Pseudo-Poincaré Inequalities and Applications to Sobolev Inequalities

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Abstract Most smoothing procedures are via averaging. Pseudo-Poincaré inequalities give a basic L^p-norm control of such smoothing procedures in terms of the gradient of the function involved. When available, pseudo-Poincaré inequalities are an efficient way to prove Sobolev type inequalities. We review this technique and its applications in various geometric setups.

1 Introduction

This paper is concerned with the question of proving the Sobolev inequality

$$\forall\, f \in \mathcal{C}_c^\infty(M), \quad \|f\|_q \leqslant S(M,p,q)\|\nabla f\|_p \tag{1.1}$$

when $M = (M, g)$ is a Riemannian manifold, perhaps with boundary ∂M, and $\mathcal{C}_c^\infty(M)$ is the space of smooth compactly supported functions on M (if M is a manifold with boundary ∂M, then points on ∂M are interior points in M and functions in $\mathcal{C}_c(M)$ do not have to vanish at such points). We say that (M, g) is *complete* when M equipped with the Riemannian distance is a complete metric space.

In (1.1), $p, q \in [1,\infty)$ and $q > p$. The norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are computed with respect to some fixed reference measure, perhaps the Riemannian measure dv on M or, more generally, a measure $d\mu$ on M of the form $d\mu = \sigma dv$, where σ is a smooth positive function on M. We set $V(x,r) = \mu(B(x,r))$, where $B(x,r)$ is the geodesic ball of center $x \in M$ and radius $r \geqslant 0$. The gradient ∇f of $f \in \mathcal{C}^\infty(M)$ at x is the tangent vector at x defined by

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$$g_x(\nabla f(x), u) = df|_x\,(u)$$

for any tangent vector $u \in T_x$. Its length $|\nabla f|$ is given by $|\nabla f|^2 = g(\nabla f, \nabla f)$.

We will not be concerned here with the (interesting) problem of finding the best constant $S(M, p, q)$ but only with the validity of the Sobolev inequality (1.1), for some constant $S(M, p, q)$.

In $\mathbb{R}^n$, equipped with the Lebesgue measure dx, (1.1) holds for any $p \in [1, n)$ with $q = np/(n-p)$. The two simplest contexts where the question of the validity of (1.1) is meaningful is when $M = \Omega$ is a subset of $\mathbb{R}^n$, or when $\mathbb{R}^n$ is equipped with a measure $\mu(dx) = \sigma(x)dx$. In the former case, it is natural to relax our basic assumption and allow domains with nonsmooth boundary. It then becomes important to pay more attention to the exact domain of validity of (1.1) as approximation by functions that are smooth up to the boundary may not be available (cf., for example, [13, 14]).

The fundamental importance of the inequality (1.1) in analysis and geometry is well established. It is beautifully illustrated in the work of V. Maz'ya. One of the fundamental references on Sobolev inequalities is Maz'ya's treaty "Sobolev Spaces" [13] which discuss (1.1) and its many variants in $\mathbb{R}^n$ and in domains in $\mathbb{R}^n$ (cf. also [1, 3, 14] and the references therein). Maz'ya's treaty anticipates on many later works including [2]. More specialized works that discuss (1.1) in the context of Riemannian manifolds and Lie groups include [11, 19, 23] among many other possible references.

The aim of this article is to discuss a particular approach to (1.1) that is based on the notion of pseudo-Poincaré inequality. This technique is elementary in nature and quite versatile. It seems it has its origin in [4, 7, 17, 18] and was really emphasized first in [7, 18], and in [2]. To put things in some perspective, recall that the most obvious approach to (1.1) is via some "representation formula" that allows us to "recover" f from its gradient through an integral transform. One is them led to study the mapping properties of the integral transform in question.

However, this natural approach is not well suited to many interesting geometric setups because the needed properties of the relevant integral transforms might be difficult to establish or might even not hold true. For instance, its seems hard to use this approach to prove the following three (well-known) fundamental results.

Theorem 1.1. *Assume that* (M, g) *is a Riemannian manifold of dimension* n *equipped with its Riemannian measure and which is of one of the following three types*:

1. *A connected simply connected noncompact unimodular Lie group equipped with a left-invariant Riemannian structure.*
2. *A complete simply connected Riemannian manifold without boundary with nonpositive sectional curvature* (*i.e., a Cartan–Hadamard manifold*)*.*
3. *A complete Riemannian manifold without boundary with nonnegative Ricci curvature and maximal volume growth.*

Then for any $p \in [1,n)$ the Sobolev inequality (1.1) *holds on M with $q = np/(n-p)$ for some constant $S(M,p,q) < \infty$.*

One remarkable thing about this theorem is the conflicting nature of the curvature assumptions made in the different cases. Connected Lie groups almost always have curvature that varies in sign, whereas the second and third cases we make opposite curvature assumptions. Not surprisingly, the original proofs of these different results have rather distinct flavors.

The result concerning unimodular Lie groups is due to Varopoulos and more is true in this case (cf. [22, 23]).

The result concerning Cartan–Hadamard manifolds is a consequence of a more general result due to Michael and Simon [15] and Hoffmann and Spruck [12]. A more direct prove was given by Croke [9] (cf. also [11, Section 8.1 and 8.2] for a discussion and further references).

The result concerning manifolds with nonnegative Ricci curvature and maximal volume growth (i.e., $V(x,r) \geqslant cr^n$ for some $c > 0$ and all $x \in M, r > 0$) was first obtained as a consequence of the Li-Yau heat kernel estimate using the line of reasoning in [22].

One of the aims of this paper is to describe proofs of these three results that are based on a common unifying idea, namely, the use of what we call pseudo-Poincaré inequalities. Our focus will be on how to prove the desired pseudo-Poincaré inequalities in the different contexts covered by this theorem. For relevant background on geodesic coordinates and Riemannian geometry see [5, 6, 10].

2 Sobolev Inequality and Volume Growth

There are many necessary conditions for (1.1) to hold and some are discussed in Maz'ya's treaty [13] in the context of Euclidean domains. For instance, if (1.1) holds for some fixed $p = p_0 \in [1, \infty)$ and $q = q_0 > p_0$ and we define m by $1/q_0 = 1/p_0 - 1/m$, then (1.1) also holds for all $p \in [p_0, m)$ with q given by $1/q = 1/p - 1/m$ (this easily follows by applying the p_0, q_0 inequality to $|f|^\alpha$ with a properly chosen $\alpha > 1$ and using the Hölder inequality). More importantly to us here is the following result (cf., for example, [2] or [19, Corollary 3.2.8]).

Theorem 2.1. *Let (M,g) be a complete Riemannian manifold equipped with a measure $d\mu = \sigma dv$, $0 < \sigma \in \mathcal{C}^\infty(M)$. Assume that* (1.1) *holds for some $1 \leqslant p < q < \infty$ and set $1/q = 1/p - 1/m$. Then for any $r \in (m, \infty)$ and any bounded open set $U \subset M$*

$$\forall\, f \in \mathcal{C}_c^\infty(U), \quad \|f\|_\infty \leqslant C_r \mu(U)^{1/m - 1/r} \|\nabla f\|_r. \tag{2.1}$$

Corollary 2.1. *If the complete Riemannian manifold* (M, g) *equipped with a measure* $d\mu = \sigma dv$ *satisfies* (1.1) *for some* $1 \leqslant p < q < \infty$*, then*

$$\inf\{s^{-m}V(x,s) : x \in M,\ s > 0\} > 0$$

with $1/q = 1/p - 1/m$.

Proof. Fix $r > m$ and apply (2.1) to the function

$$\phi_{x,s}(y) = y \mapsto (s - \rho(x,y))_+ = \max\{(s - \rho(x,y), 0\},$$

where ρ is the Riemannian distance on (M, g). Because (M, ρ) is complete, this function is compactly supported and can be approximated by smooth compactly supported functions in the norm $\|f\|_\infty + \|\nabla f\|_r$, justifying the use of (2.1). Moreover, $|\nabla\phi_{x,s}| \leqslant 1$ a.e. so that $\|\nabla\phi_{x,s}\|_r \leqslant V(x,r)^{1/r}$. This yields $s \leqslant C_r V(x,s)^{1/m-1/r}V(x,s)^{1/r} = C_r V(x,s)^{1/m}$ as desired. □

Remark 2.1. Let Ω be an unbounded Euclidean domain.

(a) If we assume that (1.1) holds but only for all traces $f|_\Omega$ of functions $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, then we can conclude that (2.1) holds for such functions. Applying (2.1) to $\psi_{x,s}(y) = (s - \|x - y\|)_+$, $x \in \Omega$, $s > 0$, yields

$$|\{z \in \Omega : \|x - z\| < s\}| \geqslant cs^m.$$

(b) If, instead, we consider the intrinsic geodesic distance $\rho = \rho_\Omega$ in Ω and assume that (1.1) holds for all ρ-Lipschitz functions vanishing outside some ρ-ball, then the same argument, properly adapted, yields $V(x,s) \geqslant cs^m$, where $V(x,s)$ is the Lebesgue measure of the ρ-ball of radius s around x in Ω.

For domains with rough boundary, the hypotheses made respectively in (a) and (b) may be very different.

3 The Pseudo-Poincaré Approach to Sobolev Inequalities

Our aim is to illustrate the following result which provide one of the most elementary and versatile ways to prove a Sobolev inequality in a variety of contexts (cf., for example, [2, Theorem 9.1]). The two main hypotheses in the following statement concern a family of linear operators A_r acting, say, on smooth compactly supported functions. The first hypothesis captures the idea that A_r is smoothing. The sup-norm of $A_r f$ is controlled in terms of the L^p-norm of f only and tends to 0 as r tends to infinity. The second hypothesis implies, in particular, that $A_r f$ is close to f if $|\nabla f|$ is in L^p and r is small.

Theorem 3.1. *Fix $m, p \geqslant 1$. Assume that for each $r > 0$ there is a linear map $A_r : \mathcal{C}_c^\infty(M) \to L^\infty(M)$ such that*

- $\forall f \in \mathcal{C}_c^\infty(M), r > 0,\ \|A_r f\|_\infty \leqslant C_1 r^{-m/p}\|f\|_p$.
- $\forall f \in \mathcal{C}_c^\infty(M), r > 0,\ \|f - A_r f\|_p \leqslant C_2 r\|\nabla f\|_p$.

Then, if $p \in [1, m)$ and $q = mp/(m-p)$, there exists a finite constant $S(M,p,q) = C(p,q)C_2 C_1^{1/m}$ such that the Sobolev inequality (1.1) *holds on M.*

Outline of the proof. The proof is entirely elementary and is given in [2]. For illustrative purpose and completeness, we explain the first step. Consider the distribution function of $|f|$, $F(s) = \mu(\{x : |f(x)| > s\})$. Then

$$F(s) \leqslant \mu(\{|f - A_r f| > s/2\}) + \mu(\{|A_r f| > s/2\}).$$

By hypothesis, if $s = 2C_1 r^{-m/p}\|f\|_p$, then $\mu(\{|A_r f| > s/2\}) = 0$ and

$$F(s) \leqslant \mu(\{|f - A_r f| > s/2\}) \leqslant 2^p C_2^p r^p s^{-p}\|\nabla f\|_p^p.$$

This gives

$$s^{p(1+1/m)}F(s) \leqslant 2^{p(1+1/m)}C_1^{p/m}C_2^p\|\nabla f\|_p^p\|f\|_p^{p/m}.$$

This is a weak form of the desired Sobolev inequality (1.1). But, as is already apparent in [13], such a weak form of (1.1) actually imply (1.1) (cf. also [2, 19]). □

Remark 3.1. When $p = 1$ and $\mu = v$ is the Riemannian volume, we get

$$s^{1+1/m}v(\{|f| > s\}) \leqslant 2^{1+1/m}C_1^{1/m}C_2\|\nabla f\|_1\|f\|_1^{1/m}.$$

For any bounded open set Ω with smooth boundary $\partial\Omega$ we can find a sequence of functions $f_n \in \mathcal{C}_c^\infty(M)$ such that $f_n \to \mathbf{1}_{\Omega_n}$ and $\|\nabla f_n\|_1 \to v_{n-1}(\partial\Omega)$. This yields the isoperimetric inequality

$$v(\Omega)^{1-1/m} \leqslant 2^{1+1/m}C_1^{1/m}C_2 v_{n-1}(\partial\Omega).$$

Of course, as was observed long ago by Maz'ya and others, the classical co-area formula and the above inequality imply

$$\forall f \in \mathcal{C}_c^\infty(M), \quad \|f\|_{m/(m-1)} \leqslant 2^{1+1/m}C_1^{1/m}C_2\|\nabla f\|_1.$$

There are many situations where one does not expect (1.1) to hold, but where one of the local versions

$$\forall f \in \mathcal{C}_c^\infty(M), \quad \|f\|_q \leqslant S(M,p,q)(\|\nabla f\|_p + \|f\|_p), \tag{3.1}$$

or ($\Subset$ indicates an open relatively compact inclusion)

$$\forall\, \Omega \Subset M,\ \ \forall\, f \in \mathcal{C}_c^\infty(\Omega), \quad \|f\|_q \leqslant S(\Omega, p, q)(\|\nabla f\|_p + \|f\|_p) \tag{3.2}$$

may hold. This is handled by the following local version of Theorem 3.1 (cf. [2] and [19, Section 3.3.2]).

Theorem 3.2. *Fix an open subset $\Omega \subset M$. Assume that for each $r \in (0, R)$ there is a linear map $A_r : \mathcal{C}_c^\infty(\Omega) \to L^\infty(M)$ such that*

- $\forall\, f \in \mathcal{C}_c^\infty(\Omega), r \in (0, R),\ \|A_r f\|_\infty \leqslant C_1 r^{-m/p}\|f\|_p.$
- $\forall\, f \in \mathcal{C}_c^\infty(\Omega), r \in (0, R),\ \|f - A_r f\|_p \leqslant C_2 r \|\nabla f\|_p.$

Then, if $p \in [1, m)$ and $q = mp/(m - p)$, there exists a finite constant $S = S(p, q)$ such that

$$\forall\, f \in \mathcal{C}_c^\infty(\Omega), \quad \|f\|_q \leqslant S C_1^{1/m}(C_2\|\nabla f\|_p + R^{-1}\|f\|_p). \tag{3.3}$$

Another useful version is as follows. For any open set Ω we let $W^{1,p}(\Omega)$ be the space of those functions in $L^p(\Omega)$ whose first order partial derivatives in the sense of distributions (in any local chart) can be represented by a locally integrable function and such that

$$\int_\Omega |\nabla f|^p dv < \infty.$$

We write $\|f\|_{\Omega,p}$ for the L^p-norm of f over Ω. Note that $\mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for $1 \leqslant p < \infty$ (cf., for example, [1, 3, 13]).

Theorem 3.3. *Fix an open subset $\Omega \subset M$. Assume that for each $r \in (0, R)$ there is a linear map $A_r : \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega) \to L^\infty(M)$ such that*

- $\forall\, f \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega), r \in (0, R),\ \|A_r f\|_\infty \leqslant C_1 r^{-m/p}\|f\|_p.$
- $\forall\, f \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega), r \in (0, R),\ \|f - A_r f\|_p \leqslant C_2 r \|\nabla f\|_p.$

Then, if $p \in [1, m)$ and $q = mp/(m - p)$, there exists a finite constant $S = S(p, q)$ such that

$$\forall\, f \in W^{1,p}(\Omega), \quad \|f\|_q \leqslant S C_1^{1/m}(C_2\|\nabla f\|_p + R^{-1}\|f\|_p). \tag{3.4}$$

4 Pseudo-Poincaré Inequalities

The term Poincaré inequality (say, with respect to a bounded domain $\Omega \subset M$) is used with at least two distinct meanings:

- The Neumann type L^p-Poincaré inequality for a bounded domain $\Omega \subset M$ is the inequality

$$\forall f \in W^{1,p}(\Omega), \ \inf_{\xi \in \mathbb{R}} \int_\Omega |f-\xi|^p dv \leqslant P_N(\Omega) \int_\Omega |\nabla f|^p dv.$$

- The Dirichlet type L^p-Poincaré inequality for a bounded domain $\Omega \subset M$ is the inequality

$$\forall f \in \mathcal{C}_c^\infty(\Omega), \ \int_\Omega |f|^p dv \leqslant P_D(\Omega) \int_\Omega |\nabla f|^p dv.$$

When $p=2$ and the boundary is smooth, the first (respectively, the second) inequality is equivalent to the statement that the lowest nonzero eigenvalue $\lambda_N(\Omega)$ (respectively, $\lambda_D(\Omega)$) of the Laplacian with the Neumann boundary condition (respectively, the Dirichlet boundary condition) is bounded below by $1/P_N(\Omega)$ (respectively, $1/P_D(\Omega)$). Note that if $M=\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$ and $\Omega = B(o,r)$, $r<2\pi$, is a geodesic ball, then $P_N(\Omega) \to 1/(n+1)$ and $P_D(\Omega) \to \infty$ as r tends to 2π.

Here, we will use the term Poincaré inequality for the collection of the Neumann type Poincaré inequalities on metric balls. More precisely, we say that the *L^p-Poincaré inequality* holds on the manifold M if there exists $P \in (0,\infty)$ such that

$$\forall B = B(x,r), \ \forall f \in W^{1,p}(B), \ \inf_{\xi \in \mathbb{R}} \int_B |f-\xi|^p dv \leqslant Pr^p \int_B |\nabla f|^p dv. \quad (4.1)$$

The notion of pseudo-Poincaré inequality was introduced in [7, 18] to describe the inequality

$$\forall f \in \mathcal{C}_c^\infty(M), \ \|f-f_r\|_p \leqslant Cr\|\nabla f\|_p, \quad (4.2)$$

where

$$f_r(x) = V(x,r)^{-1} \int_{B(x,r)} f dv.$$

Although this looks like a version of the previous Poincaré inequality, it is quite different in several respects. The most important difference is the global nature of each of the members of the pseudo-Poincaré inequality family: in (4.2) all integrals are over the whole space.

We say the *doubling volume condition* holds on M if there exists $D \in (0,\infty)$ such that

$$\forall x \in M, \ r>0, \ V(x,2r) \leqslant DV(x,r). \quad (4.3)$$

The only known strong relation between (4.1) and (4.2) is the following result from [8, 18].

Theorem 4.1. *If a complete manifold M equipped with a measure $d\mu = \sigma dv$ satisfies the conjunction of* (4.3) *and* (4.1), *then the pseudo-Poincaré inequality* (4.2) *holds on M.*

The most compelling reason for introducing the notion of pseudo-Poincaré inequality is that unimodular Lie groups always satisfy (4.2) with $C = 1$ (cf. [22] and the development in [7]). The proof is extremely simple and the result slightly stronger.

Theorem 4.2. *Let G be a connected unimodular Lie group equipped with a left-invariant Riemannian distance and Haar measure. For any group element y at distance $r(y)$ from the identity element e*

$$\forall\, f \in \mathcal{C}_c(G), \quad \|f - f_y\|_p \leqslant r(y)\|\nabla f\|_p,$$

where $f_y(x) = f(xy)$.

Proof. Indeed, let $\gamma_y : [0, r(y)] \to G$ be a (unit speed) geodesic joining e to y. Thus,

$$|f(x) - f(xy)|^p \leqslant r(y)^{p-1} \int_0^{r(y)} |\nabla f(x\gamma_y(s))|^p ds.$$

Integrating over $x \in G$ yields the desired result. □

With this simple observation and Theorem 3.1, we immediately find that any simply connected noncompact unimodular Lie group M of dimension n satisfies the Sobolev inequality

$$\|f\|_{np/(n-p)} \leqslant S(M,p)\|\nabla f\|_p.$$

This is because the volume growth function $V(x,r) = V(r)$ is always faster than cr^n (cf. [23] and the references therein). In fact, for $r \in (0,1)$, we obviously have $V(r) \simeq r^n$ and, for $r > 1$, either $V(r) \simeq r^N$ for some integer $N \geqslant n$ or $V(r)$ grows exponentially fast. This line of reasoning yields the following improved result (due to Varopoulos [22], with a different proof).

Theorem 4.3. *Let G be a connected unimodular Lie group equipped with a left-invariant Riemannian structure and Haar measure. If the volume $V(r)$ of the balls of radius r in G satisfies $V(r) \geqslant cr^m$ for some $m > 0$ and all $r > 0$, then* (1.1) *holds on G for all $p \in [1,m]$ and $q = mp/(m-p)$.*

In this article, we think of a pseudo-Poincaré inequality as an inequality of the more general form

$$\forall\, f \in \mathcal{C}_c^\infty(M), \;\; \|f - A_r f\|_p \leqslant Cr\|\nabla f\|_p, \tag{4.4}$$

where $A_r : \mathcal{C}_c^\infty(M) \to L^\infty(M)$ is a linear operator. It is indeed very useful to replace the ball averages

$$f_r = V(x,r)^{-1} \int_{B(x,r)} f d\mu$$

by other types of averaging procedures. One interesting case is the following instance.

Theorem 4.4. *Let (M, g) be a Riemannian manifold, and let Δ be the Friedrichs extension of the Laplacian defined on smooth compactly supported functions on M. Let $H_t = e^{t\Delta}$ be the associated semigroup of selfadjoint operator on $L^2(M, dv)$ (the minimal heat semigroup on M). Then*

$$\forall\, f \in \mathcal{C}_c^\infty(M), \quad \|f - H_t f\|_2 \leqslant \sqrt{t}\|\nabla f\|_2. \tag{4.5}$$

Consequently, if there are constants $C \in (0, \infty)$, $T \in (0, \infty]$ and $m > 2$ such that

$$\forall\, t \in (0, T), \quad \|H_t f\|_\infty \leqslant C t^{-m/4}\|f\|_2, \tag{4.6}$$

then there exists a constant $S = S(C, m) \in (0, \infty)$ such that the Sobolev inequality

$$\forall\, f \in \mathcal{C}_c^\infty(M), \quad \|f\|_{2m/(m-2)} \leqslant S(\|\nabla f\|_2 + T^{-1}\|f\|_2) \tag{4.7}$$

holds on M.

Proof. In order to apply Theorem 3.2 with $A_r = H_{r^2}$, it suffices to prove (4.5). But

$$H_t f - f = \int_0^t \partial_s H_s f ds$$

and

$$\langle \partial_s H_s f, H_\tau f\rangle = \langle \Delta H_s f, H_\tau f\rangle = -\|H_{(s+\tau)/2}(-\Delta)^{1/2} f\|_2^2 \geqslant -\|\nabla f\|_2^2.$$

Hence $\|H_t f - f\|_2^2 \leqslant t\|\nabla f\|_2^2$ as desired. □

Remark 4.1. One can show that (4.7) and (4.6) are, in fact, equivalent properties. This very important result was first proved by Varopoulos [21]. This equivalence holds in a much greater generality (cf. also [23]). When $m \in (0, 2)$, one can replace (4.7) by the Nash inequality

$$\forall\, f \in \mathcal{C}_c^\infty(M), \quad \|f\|_2^{2(1+2/m)} \leqslant N(\|\nabla f\|_2 + T^{-1}\|f\|_2)\|f\|_1^{4/m}$$

which is equivalent to (4.6) (for any fixed $m > 2$). See, for example, [2, 4, 19, 23] and the references therein.

5 Pseudo-Poincaré Inequalities and the Liouville Measure

Given a complete Riemannian manifold $M = (M, g)$ of dimension n (without boundary), we let $T_x M$ be the tangent space at x, $\mathbb{S}_x \subset T_x M$ the unit sphere, and SM the unit tangent bundle equipped with the Liouville measure defined by

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