On the Regularity of Nonlinear Subelliptic Equations

András Domokos and Juan J. Manfredi

Abstract We prove C^∞ regularity results for Lipschitz solutions of nondegenerate quasilinear subelliptic equations of p-Laplacian type for a class of Hörmander vector fields that include certain nonnilpotent structures.

It is a great privilege to present this contribution in honor of Professor Vladimir Maz'ya. As one of the developers of the modern theory of Sobolev spaces and their applications to partial differential equations, we are all indebted to his dedication and vision. Professor Maz'ya's contributions permeate all areas of mathematical analysis. We have chosen to present some recent developments on the regularity of solutions to the p-Laplace equation in subelliptic structures. Professor Maz'ya has written a number of influential articles on the p-Laplacian, including a seminal article in nonlinear potential theory [15] and the first boundary regularity estimate in terms of p-capacity [13]. In addition, Professor Maz'ya has written several books from which the now classic *Sobolev Spaces* [14] and (with T.O Shaposhnikova) *Theory of multipliers in spaces of differentiable functions* [16] are obligated references that have influenced several generation of analysts.

We have benefited from discussions with Professor Maz'ya in several occasions at West Lafayette, Linköping, Pittsburgh, and Washington. We would like to express our deeper appreciation for his kindness and generosity.

András Domokos
Department of Mathematics and Statistics, California State University Sacramento, Sacramento, 95819, USA e-mail: domokos@csus.edu

Juan J. Manfredi
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA e-mail: manfredi@pitt.edu

1 Introduction

Consider a domain $\Omega \subset \mathbb{R}^n$, $N \leqslant n$, and a Hörmander (or bracket-generating) system of smooth vector fields $\mathfrak{X} = \{X_1, \ldots, X_N\}$ defined on Ω. We denote by $\mathfrak{X}u = (X_1 u, \ldots, X_N u)$ the $\mathfrak{X}$-gradient, or horizontal gradient, of a function u. In this paper, we study second and higher order interior regularity for weak solutions of the following quasilinear subelliptic equation:

$$\sum_{i=1}^{N} X_i^* \left(a_i(x, \mathfrak{X}u)\right) = 0 \ \text{ in } \ \Omega \,, \tag{1.1}$$

where $a_i(x,\xi)$ are differentiable functions on $\Omega \times \mathbb{R}^N$ which for some positive constants c, l and a.e. $(x,\xi) \in \Omega \times \mathbb{R}^N$ and every $\eta \in \mathbb{R}^N$ satisfy the following properties:

$$l^{-1}|\eta|^2 \leqslant \sum_{i,j=1}^{N} \frac{\partial a_i}{\partial \xi_j}(x,\xi)\, \eta_i \eta_j \leqslant l|\eta|^2 \,, \tag{1.2}$$

$$\sum_{i,j=1}^{N} \left| \frac{\partial a_i}{\partial x_j}(x,\xi) \right| \leqslant c\,(1 + |\xi|) \ , \tag{1.3}$$

$$\sum_{i=1}^{N} |a_i(x,\xi)| \leqslant c\,(1 + |\xi|) \ . \tag{1.4}$$

The best known representative of Equation (1.1) is the subelliptic p-Laplacian equation with

$$a_i(x,\xi) = |\xi|^{p-2}\xi_i \,.$$

The assumptions (1.2)–(1.4) are satisfied if we suppose $p \geqslant 2$ and $0 < M^{-1} \leqslant |\xi| \leqslant M$ which for weak solutions u corresponds to the assumption

$$0 < M^{-1} \leqslant |\mathfrak{X}u(x)| \leqslant M \quad \text{a.e. } x \in \Omega \,. \tag{1.5}$$

In the forthcoming paper [10], we will study the quasilinear subelliptic regularity problem without assumption (1.5) and for $p \neq 2$.

As a short overview of previously proved subelliptic interior regularity results we mention Hörmander's C^∞ regularity result in the $a_i(x,\xi) = \xi_i$ linear case [11]. The Hölder continuity of weak solutions for quasilinear equations of the form

$$\sum_{i=1}^{N} X_i^* \left(a_i(x, u, \mathfrak{X}u)\right) = 0 \ \text{ in } \ \Omega \tag{1.6}$$

was proved in [4, 12], and higher regularity for

$$\sum_{i=1}^{N} X_i^* \left(a_i(x,u)X_i u\right) = 0 \quad \text{in} \quad \Omega \tag{1.7}$$

was studied in [21]. These results are valid for any system of Hörmander vector fields. However, the second order differentiability results from [5, 7, 8, 9, 18] are valid just in Heisenberg and Carnot groups and the Moser iterations leading to $C^{1,\alpha}$ regularity developed in [17] just in Heisenberg groups.

In this paper, we introduce the notion of ν-closed systems of Hörmander vector fields, which allows us to study higher order interior regularity in certain nonnilpotent structures. More precisely, this new notion includes all the previously studied nilpotent systems of vector fields including the generators of the Lie Algebra of Carnot groups, the Grušin vector fields, and extends the regularity results to the case of nonnilpotent systems of vector fields, as those generating the Lie Algebra of the rotation group SO (n) or other noncompact semisimple or solvable Lie groups. As it follows from Definition 1.1, we do not even suppose $\mathfrak{X}$ to generate the Lie algebra of a Lie group.

We denote by $\mathfrak{B}$ the set of all commutators of length up to a fixed $\nu \in \mathbb{N}$. Elements of $\mathfrak{X}$ will be considered as commutators of length 1, so we have $\mathfrak{X} \subset \mathfrak{B}$.

Definition 1.1. We say that $\mathfrak{X}$ is a *ν-closed Hörmander system* of vector fields on Ω if

(1) the set $\mathfrak{B}$ of commutators of order at most ν spans the tangent space at every $x \in \Omega$,

(2) there exists a $T_1 \in \mathfrak{B} \setminus \mathfrak{X}$ such that

$$[T_1, X_i] \in \mathfrak{X} \cup \{0\} \quad \text{for all} \quad 1 \leqslant i \leqslant N,$$

(3) if after selecting $\mathfrak{T}_k = \{T_1, \ldots, T_k\}$ we still have $\mathfrak{B} \setminus \{\mathfrak{X} \cup \mathfrak{T}_k\} \neq \varnothing$, then there exists $T_{k+1} \in \mathfrak{B} \setminus \{\mathfrak{X} \cup \mathfrak{T}_k\}$ such that

$$[T_{k+1}, Y] \in \mathfrak{X} \cup \mathfrak{T}_k \cup \{0\} \quad \text{for all} \quad Y \in \mathfrak{X} \cup \mathfrak{T}_k,$$

(4) continuing the process started in (1)-(3), we cover all $\mathfrak{B} \setminus \mathfrak{X}$.

Our starting point in finding this definition was Taylor's description [20] of a subelliptic Laplacian operator in the special unitary group SU (2). For a detailed description of the following examples we refer to [10]. In $\mathbb{R}^3$, we consider linearly independent vector fields X_1, X_2, T satisfying the commutation relations

$$[X_1, X_2] = T, \quad [X_2, T] = X_1, \quad [T, X_1] = X_2. \tag{1.8}$$

In this case, the system $\mathfrak{X} = \{X_1, X_2\}$ generates a three dimensional Lie Algebra which is isomorphic to so(3), the Lie Algebra of SO(3).

This can be immediately generalized to SO(n), which has its Lie algebra so (n) spanned by a basis $\mathfrak{B} = \{X_{jk},\ 1 \leqslant j < k \leqslant n\}$ with nonzero commutation relations

$$[X_{mk}, X_{jk}] = X_{jm}, \quad [X_{jk}, X_{jm}] = X_{mk}, \quad [X_{jm}, X_{mk}] = X_{jk}$$

for all $1 \leqslant j < m < k \leqslant n$. This allows us to consider $\mathfrak{X} = \{X_{mn},\ 1 \leqslant m \leqslant n-1\}$. Then any other vector field X_{jk}, $1 \leqslant j < k \leqslant n-1$, is a commutator of length two of elements of $\mathfrak{X}$ and satisfies the following commutation relations:

$$[X_{jm}, X_{qn}] = \left\{ \begin{array}{cl} 0 & \text{if } j \neq q \text{ and } q \neq m \\ -X_{mn} & \text{if } q = j \\ X_{mn} & \text{if } q = m \end{array} \right\} . \tag{1.9}$$

Hence we can start the selection process of Definition 1.1 with any vector field not belonging to $\mathfrak{X}$ and continue it in an arbitrary order.

As other examples for Definition 1.1 we can mention the 3-dimensional Lie algebras listed in [19]. Among them we find a nilpotent (the Heisenberg Lie Algebra), a compact semisimple (so (3)), a noncompact semisimple (sl $(2, \mathbb{R})$) and a solvable (the Lie algebra of the motion group of the Lorentzian plane).

For $k \in \mathbb{N}$ we define the following subelliptic Sobolev space:

$$\mathfrak{X}W^{k,2}(\Omega) = \Big\{ u \in L^p(\Omega) : X_{i_1} \dots X_{i_k} u \in L^2(\Omega) \ \text{ for all } \ 1 \leqslant i_j \leqslant N \Big\}.$$

Let $\mathfrak{X}W_0^{k,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $\mathfrak{X}W^{k,2}(\Omega)$ with respect to its usual norm.

A function $u \in \mathfrak{X}W_{\text{loc}}^{1,2}(\Omega)$ is a weak solution of Equation (1.1) if

$$\sum_{i=1}^{N} \int_{\Omega} a_i(x, \mathfrak{X}u(x)) \, X_i \varphi(x) dx = 0 \ \text{ for all } \ \varphi \in C_0^\infty(\Omega) \, . \tag{1.10}$$

Our main task is to prove the following theorem.

Theorem 1.1. *Assume that $\mathfrak{X}$ is a ν-closed system of Hörmander vector fields, the functions $a_i(x, \xi)$ are C^∞ in $\Omega \times \mathbb{R}^n$, and a weak solution u of Equation* (1.1) *satisfies* $0 < M^{-1} \leqslant |\mathfrak{X}u| \leqslant M$ *a.e. in Ω. Then $u \in C^\infty(\Omega)$.*

By Theorem 1.1, we generalize similar results obtained in Heisenberg and Carnot groups by Capogna [2, 3] and also show alternate proofs for the second order differentiability results in Theorems 2.1 and 2.2 below.

2 Second Order Horizontal Differentiability of Weak Solutions

We denote by $e^{tT}x$ the flow associated to a vector field T. We follow now a point of view that can be derived from the methods presented in [1, 11]. We think about a vector field X as a linear mapping $X : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ defined by

$$X\phi(x) = \left.\frac{d}{dt}\right|_{t=0} \phi(e^{tX}x) \ .$$

For $\phi \in C_0^\infty(\Omega)$ and $t > 0$ sufficiently small let us define

$$e^{tX}\phi\ (x) = \phi(e^{tX}x) \ .$$

The Taylor series expansion at $t = 0$ gives the following formal power series representation:

$$e^{tX}\phi = \sum_{k=0}^{\infty} \frac{t^k}{k!}\ X^k\phi \ . \tag{2.1}$$

If we use the operator $\mathrm{ad}Z\,(X) = [Z, X]$, then (2.1) leads to the following three lemmas. For their proofs we refer to [10].

Lemma 2.1. *Consider an arbitrary $\phi \in C_0^\infty(\Omega)$ and $x \in \Omega$. Then for sufficiently small $s > 0$ we have*

$$X\big(e^{sZ}\phi(x)\big) = \sum_{k=0}^{\infty}(-1)^k\ \frac{s^k}{k!}\ \Big((\mathrm{ad}Z)^k(X)\ \phi\Big)\ \big(e^{sZ}x\big)\,, \tag{2.2}$$

$$X\big(e^{-sZ}\phi(x)\big) = \sum_{k=0}^{\infty} \frac{s^k}{k!}\ \Big((\mathrm{ad}Z)^k(X)\ \phi\Big)\ \big(e^{-sZ}x\big)\,. \tag{2.3}$$

Lemma 2.2. *Consider a ν-closed Hörmander system $\mathfrak{X}$ of vector fields and an arbitrary $\phi \in C_0^\infty(\Omega)$ and $x \in \Omega$.*

(i) *If $T \in \mathfrak{B} \setminus \mathfrak{X}$ such that $[T, X_i] \subset \mathfrak{X} \cup \{0\}$ for all $1 \leqslant i \leqslant N$, then there exists $\Psi_i(s) = \langle \psi_{i,1}(s), \ldots, \psi_{i,N}(s)\rangle$, where $\psi_{i,k}$ are analytic functions in s, such that*

$$X_i\big(e^{\pm sT}\phi(x)\big) = X_i\phi\big(e^{\pm sT}x\big) \pm s\Psi_i(\pm s)\cdot \mathfrak{X}\phi\big(e^{\pm sT}x\big) \ . \tag{2.4}$$

(ii) *Suppose that we already selected a set of vector fields $\mathfrak{T} = \{T_1, \ldots, T_m\}$ and continue the process by selecting $Z \in \mathfrak{B}$ such that $[Z, Y] \in \mathfrak{X} \cup \mathfrak{T} \cup \{0\}$ for all $Y \in \mathfrak{X} \cup \mathfrak{T}$. Then for every $1 \leqslant i \leqslant N$ there exist two vectors Ψ_i and Φ_i of analytic functions in s such that*

$$X_i\big(e^{\pm sZ}\phi(x)\big) = X_i\phi\big(e^{\pm sZ}x\big) \pm s\Psi_i(\pm s)\cdot \mathfrak{X}\phi\big(e^{\pm sZ}x\big) \pm s\Phi_i(\pm s)\cdot \mathfrak{T}\phi\big(e^{\pm sZ}x\big) \ . \tag{2.5}$$

Example 2.1. If we return to the vector fields X_1, X_2, T with commutation relations presented in (1.8), we find that

$$\begin{aligned} X_1\big(e^{sT}\phi(x)\big) &= X_1\phi\big(e^{sT}x\big) \\ &\quad + s\frac{-\sin s}{s}X_2\phi\big(e^{sT}x\big) + s\frac{\cos s - 1}{s}X_1\phi\big(e^{sT}x\big) \ . \end{aligned} \tag{2.6}$$

For $s > 0$ we define the following difference quotients:

$$D_{Z,s,\gamma}u(x) = \frac{u(e^{sZ}x) - u(x)}{s^{\gamma}}\,,$$
$$D_{Z,-s,\gamma}u(x) = \frac{u(x) - u(e^{-sZ}x)}{s^{\gamma}}\,.$$

Lemma 2.3. (i) *Consider the vector field T from* (i) *of Lemma* 2.2. *If $u \in L^2(\Omega)$ has compact support and $X_i u \in L^2(\Omega)$ for all $1 \leqslant i \leqslant N$, then we have the following identity in the weak sense*:

$$X_i\left(D_{T,\pm s,\gamma}u(x)\right) = D_{T,\pm s,\gamma}\left(X_i u(x)\right) \pm s^{1-\gamma}\Psi_i(\pm s)\cdot \mathfrak{X}u\big(e^{\pm sT}x\big). \quad (2.7)$$

(ii) *Consider the vector field Z from* (ii) *of Lemma* 2.2. *If, in addition to* (i), *we suppose that $T_j u \in L^2(\Omega)$ for all $1 \leqslant j \leqslant m$, then we have the following identity in the weak sense*:

$$\begin{aligned} X_i\left(D_{Z,\pm s,\gamma}u(x)\right) &= D_{Z,\pm s,\gamma}\left(X_i u(x)\right) \\ &\pm s^{1-\gamma}\Psi_i(\pm s)\cdot \mathfrak{X}u\big(e^{\pm sT}x\big) \pm s^{1-\gamma}\Phi_i(\pm s)\cdot \mathfrak{T}u\big(e^{\pm sZ}x\big). \quad (2.8)\end{aligned}$$

We are able now to prove our first regularity result.

Theorem 2.1. *Consider a weak solution $u \in XW^{1,2}_{\text{loc}}(\Omega)$ of* (1.1). *Let $T \in \mathfrak{B} \setminus \mathfrak{X}$ be such that $[T, X_i] \subset \mathfrak{X} \cup \{0\}$ for all $1 \leqslant i \leqslant N$. Suppose that T is a commutator of length m of the horizontal vector fields. If $x_0 \in \Omega$ and $r > 0$ are such that $B\big(x_0, 3r\big) \subset \Omega$, then there exist numbers $k \in \mathbb{N}$ and $c > 0$ depending only on m and* dist $(x_0, \partial\Omega)$ *such that*

$$\int_{B\left(x_0, r/k\right)} |Tu(x)|^2 dx \leqslant c \int_{B\left(x_0, 2r\right)} \big(1 + |\mathfrak{X}u(x)|^2 + |u(x)|^2\big)\, dx \qquad (2.9)$$

and

$$\int_{B(x_0, r/k)} |T\mathfrak{X}u(x)|^2\; dx \leqslant c \int_{B(x_0, 2r)} \big(1 + |\mathfrak{X}u(x)|^2 + |u(x)|^2\big)\; dx\,, \quad (2.10)$$

which implies $Tu \in \mathfrak{X}W^{1,2}_{\text{loc}}(\Omega)$

Proof. Denote $\gamma = 1/m$. Let η be a cut-off function between the Carnot–Carathéodory metric balls $B(x_0, \frac{r}{2})$ and $B(x_0, r)$. For the sake of simplicity, let us just use the notation B_r. Consider the test function

$$\varphi = D_{T,-s,\gamma}\left(\eta^2 D_{T,s,\gamma}u\right)$$

to get

Springer

http://www.springer.com/978-1-4419-1342-5

Around the Research of Vladimir Maz'ya II
Partial Differential Equations
(Ed.)A. Laptev
2010, XXII, 386 p. 3 illus., Hardcover
ISBN: 978-1-4419-1342-5