $0 \leq f^+ \leq |f|$ and $0 \leq f^- \leq |f|$, the comparison test tells us that both $\int_\Omega f^+$ and $\int_\Omega f^-$ converge. Thus $\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^-$ also converges and f is integrable. □

Next we prove the change of variable formula for improper integrals. We begin with a preliminary result, which gives a useful exhaustion of any open set in $\mathbb{R}^n$ in terms of compact simple sets.