

# Chapter 1

## Plane figurate numbers

In this introductory chapter, the basic figurate numbers, **polygonal numbers**, are presented. They generalize **triangular** and **square numbers** to any regular  $m$ -gon.

Besides classical polygonal numbers, we consider in the plane **centered polygonal numbers**, in which layers of  $m$ -gons are drawn centered about a point.

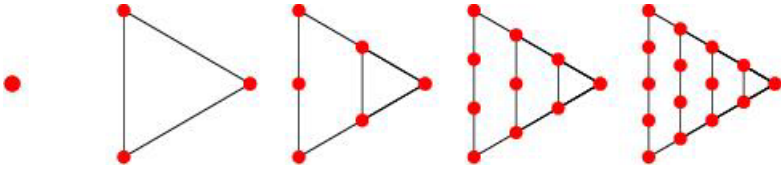
Finally, we list many other two-dimensional figurate numbers: **pronic numbers**, **trapezoidal numbers**, **polygram numbers**, **truncated plane figurate numbers**, etc.

### 1.1 Definitions and formulas

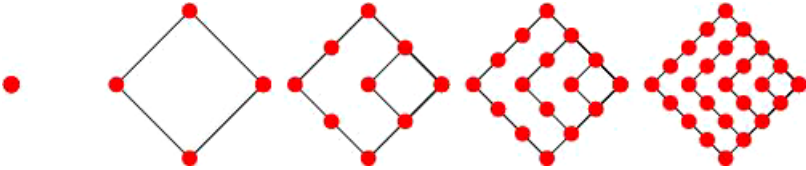
**1.1.1.** Following to ancient mathematicians, we are going now to consider the sets of points forming some geometrical figures on the plane.

Starting from a point, add to it two points, so that to obtain an equilateral triangle. Six-points equilateral triangle can be obtained from three-points triangle by adding to it three points; adding to it four points gives ten-points triangle, etc. So, by adding to a point two, three, four etc. points, then organizing the points in the form of an equilateral triangle and counting the number of points in each such triangle, one can obtain the numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ... (Sloane's A000217, [Sloa11]), which are called **triangular numbers**.

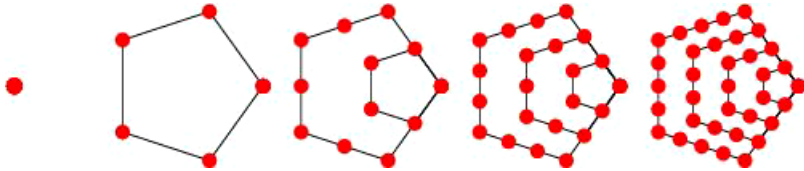
2 *Figurate Numbers*



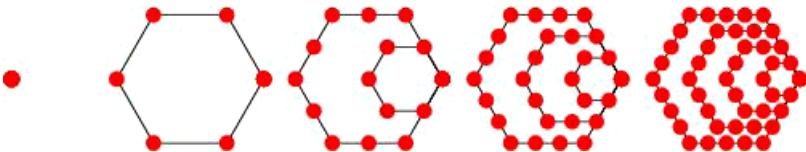
Similarly, by adding to a point three, five, seven etc. points and organizing them in the form of a square, one can obtain the numbers 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ... (Sloane's A000290), which are called **square numbers**.



By adding to a point four, seven, ten etc. points and forming from them a regular pentagon, one can construct **pentagonal numbers** 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, ... (Sloane's A000326).



Following this procedure, we can construct **hexagonal numbers** 1, 6, 15, 28, 45, 66, 91, 120, 153, 190, ... (Sloane's A000384),



**heptagonal numbers** 1, 7, 18, 34, 55, 81, 112, 148, 189, 235, ... (Sloane's A000566), **octagonal numbers** 1, 8, 21, 40, 65, 96, 133, 176, 225, 280, ... (Sloane's A000567), **nonagonal numbers** 1, 9, 24, 46, 75, 111, 154, 204, 261, 325, ... (Sloane's A001106), **decagonal**

**numbers** 1, 10, 27, 52, 85, 126, 175, 232, 297, 370, ... (Sloane's A001107), **hendecagonal numbers** 1, 11, 30, 58, 95, 141, 196, 260, 333, 415, ... (Sloane's A051682), **dodecagonal numbers** 1, 12, 33, 64, 105, 156, 217, 288, 369, 460, ... (Sloane's A051624), etc.

So, we have constructed several simplest classes of **polygonal numbers** — positive integers, corresponding to an arrangement of points on the plane, which forms a regular polygon. One speaks about  **$m$ -gonal numbers** if the arrangement forms a regular  $m$ -gon.

**1.1.2.** Polygonal numbers were a concern of Pythagorean Geometry, since Pythagoras is credited with initiating them, and originating the notion that these numbers are generated from a *gnomon* or *basic unit*. A *gnomon* is a shape which, when added to a figure, yields another figure similar to the original. So, in our case, a gnomon is the piece which needs to be added to a polygonal number to transform it to the next bigger one.

The gnomon of a triangular number is the positive integer of the general form  $n + 1$ ,  $n = 1, 2, 3, \dots$ : starting with  $n$ -th triangular number, one obtains  $(n + 1)$ -th triangular number adjoining the line with  $n + 1$  elements. For instance, the 21-point triangle, composed of gnomons, looks like this:

```

      1
     2 2
    3 3 3
   4 4 4 4
  5 5 5 5 5
 6 6 6 6 6 6

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The gnomon of the square number is the odd number of the general form  $2n + 1$ ,  $n = 1, 2, 3, \dots$ : in order to get the  $(n + 1)$ -th square from  $n$ -th square, we should adjoin  $2n + 1$  elements, one to the end of each column, one to the end of each row, and a single one to the corner. The square of size  $6 \times 6$ , composed of gnomons, looks like this:

```

 6 6 6 6 6 6
 5 5 5 5 5 6
 4 4 4 4 5 6
 3 3 3 4 5 6
 2 2 3 4 5 6
 1 2 3 4 5 6

```

The general rule for enlarging the regular polygon to the next size is to extend two adjacent sides by one point and then to add the required extra sides between those points. So, to transform  $n$ -th  $m$ -gonal number into the  $(n + 1)$ -th  $m$ -gonal number, one adjoins  $(m - 2)n + 1$  elements.

Therefore, the triangular numbers are obtained as consecutive sums of elements of the arithmetic progression  $1, 2, 3, 4, \dots, n, \dots$ ; namely, they are  $1 = 1, 3 = 1 + 2, 6 = 3 + 3, 10 = 6 + 4, \dots$ . The square numbers are obtained as consecutive sums of elements of the arithmetic progression  $1, 3, 5, 7, \dots, 2n + 1, \dots$ :  $1 = 1, 4 = 1 + 3, 9 = 4 + 5, 16 = 9 + 7, \dots$ . The pentagonal numbers are obtained by summation of the elements of the arithmetic progression  $1, 4, 7, 10, \dots, 3n + 1, \dots$ :  $1 = 1, 5 = 1 + 4, 12 = 5 + 7, 22 = 12 + 10, \dots$ . The hexagonal numbers are obtained by summation of the elements of the arithmetic progression  $1, 5, 9, 13, \dots, 4n + 1, \dots$ :  $1 = 1, 6 = 1 + 5, 15 = 6 + 9, 28 = 15 + 13, \dots$ , and so on.

Putting the points in one line, one can speak about **linear numbers**. In fact, any positive integer is a linear number. Similarly to the above construction, the linear numbers have, as a gnomon, the number 1, and are obtained as consecutive sums of elements of the sequence  $1, 1, 1, 1, \dots, 1, \dots$ :  $1 = 1, 2 = 1 + 1, 3 = 2 + 1, 4 = 3 + 1, \dots$ . In fact, the linear numbers are one-dimensional analogues of two-dimensional polygonal numbers.

**1.1.3.** The first general definition of  $m$ -gonal numbers was given by Hypsicles of Alexandria in *II*-th century BC and was quoted by Diophantys in his tract *On polygonal numbers* (see [Diop], [Heat10]): *if there are as many numbers as we please beginning with one and increasing by the same common difference, then when the common difference is 1, the sum of all the terms is a triangular number; when 2, a square; when 3, a pentagonal number; and the number of the angles is called after the number exceeding the common difference by 2, and the side after the number of terms including 1.*

In contemporary mathematical language, it has the following form:  *$n$ -th  $m$ -gonal number  $S_m(n)$  is the sum of the first  $n$  elements of the arithmetic progression*

$$1, 1 + (m - 2), 1 + 2(m - 2), 1 + 3(m - 2), \dots, m \geq 3.$$

So, by definition, it holds

$$S_m(n) = 1 + (1 + (m - 2)) + (1 + (m - 2)) + \cdots + (1 + (m - 2)(n - 1)).$$

In particular, we get

$$\begin{aligned} S_3(n) &= 1 + 2 + \cdots + n, & S_4(n) &= 1 + 3 + \cdots + (2n - 1), \\ S_5(n) &= 1 + 4 + \cdots + (3n - 2), \\ S_6(n) &= 1 + 5 + \cdots + (4n - 3), & S_7(n) &= 1 + 6 + \cdots + (5n - 4), \\ S_8(n) &= 1 + 7 + \cdots + (6n - 5). \end{aligned}$$

Above expression implies the following recurrent formula for  $m$ -gonal numbers:

$$S_m(n + 1) = S_m(n) + (1 + (m - 2)n), \quad S_m(1) = 1.$$

In particular, we get

$$\begin{aligned} S_3(n + 1) &= S_3(n) + (n + 1), & S_4(n + 1) &= S_4(n) + (2n + 1), \\ S_5(n + 1) &= S_5(n) + (3n + 1), \\ S_6(n + 1) &= S_5(n) + (4n + 1), & S_7(n + 1) &= S_7(n) + (5n + 1), \\ S_8(n + 1) &= S_8(n) + (6n + 1). \end{aligned}$$

For many applications it is convenient to add the value  $S_m(0) = 0$  to the list.

Since the sum of the first  $n$  elements of an arithmetic progression  $a_1, \dots, a_n, \dots$  is equal to  $\frac{a_1 + a_n}{2} \cdot n$ , one obtains the following general formula for  $n$ -th  $m$ -gonal number:

$$\begin{aligned} S_m(n) &= \frac{n((m - 2)n - m + 4)}{2} = \frac{(m - 2)}{2}(n^2 - n) + n \\ &= \frac{(m - 2)n^2 - (m - 4)n}{2} = \frac{1}{2}m(n^2 - n) - n^2 + 2n. \end{aligned}$$

In particular, one has

$$\begin{aligned} S_3(n) &= \frac{n(n + 1)}{2}, & S_4(n) &= \frac{n \cdot (2n)}{2} = n^2, & S_5(n) &= \frac{n(3n - 1)}{2}, \\ S_6(n) &= \frac{n(4n - 2)}{2} = n(2n - 1), & S_7(n) &= \frac{n(5n - 3)}{2}, \\ S_8(n) &= \frac{n(8n - 4)}{2} = n(4n - 2). \end{aligned}$$

These formulas for  $m$ -gonal numbers with  $3 \leq m \leq 30$ , as well as the first few elements of the corresponding sequences and the

numbers of these sequences in the Sloane's *On-Line Encyclopedia of Integer Sequences* (OEIS, [Sloa11]) classification, are given in the table below.

Name	Formula												Sloane
Triangular	$\frac{1}{2}(n^2 + n)$	1	3	6	10	15	21	28	36	45	55	66	A000217
Square	$\frac{1}{2}(n^2 - 0 \cdot n)$	1	4	9	16	25	36	49	64	81	100	121	A000290
Pentagonal	$\frac{1}{2}(3n^2 - 1 \cdot n)$	1	5	12	22	35	51	70	92	117	145	176	A000326
Hexagonal	$\frac{1}{2}(4n^2 - 2n)$	1	6	15	28	45	66	91	120	153	190	231	A000384
Heptagonal	$\frac{1}{2}(5n^2 - 3n)$	1	7	18	34	55	81	112	148	189	235	286	A000566
Octagonal	$\frac{1}{2}(6n^2 - 4n)$	1	8	21	40	65	96	133	176	225	280	341	A000567
Nonagonal	$\frac{1}{2}(7n^2 - 5n)$	1	9	24	46	75	111	154	204	261	325	396	A001106
Decagonal	$\frac{1}{2}(8n^2 - 6n)$	1	10	27	52	85	126	175	232	297	370	451	A001107
Hendecagonal	$\frac{1}{2}(9n^2 - 7n)$	1	11	30	58	95	141	196	260	333	415	506	A051682
Dodecagonal	$\frac{1}{2}(10n^2 - 8n)$	1	12	33	64	105	156	217	288	369	460	561	A051624
Tridecagonal	$\frac{1}{2}(11n^2 - 9n)$	1	13	36	70	115	171	238	316	405	505	616	A051865
Tetradecagonal	$\frac{1}{2}(12n^2 - 10n)$	1	14	39	76	125	186	259	344	441	550	671	A051866
Pentadecagonal	$\frac{1}{2}(13n^2 - 11n)$	1	15	42	82	135	201	280	372	477	595	726	A051867
Hexadecagonal	$\frac{1}{2}(14n^2 - 12n)$	1	16	45	88	145	216	301	400	513	640	781	A051868
Heptadecagonal	$\frac{1}{2}(15n^2 - 13n)$	1	17	48	94	155	231	322	428	549	685	836	A051869
Octadecagonal	$\frac{1}{2}(16n^2 - 14n)$	1	18	51	100	165	246	343	456	585	730	891	A051870
Nonadecagonal	$\frac{1}{2}(17n^2 - 15n)$	1	19	54	106	175	261	364	484	621	775	946	A051871
Icosagonal	$\frac{1}{2}(18n^2 - 16n)$	1	20	57	112	185	276	385	512	657	820	1001	A051872
Icosihenagonal	$\frac{1}{2}(19n^2 - 17n)$	1	21	60	118	195	291	406	540	693	865	1056	A051873
Icosidigonal	$\frac{1}{2}(20n^2 - 18n)$	1	22	63	124	205	306	427	568	729	910	1111	A051874
Icositrigonal	$\frac{1}{2}(21n^2 - 19n)$	1	23	66	130	215	321	448	596	765	955	1166	A051875
Icositetragonal	$\frac{1}{2}(22n^2 - 20n)$	1	24	69	136	225	336	469	624	801	1000	1221	A051876
Icosipentagonal	$\frac{1}{2}(23n^2 - 21n)$	1	25	72	142	235	351	490	652	837	1045	1276	
Icosihexagonal	$\frac{1}{2}(24n^2 - 22n)$	1	26	75	148	245	366	511	680	873	1090	1331	
Icosiheptagonal	$\frac{1}{2}(25n^2 - 23n)$	1	27	78	154	255	381	532	708	909	1135	1386	
Icosioctagonal	$\frac{1}{2}(26n^2 - 24n)$	1	28	81	160	265	396	553	736	945	1180	1441	
Icosinonagonal	$\frac{1}{2}(27n^2 - 25n)$	1	29	84	166	275	411	574	764	981	1225	1496	
Triacontagonal	$\frac{1}{2}(28n^2 - 26n)$	1	30	87	172	285	426	595	792	1017	1270	1551	

1.1.4. There are many different methods to obtain above formulas.

For example, the geometrical illustration for  $n = 4$  on the picture below shows that  $n$ -th triangular number is one half of rectangle with the edges  $n$  and  $n + 1$ . Hence,  $S_3(n) = \frac{n(n+1)}{2}$ .



On the other hand, this formula can be obtained by induction using the fact, that the triangular number with the index  $n + 1$  is obtained from the triangular number with the index  $n$  by addition of the number  $n + 1$ . For  $n = 1$  one has  $S_3(1) = 1 = \frac{1 \cdot (1+1)}{2}$ . Going from  $n$  to  $n + 1$ , one obtains  $S_3(n + 1) = S_3(n) + (n + 1) = \frac{n(n+1)}{2} + (n + 1) = \frac{(n+1)(n+2)}{2}$ .



The similar considerations can be applied to any  $m$ -gonal number. In a geometrical setting, two copies of the sum  $a_1 + a_2 + \dots + a_{n-1} + a_n$  of the first  $n$  elements of an arithmetic progression  $a_1, \dots, a_n, \dots$  corresponds to the rectangle with the edges  $a_1 + a_n$  and  $n$ . It reflects the fact, that the sums  $a_1 + a_n, a_2 + a_{n-1}, a_3 + a_{n-2}, \dots$  are equal. In our case,  $a_1 = 1$  and  $a_n = 1 + (m - 2)(n - 1)$ . It yields  $2S_m(n) = (1 + (1 + (m - 2)(n - 1)))n = ((m - 2)n - m + 4)n$ , and  $S_m(n) = \frac{((m-2)n-m+4)n}{2}$ . Let us prove the formula  $S_m(n) = \frac{n((m-2)n-m+4)}{2}$  by induction. It holds for  $n = 1$  since  $S_m(1) = 1 = \frac{1 \cdot ((m-2) \cdot 1 - m + 4)}{2}$ . Going from  $n$  to  $n + 1$ , one obtains

$$\begin{aligned} S_m(n+1) &= S_m(n) + (1 + (m-2)n) \\ &= \frac{n((m-2)n-m+4)}{2} + (1 + (m-2)n) \\ &= \frac{n^2(m-2) + n(4-m) + 2n(m-2) + 2}{2} \\ &= \frac{(n^2 + n + 1)(m-2) + (n+1)(4-m)}{2} \\ &= \frac{(n+1)((m-2)(n+1) - m + 4)}{2}. \end{aligned}$$

The special summation<sup>1</sup> has the form

$$\begin{aligned} 2S_m(n) &= \\ &1 + 1 + (m-2) + \dots + 1 + (m-2)(n-1) + \\ &1 + (m-2)(n-1) + 1 + (m-2)(n-2) + \dots + 1. \end{aligned}$$

It gives  $2S_m(n) = (2 + (m - 2)(n - 1)) + \dots + (2 + (m - 2)(n - 1)) = n(2 + (m - 2)(n - 1))$ , and  $S_m(n) = \frac{n((m-2)n-m+4)}{2}$ .

Finally, another way to derive the general formula for  $S_m(n)$  is to use the first three  $m$ -gonal numbers to find the coefficients  $A$ ,  $B$ , and  $C$  of the general 2-nd degree polynomial  $An^2 + Bn + C$ . For  $n = 1$

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<sup>1</sup>According to legend, at age 10 Gauss was told by his teacher to sum up all the numbers from 1 to 100. He reasoned that each number  $i$  could be paired up with  $101 - i$ , to form a sum of 101, and if this was done 100 times, it would result in twice the actual sum, since each number would get used twice due to the pairing. Hence, the sum would be  $1 + \dots + 100 = \frac{100 \cdot 101}{2}$ .

one has  $A + B + C = 1$ , for  $n = 2$  we obtain  $4A + 2B + C = m$ , and for  $n = 3$  it is  $9A + 3B + C = 3m - 3$ . This leads to  $A = \frac{m-2}{2}$ ,  $B = \frac{4-m}{2}$ ,  $C = 0$ , and implies above formula for  $S_m(n)$ .

**1.1.5.** The *generating function* for the sequence  $S_m(1), S_m(2), \dots, S_m(n), \dots$  of the  $m$ -gonal numbers has the form  $f(x) = \frac{x((m-3)x+1)}{(1-x)^3}$ , i.e., it holds

$$\frac{x((m-3)x+1)}{(1-x)^3} = S_m(1)x + S_m(2)x^2 + S_m(3)x^3 + \dots + S_m(n)x^n + \dots, \quad |x| < 1.$$

In particular, one gets

$$\begin{aligned} \frac{x}{(1-x)^3} &= x + 3x^2 + 6x^3 + \dots + S_3(n)x^n + \dots, \quad |x| < 1; \\ \frac{x(x+1)}{(1-x)^3} &= x + 4x^2 + 9x^3 + \dots + S_4(n)x^n + \dots, \quad |x| < 1; \\ \frac{x(2x+1)}{(1-x)^3} &= x + 5x^2 + 12x^3 + \dots + S_5(n)x^n + \dots, \quad |x| < 1; \\ \frac{x(3x+1)}{(1-x)^3} &= x + 6x^2 + 15x^3 + \dots + S_6(n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

In order to obtain the above formula, let us consider two polynomials

$$f(x) = a_0 + a_1x + \dots + a_mx^m \quad \text{and} \quad g(x) = b_0 + b_1x + \dots + b_nx^n$$

with real coefficients and  $m < n$ . It follows (see, for example, [DeMo10]), that *the rational function  $\frac{f(x)}{g(x)}$  is the generating function of the sequence  $c_0, c_1, c_2, \dots, c_n, \dots$ , which is a solution of the linear recurrent equation  $b_0c_{n+k} + b_1c_{n+k-1} + \dots + b_nc_k = 0$  of  $n$ -th order with coefficients  $b_0, b_1, \dots, b_n$ .*

In fact, one has the decomposition  $\frac{f(x)}{g(x)} = c_0 + c_1x + c_2x^2 + \dots + c_nx^n + \dots$  if  $|x| < r$ , and  $r = \min_{1 \leq i \leq n} |x_i|$ , where  $x_1, \dots, x_n$  are the roots of the polynomial  $g(x)$ . It yields that the rational function  $\frac{f(x)}{g(x)}$  is the generating function of the obtained sequence  $c_0, c_1, c_2, \dots, c_n, \dots$ . Moreover, one gets  $f(x) = g(x)(c_0 + c_1x +$

$c_2x^2 + \dots + c_nx^n + \dots$ ). In other words, it holds

$$a_0 + a_1x + a_2x^2 + \dots + a_mx^m = (b_0 + b_1x + b_2x^2 + \dots + b_nx^n) \\ \times (c_0 + c_1x + c_2x^2 + \dots + c_nx^n + \dots).$$

It is easy to check now the following equalities:

$$a_0 = b_0c_0, \quad a_1 = b_0c_1 + b_1c_0, \\ a_2 = b_0c_2 + b_1c_1 + b_2c_0, \dots, a_m = b_0c_m + \dots + b_mc_0, \\ 0 = b_0c_{m+1} + \dots + b_{m+1}c_0, \dots, 0 = b_0c_n + \dots + b_nc_0, \\ 0 = b_0c_{n+1} + \dots + b_nc_1, \dots, 0 = b_0c_{n+k} + \dots + b_kc_n, \dots$$

So, the sequence  $c_0, c_1, \dots, c_n, \dots$  is a solution of the *linear recurrent equation*  $b_0c_{n+k} + \dots + b_nc_k = 0$  of  $n$ -th order with coefficients  $b_0, \dots, b_n$ . Moreover, one can find the first  $n$  elements of this sequence using the first  $n$  above equalities:  $a_0 = b_0c_0, a_1 = b_0c_1 + b_1c_0, \dots, a_{n-1} = \sum_{k=0}^{n-1} b_kc_{n-1-k}$ .

On the other hand, let the sequence  $c_0, c_1, c_2, \dots, c_n, \dots$  be a solution of a linear recurrent equation  $b_0c_{n+k} + \dots + b_nc_k = 0$  of  $n$ -th order with coefficients  $b_0, \dots, b_n$ . Let us define numbers  $a_0, \dots, a_{n-1}$  by the formulas  $a_i = \sum_{k=0}^i b_kc_{i-k}, i = 0, 1, 2, \dots, n-1$ , using the *initial values*  $c_0, c_1, \dots, c_n$  of the given sequence. It yields the equality

$$a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} = (b_0 + b_1x + b_2x^2 + \dots + b_nx^n) \\ \times (c_0 + c_1x + c_2x^2 + \dots + c_nx^n + \dots).$$

In other words, one gets

$$\frac{a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}}{b_0 + b_1x + b_2x^2 + \dots + b_nx^n} = c_0 + c_1x + c_2x^2 + \dots + c_nx^n + \dots$$

So, the generating function of the sequence  $c_0, c_1, \dots, c_n, \dots$  has the form  $\frac{f(x)}{g(x)}$ , where

$$g(x) = b_0 + b_1x + \dots + b_nx^n, \quad \text{and} \quad f(x) = a_0 + \dots + a_{n-1}x^{n-1},$$

with  $a_0 = b_0c_0, a_1 = b_0c_1 + b_1c_0, \dots, a_{n-1} = b_0c_{n-1} + b_1c_{n-2} + \dots + b_{n-1}c_0$ .

Now we can find the generating function for the sequence of  $m$ -gonal numbers. Let us consider the recurrent equation  $S_m(n+1) = S_m(n) + (1 + (m-2)n)$ . Going from  $n$  to  $n+1$ , one

obtains  $S_m(n+2) = S_m(n+1) + (1 + (m-2)(n+1))$ . Subtracting first equality from second one, we get  $S_m(n+2) - S_m(n+1) = S_m(n+1) - S_m(n) + (m-2)$ , i.e.,  $S_m(n+2) = 2S_m(n+1) - S_m(n) + (m-2)$ .

Similarly, one obtains  $S_m(n+3) = 2S_m(n+2) - S_m(n+1) + (m-2)$ , and

$$S_m(n+3) - S_m(n+2) = 2S_m(n+2) - 2S_m(n+1) - S_m(n+1) + S_m(n),$$

i.e.,  $S_m(n+3) = 3S_m(n+2) - 3S_m(n+1) + S_m(n)$ . Hence, we get for the sequence of the  $m$ -gonal numbers the following linear recurrent equation:

$$S_m(n+3) - 3S_m(n+2) + 3S_m(n+1) - S_m(n) = 0.$$

It is a linear recurrent equation of 3-rd order with coefficients  $b_0 = 1, b_1 = -3, b_2 = 3, b_3 = -1$ . Its initial values are  $S_m(1) = 1, S_m(2) = m, S_m(3) = 3m - 3$ . Denoting  $S_m(n+1)$  by  $c_n$ , one can rewrite the above equation as

$$c_{n+3} - 3c_{n+2} + 3c_{n+1} - c_n = 0, \quad c_0 = 1, \quad c_1 = m, \quad c_2 = 3m - 3.$$

Therefore, the generating function for the sequence of  $m$ -gonal numbers has the form

$$\frac{f(x)}{g(x)} = \frac{a_0 + a_1x + a_2x^2}{b_0 + b_1x + b_2x^2 + b_3x^3},$$

where  $b_0 = 1, b_1 = -3, b_2 = 3, b_3 = -1$ , and  $a_0 = b_0c_0 = 1, a_1 = b_0c_1 + b_1c_0 = 1 \cdot m + (-3) \cdot 1 = m - 3, a_2 = b_0c_2 + b_1c_1 + b_2c_0 = 1 \cdot (3m - 3) + (-3)m + 3 \cdot 1 = 0$ . Since  $g(x) = 1 - 3x + 3x^2 - x^3 = (1-x)^3$  has three coinciding roots  $x_1 = x_2 = x_3 = 1$ , the generating function for the sequence of the  $m$ -gonal numbers obtains the form

$$\begin{aligned} \frac{1 + (m-3)x}{(1-x)^3} &= S_m(1) + S_m(2)x \\ &+ S_m(3)x^2 + \cdots + S_m(n)x^{n-1} + \cdots, \quad |x| < 1. \end{aligned}$$

In other terms, it holds

$$\begin{aligned} \frac{x(1 + (m-3)x)}{(1-x)^3} &= S_m(1)x + S_m(2)x^2 \\ &+ S_m(3)x^3 + \cdots + S_m(n)x^n + \cdots, \quad |x| < 1. \end{aligned}$$

## 1.2 Main properties of polygonal numbers

The considerations, similar to above ones, give many interesting properties of polygonal numbers.

**1.2.1.** For example, Theon of Smyrna proved in the *II*-th century AC that *the sum of two consecutive triangular numbers is a square number*, obtaining the following formula, called now the *Theon formula* (see [Theo]):

$$S_3(n) + S_3(n - 1) = S_4(n).$$

In fact, one has  $S_3(n) + S_3(n - 1) = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2 = S_4(n)$ .

Alternatively, it can be demonstrated diagrammatically:

$$\begin{array}{cccc} * & * & * & * \\ * & * & * & \cdot \\ * & * & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \end{array}$$

In the above example, constructed for  $n = 4$ , the square is formed by two interlocking triangles. Also, one can prove this formula by induction. For  $n = 2$ , it holds  $S_3(2) + S_3(1) = 3 + 1 = 4 = S_4(2)$ . Going from  $n$  to  $n + 1$ , one obtains

$$\begin{aligned} S_3(n + 1) + S_3(n) &= S_3(n) + (n + 1) + S_3(n - 1) + n \\ &= S_4(n) + 2n + 1 = n^2 + 2n + 1 \\ &= (n + 1)^2 = S_4(n + 1). \end{aligned}$$

A special summation of the form

$$S_3(n) + S_3(n - 1) = \frac{1 + 2 + 3 + \dots + n}{1 + 2 + \dots + (n - 1)} + n$$

gives  $S_3(n) + S_3(n - 1) = 1 + 3 + 5 + \dots + (2n - 1) = S_4(n)$ .

**1.2.2.** Similarly, we can construct triangular numbers, using as inner blocks some triangular numbers of smaller size. For example, a triangular number with even index can be constructed using the following formula:

$$S_3(2n) = 3S_3(n) + S_3(n - 1).$$

In fact, one has  $3S_3(n) + S_3(n - 1) = 3 \cdot \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{n}{2}(4n + 2) = \frac{2n(2n+1)}{2} = S_3(2n)$ .

The geometrical illustration of this property for  $n = 3$  is given in the picture below.



By induction, one has, for  $n = 2$ , that  $3S_3(2) + S_3(1) = 3 \cdot 3 + 1 = 10 = S_3(4)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned} 3S_3(n + 1) + S_3(n) &= S_3(n) + S_3(n - 1) + 3(n + 1) + n \\ &= S_3(2n) + (2n + 1) + (2n + 2) \\ &= S_3(2n + 1) + (2n + 2) = S_3(2n + 2) \\ &= S_3(2(n + 1)). \end{aligned}$$

Finally, a special summation of the form

$$3S_3(n) + S_3(n - 1) = \begin{matrix} 1 + \cdots + n + 1 + & 2 & + \cdots + & n & + \\ & n + (n - 1) + \cdots + & 1 & + \\ & 1 & + \cdots + & (n - 1) \end{matrix}$$

implies  $3S_3(n) + S_3(n - 1) = 1 + \cdots + n + (n + 1) + \cdots + 2n = S_3(2n)$ .

**1.2.3.** A triangular number with odd index can be constructed using the following similar formula:

$$S_3(2n + 1) = 3S_3(n) + S_3(n + 1).$$

In fact, one has  $3S_3(n) + S_3(n + 1) = \frac{3n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = \frac{n}{2}(4n + 2) = \frac{2n(2n+1)}{2} = S_3(2n + 1)$ .

The geometrical illustration of this property for  $n = 2$  is given below.



By induction, we have, for  $n = 1$ , that  $3S_3(1) + S_3(2) = 3 \cdot 1 + 3 = 6 = S_3(3)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned} 3S_3(n + 1) + S_3(n + 2) &= 3S_3(n) + S_3(n + 1) + 3(n + 1) + (n + 2) \\ &= S_3(2n + 1) + (2n + 2) + (2n + 3) \\ &= S_3(2n + 3) = S_3(2(n + 1) + 1). \end{aligned}$$

Finally, a summation of the special form

$$3S_3(n) + S_3(n + 1) =$$

$$\begin{array}{ccccccc} 1 + \cdots + n + 1 + & 2 & + \cdots + & n & + \\ & n + (n - 1) + \cdots + & 1 & + \\ & 1 & + \cdots + (n - 1) + n + (n + 1) \end{array}$$

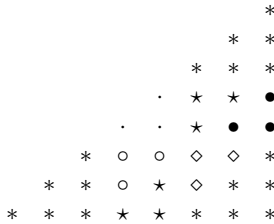
gives  $3S_3(n) + S_3(n + 1) = 1 + \cdots + n + (n + 1) + \cdots + 2n + (2n + 1) = S_3(2n + 1)$ .

**1.2.4.** The next formula needs more triangles for the construction a mosaic:

$$S_3(3n - 1) = 3S_3(n) + 6S_3(n - 1).$$

In fact, we have  $3S_3(n) + 6S_3(n - 1) = \frac{3n(n+1)}{2} + \frac{6(n-1)n}{2} = \frac{n}{2}(9n - 3) = \frac{(3n-1)3n}{2} = S_3(3n - 1)$ .

The geometrical illustration of this property for  $n = 3$  is given below.



By induction, we have, for  $n = 2$ , that  $3S_3(2) + 6S_3(1) = 15 = S_3(5)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned} 3S_3(n + 1) + 6S_3(n) &= 3S_3(n) + 6S_3(n - 1) + 3(n + 1) + 6n \\ &= S_3(3n - 1) + 3n + (3n + 1) + (3n + 2) \\ &= S_3(3n + 2) = S_3(3(n + 1) - 1). \end{aligned}$$

Finally, a special summation of the form

$$3S_3(n) + 6S_3(n + 1) =$$

$$\begin{array}{cccccccc}
 1 + \cdots + n + 1 & + & 2 & + \cdots + & n & + & & \\
 & & n + (n-1) & + \cdots + & 1 & + & & \\
 & & 1 & + \cdots + (n-1) & + & & & \\
 & & & & & & 1 & + & 2 & + \cdots + (n-1) & + \\
 & & & & & & 1 & + & 2 & + \cdots + (n-1) & + \\
 & & & & & & (n-1) & + & (n-2) & + \cdots + & 1 & + \\
 & & & & & & (n-1) & + & (n-2) & + \cdots + & 1 & + \\
 & & & & & & 1 & + & 2 & + \cdots + (n-1) & + & 
 \end{array}$$

shows, that

$$\begin{aligned}
 3S_3(n) + 6S_3(n - 1) &= 1 + \cdots + n + (n + 1) + \cdots + 2n \\
 &+ (2n + 1) + \cdots + (3n - 1) = S_3(3n - 1).
 \end{aligned}$$

**1.2.5.** The following property is called the *Diophantus’ formula* (or, sometimes, the *Plutarch formula*):<sup>2</sup>

$$S_4(2n + 1) = 8S_3(n) + 1.$$

In fact, we have  $8S_3(n) + 1 = \frac{8n(n+1)}{2} + 1 = 4n^2 + 4n + 1 = (2n + 1)^2 = S_4(2n + 1)$ .

The geometrical illustration for  $n = 2$  is given below.



By induction, we have, for  $n = 1$ , that  $8S_3(1) + 1 = 9 = S_4(3)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned}
 8S_3(n + 1) + 1 &= (8S_3(n) + 1) + 8(n + 1) = S_4(2n + 1) + 8(n + 1) \\
 &= (2n + 1)^2 + 8(n + 1) = 4n^2 + 12n + 9 = (2n + 3)^2 \\
 &= S_4(2n + 3) = S_4(2(n + 1) + 1).
 \end{aligned}$$

---

<sup>2</sup>This formula was known by Plutarch [Plut], a contemporary of Nicomachus, and Diophantus [Diop] (about 250 AC) generalized this theorem, proving by a cumbersome geometric method that  $8(m-2)S_m(n) + (m-4)^2 = ((m-2)(2n-1) + 2)^2$ , and spoke of this result as a new definition of polygonal numbers equivalent to that of Hypsicles.

Finally, a special summation of the form

$$\begin{aligned}
 8S_3(n) + 1 = & \\
 1 + 2 + \cdots + n + & \\
 1 + \cdots + (n-1) + n + & \\
 & 1 + 2 + \cdots + n + \\
 & 1 + 2 + \cdots + n & \\
 & n + (n-1) + \cdots + 1 + & \\
 & n + (n-1) + \cdots + 1 + & \\
 & 1 + 2 + \cdots + n + & \\
 & 1 + \cdots + (n-1) + & \\
 & n + & \\
 & 1 &
 \end{aligned}$$

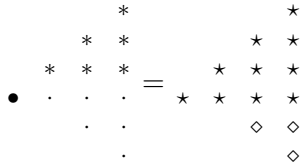
gives  $8S_3(n) + 1 = 1 + 3 + \cdots + (2n - 1) + (2n + 1) + (2n + 3) + \cdots + (4n + 1) = S_4(2n + 1)$ .

**1.2.6.** What happens, if we consider now two consecutive triangle numbers with even (or odd) indices? The answer is given by the following formula:

$$S_3(n - 1) + S_3(n + 1) = 2S_3(n) + 1.$$

In fact, we have  $S_3(n - 1) + S_3(n + 1) = \frac{(n-1)n}{2} + \frac{(n+1)(n+2)}{2} = \frac{2n^2+2n+2}{2} = 2 \cdot \frac{n(n+1)}{2} + 1 = 2S_3(n) + 1$ .

The geometrical illustration of this property for  $n = 3$  is given below.



By induction, we have, for  $n = 2$ , that  $S_3(1) + S_3(3) = 1 + 6 = 7 = 2 \cdot 3 + 1 = 2S_3(2) + 1$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned}
 S_3(n) + S_3(n + 2) &= S_3(n - 1) + S_3(n + 1) + n + (n + 2) \\
 &= 2S_3(n) + 1 + 2(n + 1) = 2(S_3(n) + (n + 1)) + 1 \\
 &= 2S_3(n + 1) + 1.
 \end{aligned}$$

The special summation of the form

$$S_3(n-1) + S_3(n) = 1 + \cdots + (n-1) + n + n + 1$$

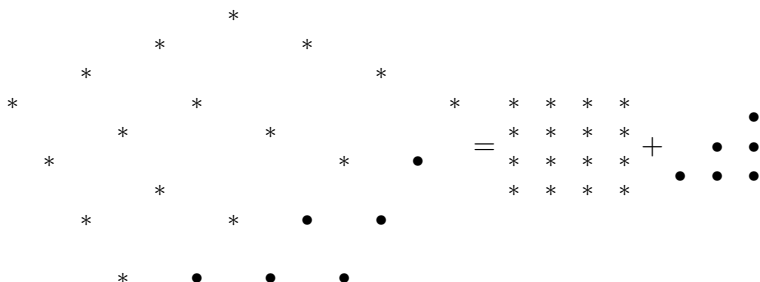
gives  $S_3(n-1) + S_3(n+1) = 2(1 + \cdots + n) + 1 = 2S_3(n) + 1$ .

**1.2.7.** The similar relations exist for other polygonal numbers. For example, one has the following property:

$$S_5(n) = S_4(n) + S_3(n-1).$$

In fact, it holds  $S_4(n) + S_3(n-1) = n^2 + \frac{(n-1)n}{2} = \frac{3n^2-n}{2} = \frac{n(3n-1)}{2} = S_5(n)$ .

One can easily obtain a geometrical interpretation of this property, noting that  $S_4(n)$  is the sum of the first  $n$  odd numbers. The corresponding picture for  $n = 4$  is given below.



By induction, we have, for  $n = 2$ , that  $S_4(2) + S_3(1) = 4 + 1 = 5 = S_5(1)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned} S_4(n+1) + S_3(n) &= S_4(n) + S_3(n-1) + (2n+1) + n \\ &= S_5(n) + (3n+1) = S_5(n+1). \end{aligned}$$

A special summation of the form

$$S_4(n) + S_3(n-1) = 1 + 3 + \cdots + (2n-1) + 1 + \cdots + (n-1)$$

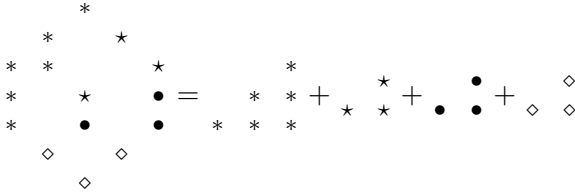
gives  $S_4(n) + S_3(n-1) = 1 + 4 + \cdots + (3n-2) = S_5(n)$ .

**1.2.8.** The following property connects triangular and hexagonal numbers:

$$S_6(n) = S_3(n) + 3S_3(n-1).$$

In fact, it is easy to see that  $S_3(n) + 3S_3(n-1) = \frac{n(n+1)}{2} + 3\frac{(n-1)n}{2} = \frac{n}{2}(4n-2) = S_6(n)$ .

The geometrical illustration of this property also is easy to obtain; below it is given for  $n = 3$ .



By induction, we have, for  $n = 2$ , that  $S_3(2) + 3S_3(1) = 3 + 3 \cdot 1 = 6 = S_6(1)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned}
 S_3(n + 1) + 3S_3(n) &= S_3(n) + 3S_3(n - 1) + (n + 1) + 3n \\
 &= S_6(n) + (4n + 1) = S_6(n + 1).
 \end{aligned}$$

A special summation of the form

$$\begin{aligned}
 &1 + 2 + \dots + (n - 1) + 1 \\
 S_3(n) + 3S_3(n - 1) &= \begin{matrix} 1 + \dots + (n - 1) + \\ 1 + \dots + (n - 1) + \\ 1 + \dots + (n - 1) \end{matrix}
 \end{aligned}$$

gives  $S_3(n) + S_3(n - 1) = 1 + 5 + \dots + (4(n - 1) + 1) = S_6(n)$ .

**1.2.9.** The next property is widely known as the *hexagonal number theorem*:

$$S_6(n) = S_3(2n - 1),$$

i.e., every hexagonal number is a triangular number.

The simplest way to get a proof of this fact is to compare two formulas:

$$S_3(2n - 1) = \frac{(2n - 1)2n}{2}, \quad \text{and} \quad \frac{n(4n - 2)}{2} = S_6(n).$$

The geometrical illustration of this property for  $n = 3$  is given below.



It is easy to see that we just rearranged the four small triangles on the previous picture into one big triangle.

Of course, one can prove it by induction: for  $n = 1$ , it holds  $S_6(1) = 1 = S_3(2 \cdot 1 - 1)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned} S_6(n + 1) &= S_6(n) + (4n + 1) = S_3(2n - 1) + (4n + 1) \\ &= \frac{(2n - 1)2n}{2} + (4n + 1) = \frac{4n^2 + 6n + 2}{2} \\ &= \frac{(2n + 1)(2n + 2)}{2} = S_3(2n + 1) = S_3(2(n + 1) - 1). \end{aligned}$$

A special summation of the form

$$S_3(2n - 1) = \begin{array}{c} 1 + 2 + 4 + \cdots + (2n - 2) + \\ 3 + 5 + \cdots + (2n - 1) \end{array}$$

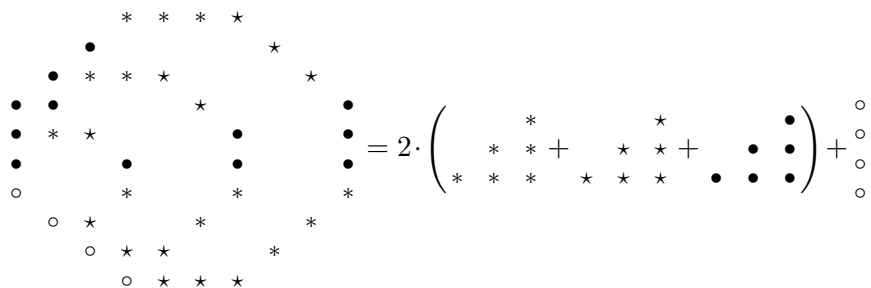
also gives  $S_3(2n - 1) = 1 + 5 + 9 + \cdots + (4n - 3) = S_6(n)$ .

**1.2.10.** The *octagonal number theorem* shows a connection between octagonal and triangular numbers:

$$S_8(n) = 6S_3(n - 1) + n.$$

In fact, one has  $6S_3(n - 1) + n = 6 \frac{(n-1)n}{2} + n = (3n^2 - 3n) + n = 3n^2 - 2n = \frac{6n^2 - 4n}{2} = S_8(n)$ .

The geometrical interpretation of this property for  $n = 4$  is given below:



By induction, we have, for  $n = 2$ , that  $S_8(2) = 8 = 6 \cdot S_3(1) + 2$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned} S_8(n + 1) &= S_8(n) + (6n + 1) = (6S_3(n - 1) + n) + (6n + 1) \\ &= 6 \frac{(n - 1)n}{2} + 6n + (n + 1) = (3n^2 + 3n) + (n + 1) \\ &= 6 \frac{n(n + 1)}{2} + (n + 1) = 6S_3(n) + (n + 1). \end{aligned}$$

A special summation of the form

$$\begin{array}{r}
 1 + 2 + \cdots + (n - 1) + \\
 1 + \cdots + (n - 2) + (n - 1) + \\
 1 + \cdots + (n - 2) + (n - 1) + \\
 6S_3(n - 1) + n = 1 + \cdots + (n - 2) + (n - 1) + \\
 1 + \cdots + (n - 2) + (n - 1) + \\
 1 + \cdots + (n - 2) + (n - 1) + \\
 n
 \end{array}$$

gives  $6S_3(n - 1) + n = 1 + 7 + \cdots + (6n - 11) + (6n - 5) = S_8(n)$ .

**1.2.11.** The following formula of Nicomachus of Alexandria was obtained in the *I*-st century BC in his *Introduction to Arithmetic* (see [Nico]): *any figurate number is equal to the sum of the figurate number of previous name, and taking in the row the same place, and the triangle number, taking in the row the previous place.*

In other words, the difference between  $n$ -th  $m$ -gonal number and  $n$ -th  $(m - 1)$ -gonal number is the  $(n - 1)$ -th triangular number. So, one has the following *Nicomachus formula*:

$$S_m(n) = S_{m-1}(n) + S_3(n - 1).$$

In fact, since  $S_m(n) = \frac{n((m-2)n-m+4)}{2}$ , one has

$$\begin{aligned}
 S_3(n - 1) + S_{m-1}(n) &= \frac{(n - 1)n}{2} + \frac{n((m - 3)n - (m - 1) + 4)}{2} \\
 &= \frac{n}{2}((m - 2)n - m + 4) = S_m(n).
 \end{aligned}$$

By induction, we have, for  $n = 2$ , that  $S_{m-1}(2) + S_3(1) = (m - 1) + 1 = m = S_m(1)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned}
 S_{m-1}(n + 1) + S_3(n) &= (S_{m-1}(n - 1) + S_3(n - 1)) + ((m - 3)n + 1) \\
 &\quad + n = S_m(n) + ((m - 2)n + 1) = S_m(n + 1).
 \end{aligned}$$

A special summation of the form

$$S_{m-1}(n) + S_3(n - 1) = \begin{array}{r} 1 + (m - 1) + \cdots + ((m - 3)(n - 1) + 1) + \\ 1 \quad + \cdots + \quad (n - 1) \end{array}$$

gives  $S_{m-1}(n) + S_3(n - 1) = 1 + m + \cdots + ((m - 2)(n - 1) + 1) = S_m(n)$ .

The geometrical illustrations of this property for  $m = 4$  and  $m = 5$  were given in the proofs of the Theon formula  $S_4(n) = S_3(n) + S_3$

$(n - 1)$  and the formula  $S_5(n) = S_4(n) + S_3(n - 1)$ , which are particular cases of the Nicomachus formula.

**1.2.12.** Bachet de Méziriac in his supplement of two books on the polygonal numbers of Diophantus (see [Dick05]) obtained the property, which allows to get any polygonal number using only triangular numbers: *any  $m$ -gonal number is equal to the sum of the triangular number, taking in the row the same place, and  $m - 3$  triangular numbers, taking in the row the previous place.* So, one has the following Bachet de Méziriac formula:<sup>3</sup>

$$S_m(n) = S_3(n) + (m - 3)S_3(n - 1).$$

One can obtain this equation, using the Diophantus' formula consecutively for  $(m - 1)$ -gonal,  $(m - 2)$ -gonal, etc., numbers:

$$S_m(n) = S_{m-1}(n) + S_3(n - 1), \quad S_m(n) = S_{m-2}(n) + 2S_3(n - 1), \dots, \\ S_m(n) = S_4(n) + (m - 4)S_3(n), \quad S_m(n) = S_3(n) + (m - 3)S_3(n - 1).$$

Of course, one can check this formula by direct computation:  $S_3(n) + (m - 3)S_3(n - 1) = \frac{n(n+1)}{2} + (m - 3)\frac{(n-1)n}{2} = \frac{n}{2}((m - 3)(n - 1) + n + 1) = \frac{n((m-2)n-m+4)}{2} = S_m(n)$ .

By induction, we have, for  $n = 2$ , that  $S_3(2) + (m - 3)S_3(1) = 3 + (m - 3) \cdot 1 = m = S_m(1)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned} S_3(n + 1) + (m - 3)S_3(n) \\ &= (S_3(n) + (m - 3)S_3(n - 1)) + (n + 1) + (m - 3)n \\ &= S_m(n) + ((m - 2)n + 1) = S_m(n + 1). \end{aligned}$$

A special summation of the form

$$S_3(n) + (m - 3)S_3(n - 1) = \begin{array}{r} 1 + 2 + \dots + n + \\ 1 + \dots + (n - 1) + \\ 1 + \dots + (n - 1) + \\ \dots \\ 1 + \dots + (n - 1) \end{array}$$

gives  $S_3(n) + (m - 3)S_3(n - 1) = 1 + (m - 1) + \dots + ((m - 2)(n - 1) + 1) = S_m(n)$ .

---

<sup>3</sup>It means that *any polygonal number is a linear combination of triangular numbers with non-zero coefficients.*

A geometrical illustration of this property for  $m = 6$  was given in the proof of the formula  $S_6(n) = S_3(n) + 3S_3(n - 1)$ , which is an particular case of the Bachet de Méziriac formula.

**1.2.13.** An additional useful equation has the following form:

$$S_m(n) = n + (m - 2)S_3(n - 1).$$

It can be obtained from the Bachet de Méziriac formula, using that  $S_3(n) = S_3(n - 1) + n$ :  $S_m(n) = S_3(n) + (m - 3)S_3(n - 1) = (S_3(n - 1) + n) + (m - 3)S_3(n - 1) = n + (m - 2)S_3(n - 1)$ .

Of course, one can check this formula by direct computation:  $n + (m - 2)S_3(n - 1) = n + (m - 2)\frac{(n-1)n}{2} = \frac{n}{2}((m - 2)(n - 1) + 2) = \frac{n((m-2)n-m+4)}{2} = S_m(n)$ .

By induction, we have, for  $n = 2$ , that  $2 + (m - 2)S_3(1) = 2 + (m - 2) \cdot 1 = m = S_m(1)$ , and, going from  $n$  to  $n + 1$ , we obtain

$$\begin{aligned} (n + 1) + (m - 2)S_3(n) &= (n + (m - 2)S_3(n - 1)) + 1 + (m - 2)n \\ &= S_m(n) + ((m - 2)n + 1) = S_m(n + 1). \end{aligned}$$

A special summation of the form

$$\begin{aligned} n + (m - 2)S_3(n - 1) &= \begin{array}{l} 1 + 2 + \dots + (n - 1) + \\ 1 + \dots + (n - 2) + (n - 1) + \\ 1 + \dots + (n - 2) + (n - 1) + \\ \dots \\ 1 + \dots + (n - 2) + (n - 1) + \\ \phantom{1 + \dots + (n - 2) +} + \phantom{(n - 1) +} n \end{array} \end{aligned}$$

gives  $n + (m - 2)S_3(n - 1) = 1 + (m - 1) + \dots + ((m - 2)(n - 2) + 1) + ((m - 2)(n - 1) + 1) = S_m(n)$ .

The geometrical illustration of this property for  $m = 8$  was given in the proof of the octagonal number theorem  $S_8(n) = n + 6S_3(n - 1)$  which is the most known case of it.

## 1.3 Square triangular numbers

**1.3.1.** A **square triangular number** (or **triangular square number**) is a number which is both, a triangular number and a perfect square. The first few square triangular numbers are 1, 36, 1225,

41616, 1413721, ... (Sloane's A001110). The indices of the corresponding square numbers are 1, 6, 35, 204, 1189, ... (Sloane's A001109), and the indices of the corresponding triangular numbers are 1, 8, 49, 288, 1681, ... (Sloane's A001108).

Since every triangular number is of the form  $\frac{1}{2}u(u+1)$ , and every square number is of the form  $v^2$ , in order to find all square triangular numbers, one seeks positive integers  $u$  and  $v$ , such that  $\frac{1}{2}u(u+1) = v^2$ .

The problem of finding square triangular numbers reduces to solving the Pell's equation  $x^2 - 2y^2 = 1$  in the following way:

$$\begin{aligned} \frac{1}{2}u(u+1) = v^2 &\Leftrightarrow u^2 + u = 2v^2 \Leftrightarrow 4u^2 + 4u + 1 \\ &= 8v^2 + 1 \Leftrightarrow (2u+1)^2 - 2(2v)^2 = 1. \end{aligned}$$

So, having a solution  $(u, v)$  of the equation  $\frac{1}{2}u(u+1) = v^2$  and taking  $x = 2u+1$  and  $y = 2v$ , one obtains a solution  $(x, y)$  of the Pell's equation  $x^2 - 2y^2 = 1$ . On the other hand, from a solution  $(x, y)$  of the Pell's equation  $x^2 - 2y^2 = 1$  one can obtain a solution  $(u, v)$  of the equation  $\frac{1}{2}u(u+1) = v^2$  with  $u = \frac{x-1}{2}$ , and  $v = \frac{y}{2}$ . Since the method of finding all solutions of a Pell's equation is well-known, one obtains the method of finding all triangular numbers, which are also square numbers.

**1.3.2.** There are infinitely many triangular numbers that are also square numbers. It was shown by Euler in 1730 (see [Dick05]). More exactly, Euler proved the following theorem: *all positive integer solutions of the equation  $\frac{1}{2}u(u+1) = v^2$  can be obtained by the formulas*

$$\begin{aligned} u_n &= \frac{(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n - 2}{4}, \quad \text{and} \\ v_n &= \frac{(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n}{4\sqrt{2}}, \end{aligned}$$

where  $n$  is any positive integer.

In fact (see, for example, [Buch09]), all non-negative integer solutions of the Pell's equation  $x^2 - Dy^2 = 1$  can be obtained by the formulas  $x = P_{kt-1}$ ,  $y = Q_{kt-1}$ , where  $P_i$  and  $Q_i$  are numerators and

denominators, respectively, of convergents in the decomposition

$$\begin{aligned}\sqrt{D} &= a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} \\ &= [a_0, a_1, \dots, a_k, a_1, \dots, a_k, a_1, \dots, a_k, \dots] = [a_0, (a_1, \dots, a_k)]\end{aligned}$$

of  $\sqrt{D}$  into a continued fraction,  $k$  is the length of the period of this decomposition, and  $t$  is any positive integer for which  $kt$  is even. For  $D = 2$ , it holds

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \dots}} = [1, 2, 2, 2, 2, \dots] = [1, (2)],$$

i.e.,  $k = 1$ , and all positive integer solutions of the equation  $x^2 - 2y^2 = 1$  have the form  $x = P_{2n-1}$ ,  $y = Q_{2n-1}$ ,  $n \in \mathbb{N}$ .

The numerators and denominators of the convergents of a continued fraction  $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} = [a_0, a_1, \dots, a_n, \dots]$  have the following property (see [Buch09]):

$$P_n = a_n P_{n-1} + P_{n-2}, \quad \text{and} \quad Q_n = a_n Q_{n-1} + Q_{n-2}$$

for any positive integer  $n$ ,  $n \geq 2$ . In particular, for the decomposition  $\sqrt{2} = [1, (2)]$  of  $\sqrt{2}$  into the continued fraction, we have:

- $P_0 = 1$ ,  $Q_0 = 1$  (since  $\frac{P_0}{Q_0} = 1 = \frac{1}{1}$ );
- $P_1 = 3$ ,  $Q_1 = 2$  (since  $\frac{P_1}{Q_1} = 1 + \frac{1}{2} = \frac{3}{2}$ );
- $P_n = 2P_{n-1} + P_{n-2}$ ,  $Q_n = 2Q_{n-1} + Q_{n-2}$  for  $n \geq 2$  (since  $a_n = 2$  for  $n \geq 1$ ).

Now it is easy to obtain the sequence  $P_1 = 3, Q_1 = 2, P_3 = 17, Q_3 = 12, P_5 = 99, Q_5 = 70, P_7 = 577, Q_7 = 408, \dots$ , giving the sequence  $(3, 2), (17, 12), (99, 70), (577, 408), (3363, 2378), \dots$  of all positive integer solutions  $(x_n, y_n), n \in \mathbb{N}$ , of the Pell's equation  $x^2 - 2y^2 = 1$ . Using above relations  $u_n = \frac{x_n - 1}{2} = \frac{P_{2n-1} - 1}{2}$  and  $v_n = \frac{y_n}{2} = \frac{Q_{2n-1}}{2}$ , one obtains the sequence

$$(1, 1), (8, 6), (49, 35), (288, 204), (1681, 1189), \dots$$

of all positive integer solutions  $(u_n, v_n), n \in \mathbb{N}$ , of the equation  $\frac{1}{2}u(u+1) = v^2$ .

It is easy to show (for example, by induction), that the numerators  $P_n$  and denominators  $Q_n$  of the convergents of the continued fraction  $1 + \frac{1}{2 + \frac{1}{2 + \dots}}$  =  $[1, (2)] = \sqrt{2}$  have the following form:

$$P_n = \frac{(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1}}{2}, \text{ and}$$

$$Q_n = \frac{(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}}{2\sqrt{2}}.$$

In fact, for  $n = 1$ , one has

$$\frac{(1 + \sqrt{2})^2 + (1 - \sqrt{2})^2}{2} = 3 = P_1, \text{ and}$$

$$\frac{(1 + \sqrt{2})^2 - (1 - \sqrt{2})^2}{2\sqrt{2}} = 2 = Q_1.$$

For  $n = 2$ , one gets

$$\frac{(1 + \sqrt{2})^3 + (1 - \sqrt{2})^3}{2} = 17 = P_3, \text{ and}$$

$$\frac{(1 + \sqrt{2})^3 - (1 - \sqrt{2})^3}{2\sqrt{2}} = 12 = Q_3.$$

Going from  $n - 2$  and  $n - 1$  to  $n$ , one has

$$\begin{aligned} P_n &= a_n P_{n-1} + P_{n-2} \\ &= 2 \cdot \frac{(1 + \sqrt{2})^n + (1 - \sqrt{2})^n}{2} + \frac{(1 + \sqrt{2})^{n-1} + (1 - \sqrt{2})^{n-1}}{2} \\ &= \frac{(1 + \sqrt{2})^{n-1}(2(1 + \sqrt{2}) + 1) + 2(1 - \sqrt{2})^{n-1}(2(1 - \sqrt{2}) + 1)}{2} \\ &= \frac{(1 + \sqrt{2})^{n-1}(1 + \sqrt{2})^2 + (1 - \sqrt{2})^{n-1}(1 - \sqrt{2})^2}{2}, \end{aligned}$$

i.e.,  $P_n = \frac{(1 + \sqrt{2})^{n+1} + (1 - \sqrt{2})^{n+1}}{2}$  holds. Similarly, we get

$$\begin{aligned} Q_n &= a_n Q_{n-1} + Q_{n-2} \\ &= 2 \cdot \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}} + \frac{(1 + \sqrt{2})^{n-1} - (1 - \sqrt{2})^{n-1}}{2\sqrt{2}} \\ &= \frac{(1 + \sqrt{2})^{n-1}(2(1 + \sqrt{2}) + 1) - (1 - \sqrt{2})^{n-1}(2(1 - \sqrt{2}) + 1)}{2\sqrt{2}} \\ &= \frac{(1 + \sqrt{2})^{n-1}(1 + \sqrt{2})^2 - (1 - \sqrt{2})^{n-1}(1 - \sqrt{2})^2}{2\sqrt{2}}, \end{aligned}$$

i.e.,  $Q_n = \frac{(1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1}}{2\sqrt{2}}$  holds.

Using the above equalities for odd indices, we obtain

$$P_{2n-1} = \frac{(1 + \sqrt{2})^{2n} + (1 - \sqrt{2})^{2n}}{2}, \quad \text{and}$$

$$Q_{2n-1} = \frac{(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}}{2\sqrt{2}}.$$

These formulas with the identities  $(1 \pm \sqrt{2})^2 = 3 \pm 2\sqrt{2}$  yield now<sup>4</sup> the formulas for  $u_n$  and  $v_n$ :

$$u_n = \frac{P_{2n-1} - 1}{2}$$

$$= \frac{(1 + \sqrt{2})^{2n} + (1 - \sqrt{2})^{2n} - 2}{4} = \frac{(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n - 2}{4},$$

$$v_n = \frac{Q_{2n-1}}{2} = \frac{(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}}{4\sqrt{2}}$$

$$= \frac{(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n}{4\sqrt{2}}.$$

For small values of  $n$  we obtain the following results:

- \* for  $n = 1$ , one has  $u_1 = 1$ ,  $v_1 = 1$ , and  $S_{4,3}(1) = S_3(1) = S_4(1) = 1$ ;
- \* for  $n = 2$ , one has  $u_2 = 8$ ,  $v_2 = 6$ , and  $S_{4,3}(2) = S_3(8) = S_4(6) = 36$ ;
- \* for  $n = 3$ , one has  $u_3 = 49$ ,  $v_3 = 35$ , and  $S_{4,3}(3) = S_3(49) = S_4(35) = 1225$ ;
- \* for  $n = 4$ , one has  $u_4 = 288$ ,  $v_4 = 204$ , and  $S_{4,3}(4) = S_3(288) = S_4(204) = 41616$ ;
- \* for  $n = 5$ , one has  $u_5 = 1681$ ,  $v_5 = 1189$ , and  $S_{4,3}(5) = S_3(1681) = S_4(1189) = 1413721$ ;
- \* for  $n = 6$ , one has  $u_6 = 9800$ ,  $v_6 = 6930$ , and  $S_{4,3}(6) = S_3(9800) = S_4(6930) = 48024900$ ;

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<sup>4</sup>Of course, the same result can be obtained using the following well-known rule: if  $(x_0, y_0)$  is the smallest positive integer solution of the Pell's equation  $x^2 - Dy^2 = 1$ , then all positive integer solutions  $(x, y)$  of the equation are given by  $x + y\sqrt{D} = \pm(x_0 + y_0\sqrt{D})^n$ ,  $n \in \mathbb{N}$ ; therefore,  $x = \frac{(x_0 + y_0\sqrt{D})^n + (x_0 - y_0\sqrt{D})^n}{2}$ , and  $y = \frac{(x_0 + y_0\sqrt{D})^n - (x_0 - y_0\sqrt{D})^n}{2\sqrt{D}}$ ,  $n \in \mathbb{N}$ . In our case  $D = 2$ ,  $(x_0, y_0) = (3, 2)$ , and we have  $x = \frac{(2+3\sqrt{2})^n + (2-3\sqrt{2})^n}{2}$ ,  $y = \frac{(2+3\sqrt{2})^n - (2-3\sqrt{2})^n}{2\sqrt{2}}$ .

\* for  $n = 7$ , one has  $u_7 = 57121$ ,  $v_7 = 40391$ , and  $S_{4,3}(7) = S_3(57121) = S_4(40391) = 1631432881$ .

**1.3.3.** Using the above formulas, one obtains now, that  $n$ -th square triangular number  $S_{4,3}(n)$  has the following form:

$$S_{4,3}(n) = \frac{1}{32}((17 + 12\sqrt{2})^n + (17 - 12\sqrt{2})^n - 2).$$

In fact,  $S_{4,3}(n) = v_n^2$ . Noting that  $(3 \pm 2\sqrt{2})^2 = 17 \pm 12\sqrt{2}$ , one gets

$$\begin{aligned} S_{4,3}(n) &= \left( \frac{(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n}{4\sqrt{2}} \right)^2 \\ &= \frac{1}{32}((17 + 12\sqrt{2})^n + (17 - 12\sqrt{2})^n - 2). \end{aligned}$$

**1.3.4.** As  $n$  becomes larger, the ratio  $\frac{u_n}{v_n}$  approaches  $\sqrt{2} = 1.4142\dots$ :

$$\begin{aligned} \frac{u_1}{v_1} &= \frac{1}{1} = 1; & \frac{u_2}{v_2} &= \frac{8}{6} = 1.3333\dots; & \frac{u_3}{v_3} &= \frac{49}{35} = 1.4; \\ \frac{u_4}{v_4} &= \frac{288}{204} = 1.4117\dots; & \frac{u_5}{v_5} &= \frac{1681}{1189} = 1.4137\dots; \\ \frac{u_6}{v_6} &= \frac{9800}{6930} = 1.4141\dots; & \frac{u_7}{v_7} &= \frac{57121}{40391} = 1.4142\dots \end{aligned}$$

In general, it holds  $\frac{u_n}{v_n} = \frac{P_{2n-1}-2}{Q_{2n-1}} = \frac{P_{2n-1}}{Q_{2n-1}} - \frac{2}{Q_{2n-1}}$ . Since  $Q_t \rightarrow \infty$  and  $\frac{P_t}{Q_t} \rightarrow \sqrt{2}$  for  $t \rightarrow \infty$ , we get  $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \sqrt{2}$ .

Similarly, the ratio  $\frac{S_{4,3}(n+1)}{S_{4,3}(n)}$  of successive square triangular numbers approaches  $17 + 12\sqrt{2} = 33.9705\dots$  (see [Wiki11]):

$$\begin{aligned} \frac{36}{1} &= 36; & \frac{1225}{36} &= 34.0277\dots; & \frac{41616}{1225} &= 33.9722\dots; \\ \frac{1413721}{41616} &= 33.9706\dots; & \frac{48024900}{1413721} &= 33.9705\dots \end{aligned}$$

**1.3.5.** There is an another way to obtain the consecutive members of the sequence of square triangular numbers (see [CoGu96]):  $n$ -th square triangular number  $S_{4,3}(n)$  is equal to  $(P_n Q_n)^2$ , where  $\frac{P_n}{Q_n}$  is  $n$ -th convergent in the decomposition of  $\sqrt{2}$  into the continued fraction.

In order to prove this fact, let us show that the numerators  $P_n$  and denominators  $Q_n$  of the convergents in the decomposition  $\sqrt{2} = [1, (2)]$  have the following property:

$$Q_{2n-1} = 2P_{n-1}Q_{n-1} \quad \text{for } n \in \mathbb{N}.$$

In fact, for  $n = 1$ , one has  $Q_1 = 2 = 2 \cdot 1 \cdot 1 = 2P_0Q_0$ . For  $n > 1$ , it holds

$$\begin{aligned} & 2 \frac{(1 + \sqrt{2})^n + (1 - \sqrt{2})^n}{2} \cdot \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}} \\ &= \frac{(1 + \sqrt{2})^{2n} - (1 - \sqrt{2})^{2n}}{2\sqrt{2}}. \end{aligned}$$

Hence, we get  $S_{4,3}(n) = v_n^2 = \left(\frac{Q_{2n-1}}{2}\right)^2 = (P_nQ_n)^2$ . Remind that  $P_0 = 1$ ,  $Q_0 = 1$ ,  $P_1 = 3$ ,  $Q_1 = 2$ , and  $P_n = 2P_{n-1} + P_{n-2}$ ,  $Q_n = 2Q_{n-1} + Q_{n-2}$  for any  $n \geq 2$ . So, we start with the fractions  $\frac{P_0}{Q_0} = \frac{1}{1}$  and  $\frac{P_1}{Q_1} = \frac{3}{2}$ . Moreover, for any  $n \geq 2$  we construct  $n$ -th convergent  $\frac{P_n}{Q_n}$  by doubling the  $(n-1)$ -th convergent and “adding” it to the  $(n-2)$ -th convergent. The first few elements of the described sequence are:

$$\frac{1}{1}, \frac{3}{2}, \frac{2 \cdot 3 + 1}{2 \cdot 2 + 1} = \frac{7}{5}, \frac{2 \cdot 7 + 3}{2 \cdot 5 + 2} = \frac{17}{12}, \frac{2 \cdot 17 + 7}{2 \cdot 12 + 5} = \frac{41}{29},$$

$$\frac{2 \cdot 41 + 17}{2 \cdot 29 + 12} = \frac{99}{70}, \frac{2 \cdot 99 + 41}{2 \cdot 70 + 29} = \frac{239}{169}.$$

The numerators and denominators of the obtained fractions permit to construct the first few square triangular numbers:

$$\begin{aligned} (1 \cdot 1)^2 &= 1, \quad (3 \cdot 2)^2 = 36, \quad (7 \cdot 5)^2 = 1225, \quad (17 \cdot 12)^2 = 41616, \\ (41 \cdot 29)^2 &= 1413721, \quad (99 \cdot 70)^2 = 48024900, \\ (239 \cdot 169)^2 &= 1631432881. \end{aligned}$$

**1.3.6.** Conversely, if one knows the value of  $n$ -th square triangular number  $S_{4,3}(n)$ , which is the  $v_n$ -th perfect square  $S_4(v_n) = v_n^2$  and the  $u_n$ -th triangular number  $S_3(u_n) = \frac{u_n(u_n+1)}{2}$ , one can easily obtain the

indices  $v_n$  and  $u_n$  of corresponding square and triangular numbers:

$$v_n = v_n(S_{4,3}(n)) = \sqrt{S_{4,3}(n)}, \quad \text{and}$$

$$u_n = u_n(S_{4,3}(n)) = \left\lfloor \sqrt{2S_{4,3}(n)} \right\rfloor.$$

The first equality is obvious, while the second one follows from the equality  $u_n(u_n + 1) = 2S_{4,3}(n)$  and the chain of simple inequalities:

$$\begin{aligned} u_n &= \sqrt{u_n^2} < \sqrt{u_n(u_n + 1)} \\ &= \sqrt{u_n^2 + u_n} < \sqrt{u_n^2 + 2u_n + 1} = \sqrt{(u_n + 1)^2} = u_n + 1. \end{aligned}$$

Hence, one has  $u_n < \sqrt{2S_{4,3}(n)} < u_n + 1$ , i.e.,  $u_n$  is the greatest integer less than or equal to  $\sqrt{2S_{4,3}(n)}$ . So, we get  $u_n = \lfloor \sqrt{2S_{4,3}(n)} \rfloor$ .

**1.3.7.** The above result of Euler, giving all indices  $v_n$  and  $u_n$  of the square and triangular numbers, which correspond to  $n$ -th square triangular number  $S_{4,3}(n) = S_4(v_n) = S_3(u_n)$ , is very beautiful, but almost useless to apply for big values of  $n$ . However, there are other ways for calculation indices  $v_n$  and  $u_n$ . In particular, *all positive integer solutions  $u_n$  and  $v_n$  of the equation  $\frac{1}{2}u(u+1) = v^2$  can be obtained by the following recurrent formulas:*

$$\begin{aligned} u_{n+1} &= 6u_n - u_{n-1} + 2, \quad u_1 = 1, \quad u_2 = 8, \quad \text{and} \\ v_{n+1} &= 6v_n - v_{n-1}, \quad v_1 = 1, \quad v_2 = 6. \end{aligned}$$

In fact, it holds  $v_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{4\sqrt{2}}$ . Let  $v_n = a - b$ , where  $a = \frac{(3+2\sqrt{2})^n}{4\sqrt{2}}$ , and  $b = \frac{(3-2\sqrt{2})^n}{4\sqrt{2}}$ . It implies  $v_{n+1} = a(3+2\sqrt{2}) - b(3-2\sqrt{2}) = 3(a-b) + 2\sqrt{2}(a+b)$ , and

$$\begin{aligned} v_{n-1} &= \frac{a}{3+2\sqrt{2}} - \frac{b}{3-2\sqrt{2}} = a(3-2\sqrt{2}) - b(3+2\sqrt{2}) \\ &= 3(a-b) - 2\sqrt{2}(a+b). \end{aligned}$$

Hence, one gets  $v_{n-1} + v_{n+1} = 6(a-b) = 6v_n$ . In other words,  $v_{n+1} = 6v_n - v_{n-1}$  holds.

Similarly, one has  $u_n = \frac{(3+\sqrt{2})^n + (3-2\sqrt{2})^n - 2}{4}$ . Let  $u_n = c + d - \frac{1}{2}$ , where  $c = \frac{(3+\sqrt{2})^n}{4}$ , and  $d = \frac{(3-2\sqrt{2})^n}{4}$ . It implies  $u_{n+1} = c(3+2\sqrt{2}) +$

$d(3 - 2\sqrt{2}) - \frac{1}{2} = 3(c + d) + 2\sqrt{2}(c - d) - \frac{1}{2}$ , and

$$\begin{aligned} u_{n-1} &= \frac{c}{3 + 2\sqrt{2}} + \frac{d}{3 - 2\sqrt{2}} - \frac{1}{2} \\ &= c(3 - 2\sqrt{2}) + d(3 + 2\sqrt{2}) - \frac{1}{2} = 3(c + d) - 2\sqrt{2}(c - d) - \frac{1}{2}. \end{aligned}$$

Hence, one gets  $u_{n+1} + u_{n-1} = 6(c + d) - 1 = 6(c + d - \frac{1}{2}) + 2 = 6u_n + 2$ . In other words,  $u_{n+1} = 6u_n - u_{n-1} + 2$  holds (see [Weis11]).

So, starting from  $v_1 = 1$  and  $v_2 = 6$ , one obtains  $v_3 = 35$ ,  $v_4 = 204$ ,  $v_5 = 1189$ ,  $v_6 = 6930$ ,  $v_7 = 40391$ , etc. Starting from  $u_1 = 1$  and  $u_2 = 8$ , one obtains  $u_3 = 49$ ,  $u_4 = 288$ ,  $u_5 = 1681$ ,  $u_6 = 9800$ ,  $u_7 = 57121$ , etc.

**1.3.8.** Now we can prove that *the sequence of square triangular numbers can be obtained by the following recurrent equation:*

$$S_{4,3}(n) = 34S_{4,3}(n-1) - S_{4,3}(n-2) + 2, \quad S_{4,3}(1) = 1, \quad S_{4,3}(2) = 36.$$

In fact, we have  $S_{4,3}(n) = v_n^2 = (a - b)^2$ ,  $S_{4,3}(n-1) = v_{n-1}^2 = (3(a-b) - 2\sqrt{2}(a+b))^2$ ,  $S_{4,3}(n+1) = v_{n+1}^2 = (3(a-b) + 2\sqrt{2}(a+b))^2$ .

Since  $ab = \frac{(3+2\sqrt{2})^n}{4\sqrt{2}} \cdot \frac{(3-2\sqrt{2})^n}{4\sqrt{2}} = \frac{1}{32}$ , we obtain

$$\begin{aligned} S_{4,3}(n+1) + S_{4,3}(n-1) &= 18(a-b)^2 + 16(a+b)^2 \\ &= 34(a-b)^2 + 64ab = 34S_{4,3}(n) + 2. \end{aligned}$$

In other words, the equation  $S_{4,3}(n) = 34S_{4,3}(n-1) - S_{4,3}(n-2) + 2$  holds (see [Weis11]).

**1.3.9.** An infinite subsequence  $S_{4,3}(k)$  of square triangular numbers can be generated (see [PSJW62]) by a more simple recurrent formula

$$S_{4,3}(k+1) = 4S_{4,3}(k)(8S_{4,3}(k) + 1), \quad S_{4,3}(1) = 1.$$

In fact, let  $S_3(u) = \frac{1}{2}u(u+1) = v^2 = S_4(v)$ . Then one gets

$$\begin{aligned} 4S_3(u)(8S_3(u) + 1) &= 2u(u+1)(4u(u+1) + 1) \\ &= \frac{4u(u+1)(4u(u+1) + 1)}{2} = S_3(4u(u+1)). \end{aligned}$$

On the other hand,

$$4S_4(v)(8S_4(v)+1) = 4v^2(4u(u+1)+1) = 4v^2(2u+1)^2 = S_4(v(2u+1)).$$

So, starting from the unity, we obtain by above recurrent formula an infinite subsequence of square triangular numbers 1, 36, 41616, 55420693056, 982865030092057414584576, ...

**1.3.10.** The above recurrent formula gives a more simple proof of the fact, that there are infinitely many square triangular numbers. Another proof of this fact can be obtained using the theory of *Pythagorean triples* (see [Sier03]).

At first, let us obtain a bijection between the set  $(u, v)$  of positive integer solutions of the equation  $\frac{1}{2}u(u+1) = v^2$  and the set  $(x, z)$  of positive integer solutions of the equation  $x^2 + (x+1)^2 = z^2$ . In fact, let  $(x, z)$  be a positive integer solution of the equation  $x^2 + (x+1)^2 = z^2$ . Then the pair  $(u, v)$  with  $u = z - x - 1$  and  $v = \frac{1}{2}(2x + 1 - z)$  gives a solution of the equation  $\frac{1}{2}u(u+1) = v^2$ :

$$\begin{aligned} \frac{1}{2}(z-x-1)(z-x) &= \left(\frac{1}{2}(2x+1-z)\right)^2 \Leftrightarrow \\ 2(z-x-1)(z-x) &= (2x+1-z)^2 \Leftrightarrow \\ 2z^2 + 2x^2 - 4xz - 2z + 2x &= 4x^2 + z^2 + 1 - 4xz + 4x - 2z \\ \Leftrightarrow z^2 &= 2x^2 + 2x + 1 \Leftrightarrow z^2 = x^2 + (x+1)^2. \end{aligned}$$

It is easy to see that it is a positive integer solution. As  $z^2 = x^2 + (x+1)^2$ , then  $z^2 > (x+1)^2$ , and  $z > x+1$ , i.e.,  $u = z - x - 1 \in \mathbb{N}$ . Similarly,  $z^2 = 2x^2 + 2x + 1 < 4x^2 + 2x + 1 = (2x+1)^2$ , and  $z < 2x+1$ , i.e.,  $2x+1-z > 0$ ; as  $z^2$  is a sum of two integers  $x^2$  and  $(x+1)^2$  of different parity, it is an odd number, and, hence,  $z$  itself is an odd number. Therefore,  $2x+1-z$  is an even number, and  $v = \frac{1}{2}(2x+1-z) \in \mathbb{N}$ .

Conversely, let the pair  $(u, v)$  be a positive integer solution of the equation  $\frac{1}{2}u(u+1) = v^2$ . Then the pair  $(x, z)$  of positive integers  $x = u + 2v$  and  $z = 2u + 2v + 1$  gives a solution of the equation  $x^2 + (x+1)^2 = z^2$ :

$$\begin{aligned} (u+2v)^2 + (u+2v+1)^2 &= (2u+2v+1)^2 \Leftrightarrow \\ 2u^2 + 8v^2 + 8uv + 1 + 2u + 4v &= 4u^2 + 4v^2 + 8uv + 1 + 4u + 4v \\ \Leftrightarrow 4v^2 = 2u^2 + 2u &\Leftrightarrow v^2 = \frac{1}{2}u(u+1). \end{aligned}$$

Now we are going to show that the equation  $x^2 + (x + 1)^2 = z^2$  has infinitely many positive integer solutions  $(x_n, z_n)$ ,  $n \in \mathbb{N}$ . Namely, starting from the pair  $(3, 5)$ , we note, that for a given solution  $(x, z)$  of the equation  $x^2 + (x + 1)^2 = z^2$  the pair  $(3x + 2z + 1, 4x + 3z + 2)$  also is a solution of this equation:

$$\begin{aligned} & (3x + 2z + 1)^2 + (3x + 2z + 2)^2 \\ &= (4x + 3z + 2)^2 \Leftrightarrow 18x^2 + 8z^2 + 5 + 24xz + 18x + 12z \\ &= 16x^2 + 9z^2 + 4 + 24xz + 16x + 12z \Leftrightarrow 2x^2 + 2x + 1 \\ &= z^2 \Leftrightarrow x^2 + (x + 1)^2 = z^2. \end{aligned}$$

Obviously,  $3x + 2z + 1 > x$  and  $4x + 3z + 2 > z$ , so, we get an infinite recurrent sequence  $(x_n, z_n)$ ,  $n \in \mathbb{N}$ , of solutions of the equation  $x^2 + (x + 1)^2 = z^2$ :

$$x_{n+1} = 3x_n + 2z_n + 1, \quad \text{and} \quad z_{n+1} = 4x_n + 3z_n + 2, \quad x_1 = 3, \quad z_1 = 5.$$

In fact, these pairs give all positive integer solutions of the equation  $x^2 + (x + 1)^2 = z^2$ . In order to prove it, let us consider a solution  $(x, z)$  of our equation with  $x > 3$  and check that the pair  $(3x - 2z + 1, 3z - 4x - 2)$  also gives a positive integer solution of this equation, but in this case it holds  $3z - 4x - 2 < z$ . So, we should check the inequalities  $3x - 2z + 1 > 0$  and  $0 < 3z - 4x - 2 < z$ , or, which is the same, the inequalities  $2z < 3x + 1$ ,  $3z > 4x + 2$ , and  $2z < 4x + 2$ . As  $3x + 1 < 4x + 2$  for  $x > 3$ , one should check just first two above inequalities. Since  $z^2 = 2x^2 + 2x + 1$ , so  $4z^2 = 8x^2 + 8x + 1 < 9x^2 + 6x + 1 = (3x + 1)^2$ , and  $2z < 3x + 1$ . Similarly,  $9z^2 = 18x^2 + 18x + 9 > 16x^2 + 16x + 4 = (4x + 2)^2$ , and  $3z > 4x + 2$ . So, using the operation  $g(x, z) = (3x - 2z + 1, 3z - 4x - 2)$ , which delimitates the value  $z$ , we necessary come to the pair  $(3, 5)$ ; it means that for some positive integer  $n$ , one gets  $g^n(x, z) = (3, 5)$ . Furthermore, the operation  $f(x, z) = (3x + 2z + 1, 4x + 3z + 2)$  is connected with the operation  $g$  by the formula  $f \cdot g(x, z) = f(3x - 2z + 1, 3z - 4x - 2) = (x, z)$ , and, hence, we obtain, that, for any positive integer  $k$ , it holds  $f^k \cdot g^k(x, z) = (x, z)$ . So, it is proven, that for any positive integer solution  $(x, z)$  of the equation

$x^2 + (x + 1)^2 = z^2$ , there exists some positive integer  $n$  (for which  $g^n(x, z) = (3, 5)$ ), such that  $f^n(3, 5) = (x, z)$ . In other words, all positive integer solutions of our equation belong to the following tree with the root  $(3, 5)$ :

$$(3, 5) \rightarrow (20, 29) \rightarrow (119, 169) \rightarrow (696, 985) \rightarrow \dots$$

As it was proven above, any solution  $(x_n, z_n)$  of the equation  $x^2 + (x + 1)^2 = z^2$  corresponds to a solution  $(u_n, v_n)$  of the equation  $\frac{1}{2}u(u+1) = v^2$  with  $u_n = z_n - x_n - 1$  and  $v_n = \frac{1}{2}(2x_n + 1 - z_n)$ . So,  $u_1 = z_1 - x_1 - 1 = 5 - 3 - 1 = 1$ ,  $v_1 = \frac{1}{2}(2x_1 + 1 - z_1) = \frac{1}{2}(2 \cdot 3 + 1 - 5) = 1$ , and  $u_{n+1} = z_{n+1} - x_{n+1} - 1 = (4x_n + 3z_n + 2) - (3x_n + 2z_n + 1) = x_n + z_n + 1$ ,  $v_{n+1} = \frac{1}{2}(2x_{n+1} + 1 - z_{n+1}) = \frac{1}{2}(2(3x_n + 2z_n + 1) + 1 - (4x_n + 2z_n + 2)) = x_n + \frac{z_n + 1}{2}$ . Since the sequences  $x_n$  and  $z_n$  are increasing, so are the sequences  $u_n$  and  $v_n$ , and, therefore, we can get all positive integer solutions of the equation  $\frac{1}{2}u(u + 1) = v^2$  in the form of an infinite sequence  $(u_n, v_n)$ ,  $n \in \mathbb{N}$ , depending on sequence  $(x_n, y_n)$ ,  $n \in \mathbb{N}$ :

$$u_1 = 1, \quad v_1 = 1, \quad \text{and} \quad u_{n+1} = x_n + z_n, \quad v_{n+1} = x_n + \frac{z_n + 1}{2},$$

where

$$\begin{aligned} x_1 &= 3, \quad z_1 = 5, \quad \text{and} \quad x_n = 3x_{n-1} + 2z_{n-1} + 1, \\ z_n &= 4x_{n-1} + 3z_{n-1} + 2 \quad \text{for } n \geq 2. \end{aligned}$$

So, we have proved, that there exist infinitely many triangular numbers  $S_3(u_n)$ , which are simultaneously the square numbers  $S_4(v_n)$ . Moreover, we get a method to find indices  $u_n$  and  $v_n$  of such numbers, using the tree  $(3, 5) \rightarrow (20, 29) \rightarrow (119, 169) \rightarrow (696, 985) \rightarrow \dots$  of all positive integer solutions of the Diophantine equation  $x^2 + (x + 1)^2 = z^2$ . So,  $u_1 = 1$  and  $v_1 = 1$ . Using the pair  $(x_1, z_1) = (3, 5)$ , we obtain  $u_2 = 3 + 5 = 8$ , and  $v_2 = 3 + \frac{5+1}{2} = 6$ . Using the pair  $(x_2, z_2) = (20, 29)$ , we obtain  $u_3 = 20 + 29 = 49$ , and  $v_3 = 20 + \frac{29+1}{2} = 35$ . Using the pair  $(x_3, z_3) = (119, 169)$  we get  $u_4 = 119 + 169 = 288$ , and  $v_4 = 119 + \frac{169+1}{2} = 3204$ . Using the pair  $(x_4, z_4) = (696, 985)$ , we get  $u_5 = 696 + 985 = 1681$ , and  $v_5 = 696 + \frac{985+1}{2} = 1189$ , and so on (see also **4.3.1**).

**1.3.11.** The generating function for the sequence of the square triangular numbers has the form  $f(x) = \frac{x(1+x)}{(1-x)(1-34x+x^2)}$  (see [SIP195]).

More exactly, it holds

$$\frac{x(1+x)}{(1-x)(1-34x+x^2)} = S_{4,3}(1)x + S_{4,3}(2)x^2 + S_{4,3}(3)x^3 + \dots + S_{4,3}(n)x^n + \dots, |x| < 4\sqrt{18} - 17.$$

In fact, let us consider the recurrent equation  $S_{3,4}(n+2) = 34S_{3,4}(n+1) - S_{3,4}(n) + 2$ . Going from  $n$  to  $n+1$ , one obtains  $S_{3,4}(n+3) = 34S_{3,4}(n+2) - S_{3,4}(n+1) + 2$ . Subtracting first equality from the second one, we get

$$S_{3,4}(n+3) - S_{3,4}(n+2) = 34S_{3,4}(n+2) - 34S_{3,4}(n+1) - S_{3,4}(n+1) + S_{3,4}(n).$$

Hence, we get the following linear recurrent equation:

$$S_{3,4}(n+3) - 35S_{3,4}(n+2) + 35S_{3,4}(n+1) - S_{3,4}(n) = 0.$$

It is a linear recurrent equation of 3-rd order with coefficients  $b_0 = 1, b_1 = -35, b_2 = 35$ , and  $b_3 = -1$ . Its initial values are  $S_{3,4}(1) = 1, S_{3,4}(2) = 36$ , and  $S_{3,4}(3) = 1225$ . Denoting  $S_{3,4}(n+1)$  by  $c_n$ , one obtains a linear recurrent equation

$$c_{n+3} - 35c_{n+2} + 35c_{n+1} - c_n = 0, \quad c_0 = 1, \quad c_1 = 36, \quad c_2 = 1225.$$

So, the generating function for the sequence of the square triangular numbers has the form

$$\frac{f(x)}{g(x)} = \frac{a_0 + a_1x + a_2x^2}{b_0 + b_1x + b_2x^2 + b_3x^3},$$

where  $b_0 = 1, b_1 = -35, b_2 = 35, b_3 = -1$ , and  $a_0 = b_0c_0 = 1, a_1 = b_0c_1 + b_1c_0 = 1 \cdot 36 + (-35) \cdot 1 = 1, a_2 = b_0c_2 + b_1c_1 + b_2c_0 = 1 \cdot 1225 + (-35) \cdot 36 + 35 \cdot 1 = 0$ .

The polynomial  $g(x) = 1 - 35x + 35x^2 - x^3 = (1-x)(1-34x+x^2)$  has three real roots  $x_1 = 1, x_2 = 17 + 4\sqrt{18}$ , and  $x_3 = 17 - 4\sqrt{18}$ . So, one has  $\min\{|x_1|, |x_2|, |x_3|\} = 4\sqrt{18} - 17 = 0.029\dots$ . Hence, the generating function for the sequence of the square triangular numbers has the form

$$\frac{1+x}{(1-x)(1-34x+x^2)} = S_{3,4}(1) + S_{3,4}(2)x + S_{3,4}(3)x^2 + \dots + S_{4,3}(n+1)x^n + \dots, |x| < 4\sqrt{18} - 17.$$

## 1.4 Other highly polygonal numbers

Square triangular numbers are the most known **highly polygonal numbers**, i.e., positive integers, which are polygonal in two or more ways. However, there are many similar classes of such polygonal numbers (see [Weis11]).

**1.4.1. A pentagonal triangular number** is a number which is simultaneously pentagonal and triangular. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(3u - 1) = \frac{1}{2}v(v + 1).$$

Completing the square gives  $(6u - 1)^2 - 3(2v + 1)^2 = -2$ . Substituting  $x = 6u - 1$  and  $y = 2v + 1$  gives the Pell-like Diophantine equation  $x^2 - 3y^2 = -2$ , which has positive integer solutions  $(x, y) = (5, 3), (19, 11), (71, 41), (265, 153), \dots$  (see, for example, [Nage51]). In terms of  $(u, v)$ , these solutions give  $(u, v) = (1, 1), (\frac{10}{3}, 5), (12, 20), (\frac{133}{3}, 76), (165, 285), \dots$ , of which the integer solutions are

$$(u, v) = (1, 1), (12, 20), (165, 285), (2296, 3976), (31977, 55385), \dots$$

(Sloane's A046174 and A046175). They correspond to the pentagonal triangular numbers (Sloane's A014979)

$$1, 210, 40755, 7906276, 1533776805, \dots$$

**1.4.2. A pentagonal square number** is a number which is simultaneously pentagonal and square. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(3u - 1) = v^2.$$

Completing the square gives  $(6u - 1)^2 - 24v^2 = 1$ . Substituting  $x = 6u - 1$  and  $y = 2v$  gives the Pell's equation  $x^2 - 6y^2 = 1$ . It has positive integer solutions  $(x, y) = (5, 2), (49, 20), (495, 198), \dots$ . In terms of  $(u, v)$ , these give  $(u, v) = (1, 1), (\frac{25}{3}, 10), (81, 99), (\frac{2401}{3}, 980), (7921, 9701), \dots$ , of which the integer solutions are

$$(u, v) = (1, 1), (81, 99), (7921, 9701), (776161, 950599), (76055841, 93149001), \dots$$

(Sloane's A046172 and A046173). They correspond to the pentagonal square numbers (Sloane's A036353)

$$1, 9801, 94109401, 903638458801, 8676736387298001, \dots$$

**1.4.3. A pentagonal square triangular number** is a number that is simultaneously pentagonal, square and triangular. This requires a solution to the system of Diophantine equations

$$\begin{cases} \frac{1}{2}l(3l-1) = v^2 \\ \frac{1}{2}u(u+1) = v^2. \end{cases}$$

Solutions of this system can be found by checking pentagonal triangular numbers up to some limit to see if any are also square, other than the trivial case 1. Using this approach shows that none of the first 9690 pentagonal triangular numbers are square. So, there is no other pentagonal square triangular number less than  $10^{22166}$ . It is almost certain, therefore, that no other solution exists, although no proof of this fact appeared in print yet (see [Weis11]).

**1.4.4.** Formally, an **hexagonal triangular number** is a number which is both hexagonal and triangular. However, it was proven, that  $S_6(n) = S_3(2n-1)$ , i.e., any hexagonal number is triangular, and the situation is trivial.

**1.4.5.** An **hexagonal square number** is a number which is both hexagonal and square. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$u(2u-1) = v^2.$$

Completing the square and rearranging gives  $(4u-1)^2 - 8v^2 = 1$ . Substituting  $x = 4u-1$  and  $y = 2v$  gives the Pell's equation  $x^2 - 2y^2 = 1$ . It has positive integer solutions  $(x, y) = (3, 2), (17, 12), (99, 70), (577, 408), \dots$ . In terms of  $(u, v)$ , these give  $(u, v) = (1, 1), (\frac{9}{2}, 6), (25, 35), (\frac{289}{2}, 204), \dots$ , of which the integer solutions are

$$(u, v) = (1, 1), (25, 35), (841, 1189), (28561, 40391), (970225, 1372105), \dots$$

(Sloane's A008844 and A046176). They correspond to the hexagonal square numbers (Sloane's A046177)

$$1, 1225, 1413721, 1631432881, 1882672131025, \dots$$

**1.4.6. An hexagonal pentagonal number** is a number which is simultaneously pentagonal and hexagonal. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(3u - 1) = v(2v - 1).$$

Completing the square and rearranging gives  $(6u - 1)^2 - 3(4v - 1)^2 = -2$ . Substituting  $x = 6u - 1$  and  $y = 4v - 1$  gives the Pell-like equation  $x^2 - 3y^2 = -2$ . Its first few positive integer solutions are  $(x, y) = (1, 1), (5, 3), (19, 11), (71, 74), (265, 153), (989, 571), \dots$ . In terms of  $(u, v)$ , these give the solutions  $(u, v) = (\frac{1}{3}, \frac{1}{2}), (1, 1), (\frac{10}{3}, 3), (12, \frac{21}{2}), (\frac{133}{3}, \frac{77}{2}), (165, 143), \dots$ , of which the integer solutions are

$$(u, v) = (1, 1), (165, 143), (31977, 27693), (6203341, 5372251), \\ (1203416145, 1042188953), \dots$$

(Sloane's A046178 and A046179). They correspond to the hexagonal pentagonal numbers (Sloane's A046180)

$$1, 40755, 1533776805, 57722156241751, 2172315626468283465, \dots$$

**1.4.7. An heptagonal triangular number** is a number which is simultaneously heptagonal and triangular. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(5u - 3) = \frac{1}{2}v(v + 1).$$

Completing the square and rearranging gives  $(10u - 3)^2 - 5(2v + 1)^2 = 4$ . Substituting  $x = 10u - 3$  and  $y = 2v + 1$  gives the Pell-like equation  $x^2 - 5y^2 = 4$ . It has the positive integer solutions  $(x, y) = (3, 1), (7, 3), (18, 8), (47, 21), (322, 144), \dots$ . The integer solutions in  $u$  and  $v$  are given by

$$(u, v) = (1, 1), (5, 10), (221, 493), (1513, 3382), (71065, 158905), \dots$$

(Sloane's A046193 and A039835). They correspond to the heptagonal triangular numbers (Sloane's A046194)

$$1, 55, 121771, 5720653, 12625478965, \dots$$

**1.4.8.** An **heptagonal square number** is a number which is simultaneously heptagonal and square. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(5u - 3) = v^2.$$

Completing the square and rearranging gives  $(10u - 3)^2 - 40v^2 = 9$ . Substituting  $x = 10u - 3$  and  $y = 2v$  gives the Pell-like equation  $x^2 - 10y^2 = 9$ . It has positive integer solutions  $(x, y) = (7, 2), (13, 4), (57, 18), (253, 80), (487, 154), \dots$ . The integer solutions in  $u$  and  $v$  are given then by

$$(u, v) = (1, 1), (6, 9), (49, 77), (961, 1519), (8214, 12987), \dots$$

(Sloane's A046195 and A046196). They correspond to the heptagonal square numbers (Sloane's A036354)

$$1, 81, 5929, 2307361, 168662169, \dots$$

**1.4.9.** An **heptagonal pentagonal number** is a number which is simultaneously heptagonal and pentagonal. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(5u - 3) = \frac{1}{2}v(3v - 1).$$

Completing the square and rearranging gives  $3(10u - 3)^2 - 5(6v - 1)^2 = 22$ . Substituting  $x = 10u - 3$  and  $y = 6v - 1$  gives the Pell-like equation  $3x^2 - 5y^2 = 22$ . It has the positive integer solutions  $(x, y) = (3, 1), (7, 5), (17, 13), (53, 41), (133, 103), \dots$ . The integer solutions in  $u$  and  $v$  are given then by

$$(u, v) = (1, 1), (42, 54), (2585, 3337), (160210, 206830), \\ (9930417, 12820113), \dots$$

(Sloane's A046198 and A046199). They correspond to the heptagonal pentagonal numbers (Sloane's A048900)

$$1, 4347, 16701685, 64167869935, 246532939589097, \dots$$

**1.4.10.** An **heptagonal hexagonal number** is a number which is simultaneously heptagonal and hexagonal. Such numbers

correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(5u - 3) = v(2v - 1).$$

Completing the square and rearranging gives  $(10u - 3)^2 - 5(4v - 1)^2 = 4$ . Substituting  $x = 10u - 3$  and  $y = 4v - 1$  gives the Pell-like equation  $x^2 - 5y^2 = 4$ . It has positive integer solutions  $(x, y) = (3, 1), (7, 3), (18, 8), (47, 21), (123, 55), \dots$ . The integer solutions in  $u$  and  $v$  are given by

$$(u, v) = (1, 1), (221, 247), (71065, 79453), (22882613, 25583539), \\ (7368130225, 8237820025), \dots$$

(Sloane's A048902 and A048901). They correspond to the heptagonal hexagonal numbers (Sloane's A048903)

$$1, 121771, 12625478965, 1309034909945503, \\ 135723357520344181225, \dots$$

**1.4.11.** An **octagonal triangular number** is a number which is simultaneously octagonal and triangular. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$u(3u - 2) = \frac{v(v + 1)}{2}.$$

Completing the square and rearranging gives  $8(3u - 1)^2 - 3(2v + 1)^2 = 5$ . Substituting  $x = 2(2u - 1)$  and  $y = 2v + 1$  gives the Pell-like equation  $2x^2 - 3y^2 = 5$ . It has the positive integer solutions  $(x, y) = (2, 1), (4, 3), (16, 13), (38, 31), (158, 129), (376, 307), \dots$ . These give the solutions  $(u, v) = (\frac{2}{3}, 0), (1, 1), (3, 6), (\frac{20}{3}, 15), (\frac{80}{3}, 64), (63, 153), \dots$ , of which the integer solutions are

$$(u, v) = (1, 1), (3, 6), (63, 153), (261, 638), (6141, 15041), \dots$$

(Sloane's A046181 and A046182). They correspond to the octagonal triangular numbers (Sloane's A046183)

$$1, 21, 11781, 203841, 113123361, \dots$$

**1.4.12.** An **octagonal square number** is a number which is simultaneously octagonal and square. Such numbers correspond to the quadratic Diophantine equation

$$u(3u - 2) = v^2.$$

Completing the square and rearranging gives  $(3u - 1)^2 - 3v^2 = 1$ . Substituting  $x = 3u - 1$  and  $y = v$  gives the Pell's equation  $x^2 - 3y^2 = 1$ . Its first few positive integer solutions are  $(x, y) = (2, 1), (7, 4), (26, 15), (97, 56), (362, 209), \dots$ . These give the solutions  $(u, v) = (1, 1), (\frac{8}{3}, 4), (9, 15), (\frac{98}{3}, 56), (121, 209), \dots$ , of which the integer solutions are

$$(u, v) = (1, 1), (9, 15), (121, 209), (1681, 2911), (23409, 40545), \dots$$

(Sloane's A046184 and A028230). They correspond to the octagonal square numbers (Sloane's A036428)

$$1, 225, 43681, 8473921, 1643897025, \dots$$

**1.4.13.** An **octagonal pentagonal number** is a number which is simultaneously octagonal and pentagonal. Such numbers correspond to the Diophantine equation

$$u(3u - 2) = \frac{1}{2}v(3v - 1).$$

Completing the square and rearranging gives  $(6v - 1)^2 - 8(3u - 1)^2 = -7$ . Substituting  $x = 6v - 1$  and  $y = 2(3u - 1)$  gives the Pell-like equation  $x^2 - 2y^2 = -7$ . Its first few positive integer solutions are  $(x, y) = (1, 2), (5, 4), (11, 8), (31, 22), (65, 46), \dots$ . These give the solutions  $(u, v) = (\frac{1}{3}, \frac{2}{3}), (1, 1), (2, \frac{5}{3}), (\frac{16}{3}, 4), (11, 8), \dots$ , of which the integer solutions are

$$(u, v) = (1, 1), (11, 8), (1025, 725), (12507, 8844), \\ (1182657, 836265), \dots$$

(Sloane's A046187 and A046188). They correspond to the octagonal pentagonal numbers (Sloane's A046189)

$$1, 176, 1575425, 234631320, 2098015778145, \dots$$

**1.4.14.** An **octagonal hexagonal number** is a number which is simultaneously octagonal and hexagonal. Such numbers correspond

to the positive integer solutions of the Diophantine equation

$$u(3u - 2) = v(2v - 1).$$

Completing the square and rearranging gives  $8(3u - 1)^2 - 3(4u - 1)^2 = 5$ . Substituting  $x = 2(3u - 1)$  and  $y = 4v - 1$  gives the Pell-like equation  $2x^2 - 3y^2 = 5$ . The first few positive integer solutions are  $(x, y) = (2, 1), (4, 3), (16, 13), (38, 31), (158, 129), (376, 307), \dots$ . These give the solutions  $(u, v) = (\frac{2}{3}, \frac{1}{2}), (1, 1), (3, \frac{7}{2}), (\frac{20}{3}, 8), (\frac{80}{3}, \frac{65}{2}), (63, 77), \dots$ , of which the integer solutions are

$$(u, v) = (1, 1), (63, 77), (6141, 7521), (601723, 736957), \\ (58962681, 72214241), \dots$$

(Sloane's A046190 and A046191). They correspond to the octagonal hexagonal numbers (Sloane's A046192)

$$1, 11781, 113123361, 1086210502741, 10429793134197921, \dots$$

**1.4.15.** An **octagonal heptagonal number** is a number which is simultaneously octagonal and heptagonal. Such numbers correspond to the Diophantine equation

$$\frac{1}{2}u(5u - 3) = v(3v - 2).$$

Completing the square and rearranging gives  $3(10u - 3)^2 - 40(3v - 1)^2 = -13$ . Substituting  $x = 10u - 3$  and  $y = 2(3v - 1)$  gives the Pell-like equation  $3x^2 - 10y^2 = -13$ . Its first few positive integer solutions are  $(x, y) = (3, 2), (7, 4), (73, 40), (157, 86), \dots$ . The integer solutions in  $u$  and  $v$  are given then by

$$(u, v) = (1, 1), (345, 315), (166145, 151669), (80081401, 73103983), \\ (38599068993, 35235967977), \dots$$

(Sloane's A048904 and A048905). They correspond to the octagonal heptagonal numbers (Sloane's A048906)

$$1, 297045, 69010153345, 16032576845184901, \\ 3724720317758036481633, \dots$$

**1.4.16.** A **nonagonal triangular number** is a number which is simultaneously nonagonal and triangular. Such numbers correspond

to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(7u - 5) = \frac{1}{2}v(v + 1).$$

Completing the square and rearranging gives  $(14u - 5)^2 - 7(2v + 1)^2 = 18$ . Substituting  $x = 14u - 5$  and  $y = 2v + 1$  gives the Pell-like equation  $x^2 - 7y^2 = 18$ . It has positive integer solutions  $(x, y) = (5, 1), (9, 3), (19, 7), (61, 23), (135, 51), (299, 113), (971, 367), \dots$  The integer solutions in  $u$  and  $v$  are given then by

$$(u, v) = (1, 1), (10, 25), (154, 406), (2449, 6478), (39025, 103249), \dots$$

(Sloane's A048907 and A048908). They correspond to the nonagonal triangular numbers (Sloane's A048909)

$$1, 325, 82621, 20985481, 5330229625, \dots$$

**1.4.17. A nonagonal square number** is a number which is simultaneously nonagonal and square. Such numbers correspond to the Diophantine equation

$$\frac{1}{2}v(7v - 5) = u^2.$$

Completing the square and rearranging gives  $(14u - 5)^2 - 56v^2 = 25$ . Substituting  $x = 14u - 5$  and  $y = 2v$  gives the Pell-like equation  $x^2 - 14y^2 = 25$ . It has positive integer solutions  $(x, y) = (9, 2), (23, 6), (75, 20), (247, 66), (681, 182), (2245, 600), \dots$  The corresponding integer solutions in  $u$  and  $v$  are

$$(u, v) = (1, 1), (2, 3), (18, 33), (49, 91), (529, 989), \dots$$

(Sloane's A048910 and A048911). They give the nonagonal square numbers (Sloane's A036411)

$$1, 9, 1089, 8281, 978121, \dots$$

**1.4.18. A nonagonal pentagonal number** is a number which is simultaneously nonagonal and pentagonal. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(7u - 5) = \frac{1}{2}v(3v - 1).$$

Completing the square and rearranging gives  $3(14v - 5)^2 - 7(6u - 1)^2 = 68$ . Substituting  $x = 14v - 5$  and  $y = 6u - 1$  gives the Pell-like equation  $3x^2 - 7y^2 = 68$ . Its positive integer solutions  $(x, y)$  correspond to positive integer solutions in  $u$  and  $v$  when

$$(u, v) = (1, 1), (14, 21), (7189, 10981), (165026, 252081), \\ (86968201, 132846121), \dots$$

(Sloane's A048913 and A048914). It gives the nonagonal pentagonal numbers (Sloane's A048915)

$$1, 651, 180868051, 95317119801, 26472137730696901, \dots$$

**1.4.19. A nonagonal hexagonal number** is a number which is simultaneously nonagonal and hexagonal. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(7u - 5) = v(2v - 1).$$

Completing the square and rearranging gives  $(14v - 5)^2 - 7(4u - 1)^2 = 18$ . Substituting  $x = 14v - 5$  and  $y = 4u - 1$  gives the Pell-like equation  $x^2 - 7y^2 = 18$ . It has positive integer solutions  $(x, y) = (5, 1), (9, 3), (19, 17), (61, 23), (135, 51), (509, 193), \dots$ . The corresponding integer solutions in  $u$  and  $v$  are

$$(u, v) = (1, 1), (10, 13), (39025, 51625), (621946, 822757), \\ (2517635809, 3330519121), \dots$$

(Sloane's A048916 and A048917). They give the nonagonal hexagonal numbers (Sloane's A048918)

$$1, 325, 5330229625, 1353857339341, 22184715227362706161, \dots$$

**1.4.20. A nonagonal heptagonal number** is a number which is simultaneously nonagonal and heptagonal. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(7u - 5) = \frac{1}{2}v(5v - 4).$$

Completing the square and rearranging gives  $(14v - 5)^2 - 7(10u - 3)^2 = 62$ . Substituting  $x = 14v - 5$  and  $y = 10u - 3$  gives the Pell-like equation  $x^2 - 7y^2 = 62$ . Its positive integer solutions  $(x, y)$

correspond to positive integer solutions in  $u$  and  $v$  when

$$(u, v) = (1, 1), (88, 104), (12445, 14725), (1767052, 2090804), (250908889, 296879401), \dots$$

(Sloane's A048919 and A048920). It gives the nonagonal heptagonal numbers (Sloane's A048921)

$$1, 26884, 542041975, 10928650279834, 220343446399977901, \dots$$

**1.4.21. A nonagonal octagonal number** is a number which is simultaneously nonagonal and octagonal. Such numbers correspond to the positive integer solutions of the Diophantine equation

$$\frac{1}{2}u(7u - 5) = v(3v - 2).$$

Completing the square and rearranging gives  $(14v - 5)^2 - 56(3u - 1)^2 = 19$ . Substituting  $x = 14v - 5$  and  $y = 3u - 1$  gives the Pell-like equation  $3x^2 - 56y^2 = 19$ . Its positive integer solutions  $(x, y)$  correspond to the following positive integer solutions in  $u$  and  $v$ :

$$(u, v) = (1, 1), (425, 459), (286209, 309141), (192904201, 208360351), (130017145025, 140434567209), \dots$$

(Sloane's A048922 and A048923). It gives the nonagonal octagonal numbers (Sloane's A048924)

$$1, 631125, 286703855361, 130242107189808901, 59165603001256545014625, \dots$$

**1.4.22.** The table below summarizes the set of  $m$ -gonal  $k$ -gonal numbers for small values of  $m$  and  $k$ .

$m$	$k$	Sequence	Sloane
4	3	1, 36, 1225, 41616, 1413721, ...	A001110
5	3	1, 210, 40755, 7906276, 1533776805, ...	A014979
5	4	1, 9801, 94109401, 903638458801, 8676736387298001, ...	A036353
6	3	1, 6, 15, 28, 45, ... (i.e., all hexagonal numbers)	A000384
6	4	1, 1225, 1413721, 1631432881, 1882672131025, ...	A046177
6	5	1, 40755, 1533776805, 57722156241751, 2172315626468283465, ...	A046180
7	3	1, 55, 121771, 5720653, 12625478965, ...	A046194
7	4	1, 81, 5929, 2307361, 168662169, ...	A036354
7	5	1, 4347, 16701685, 64167869935, 246532939589097, ...	A048900
7	6	1, 121771, 12625478965, 1309034909945503, 135723357520344181225, ...	A048903
8	3	1, 21, 11781, 203841, 113123361, ...	A046183
8	4	1, 225, 43681, 8473921, 1643897025, ...	A036428
8	5	1, 176, 1575425, 234631320, 2098015778145, ...	A046189
8	6	1, 11781, 113123361, 1086210502741, 10429793134197921, ...	A046192
8	7	1, 297045, 69010153345, 16032576845184901, 3724720317758036481633, ...	A048906
9	3	1, 325, 82621, 20985481, 5330229625, ...	A048909
9	4	1, 9, 1089, 8281, 978121, ...	A036411
9	5	1, 651, 180868051, 95317119801, 26472137730696901, ...	A048915
9	6	1, 325, 5330229625, 1353857339341, 22184715227362706161, ...	A048918
9	7	1, 26884, 542041975, 10928650279834, 220343446399977901, ...	A048921
9	8	1, 631125, 286703855361, 130242107189808901, 59165603001256545014625, ...	A048924

## 1.5 Amount of a given number in all polygonal numbers

**1.5.1.** The problem about amount of a given number in all polygonal numbers was formulated by Diophantus in his treatise *On Polygonal Numbers* ([Diop], [Dick05]), of which only a part survives: *find how many times a given number  $N$  is contained among all polygonal numbers.*

In other words, for a given positive integer  $N$ , one must find all positive integers  $m$  and  $n$  for which  $N = S_m(n)$ . Obviously,  $N = S_N(2)$ , and without loss of generality we can restrict the consideration to  $m \geq 3$  and  $n \geq 2$ . Since  $S_m(n) = \frac{n((m-2)n-m+4)}{2}$ , one obtains a chain of the following equalities:

$$\begin{aligned} N &= \frac{(m-2)n^2 - (m-2)n + 2n}{2}, \\ 2N &= (m-2)n^2 - (m-2)n + 2n, \\ 2N - 2n &= (m-2)(n^2 - n), \quad m-2 = \frac{2N-2n}{n(n-1)}. \end{aligned}$$

Decomposition of the fraction  $\frac{2N-2n}{n(n-1)}$  into a sum of elementary fractions  $\frac{2N-2n}{n(n-1)} = \frac{2N-2}{n-1} - \frac{2N}{n}$  gives

$$m-2 = \frac{2N-2}{n-1} - \frac{2N}{n}.$$

The positive integer numbers  $n$  and  $n-1$  are relatively prime. So, in above formula for the positive integer  $m-2$ , the number  $2N-2$  is divided by  $n-1$ , and the number  $2N$  is divided by  $n$ .

Hence, for finding all polygonal numbers which coincide with  $N$ , one can use the following algorithm:

- find all positive integer divisors of the number  $2N$ ;
- find all positive integer divisors of the number  $2N-2$ ;
- from first sequence choose numbers, which are on one unity greater than some number from second sequence: these numbers correspond to  $n$ ;
- find  $m = \frac{2N-2}{n-1} - \frac{2N}{n} + 2$ .

Let, for example,  $N = 7$ . All positive integer divisors of  $2N = 14$  are 1, 2, 7, 14. All positive integer divisors of  $2N - 2 = 12$  are 1, 2, 3, 4, 6, 12. Let us choose from the first set numbers  $2 = 1 + 1$  and  $7 = 6 + 1$ . So, one has  $n \in \{2, 7\}$ . For  $n = 2$ , one obtains  $m = \frac{12}{1} - \frac{14}{2} + 2 = 12 - 7 + 2 = 7$ , and  $S_m(n) = S_7(2) = 7$ ; for  $n = 7$ , it holds  $m = \frac{12}{6} - \frac{14}{7} + 2 = 2$ , and one can put  $S_m(n) = S_2(7) = 7$  (remains, that  $S_2(n) = n$  are linear numbers).

If  $N = 105$ , then  $2N = 210$ , and  $2N - 2 = 208$ . All positive integer divisors of 210 are 1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210. All positive integer divisors of 208 are 1, 2, 4, 8, 13, 16, 26, 52, 104, 208. Then one gets  $n \in \{2, 3, 5, 14, 105\}$ . For  $n = 2$ , one has  $m = \frac{208}{1} - \frac{210}{2} + 2 = 105$ ; so  $N$  is  $S_{105}(2)$ . For  $n = 3$ , it holds  $m = \frac{208}{2} - \frac{210}{3} + 2 = 36$ ; so  $N$  is  $S_{36}(3)$ . For  $n = 5$ , one gets  $m = \frac{208}{4} - \frac{210}{5} + 2 = 12$ ; so  $N$  is  $S_{12}(5)$ . For  $n = 14$ ,  $m = \frac{208}{13} - \frac{210}{14} + 2 = 3$ ; so  $N$  is  $S_3(14)$ . For  $n = 105$ ,  $m = \frac{208}{104} - \frac{210}{105} + 2 = 2$ ; so  $N$  is  $S_2(105)$ .

**1.5.2.** Call a positive integer  **$k$ -highly polygonal number**, if it is  $m$ -polygonal in  $k$  or more ways out of  $m = 3, 4, \dots$  up to some limit.

Then the first few **2-highly polygonal numbers** up to  $m = 16$  are 1, 6, 9, 10, 12, 15, 16, 21, 28, 36, ... (Sloane's A090428). In fact, 1 is  $m$ -gonal for any  $m$ , 6 is 3- and 6-gonal, 9 is 4- and 9-gonal, 10 is 3- and 10-gonal, 12 is 5- and 12-gonal, 15 is 3-, 6- and 15-gonal, 16 is 4- and 16-gonal, 21 is 3- and 8-gonal, 28 is 3- and 6-gonal, 36 is 3-, 4- and 13-gonal, etc.

Similarly, the first few **3-highly polygonal numbers** up to  $m = 16$  are 1, 15, 36, 45, 325, 561, 1225, 1540, 3025, 4186, ... (Sloane's A062712). In fact, 1 is  $m$ -gonal for any  $m$ , 15 is 3-, 6- and 15-gonal, 36 is 3-, 4- and 13-gonal, 45 is 3-, 6- and 16-gonal, 325 is 3-, 6- and 9-gonal, 561 is 3-, 6- and 12-gonal, 1225 is 3-, 4- and 6-gonal, 1540 is 3-, 6- and 10-gonal, 3025 is 4-, 12- and 15-gonal, 4186 is 3-, 6- and 13-gonal, etc.

There are no **4-highly polygonal numbers** of this type (i.e., up to  $m = 16$ ) less than  $10^{12}$ , except 1 (see [Weis11]).

**1.5.3.** Consider now one more simple question: how to check whether a number  $N$  is an  $m$ -gonal number  $S_m(n)$  for some  $n$ ? An arbitrary number  $N$  can be checked for  $m$ -gonality as follows. It is

easy to prove the following *Diophantus' identity*:

$$8(m-2)S_m(n) + (m-4)^2 = (2n(m-2) - (m-4))^2.$$

In fact, since  $S_m(n) = \frac{n((m-2)n-m+4)}{2}$ , one obtains

$$\begin{aligned} 8(m-2)S_m(n) + (m-4)^2 &= 4(m-2)n((m-2)n-m+4) + (m-4)^2 \\ &= 4(m-2)^2n^2 - 4(m-2)n(m-4) + (m-4)^2 \\ &= (2n(m-2) - (m-4))^2. \end{aligned}$$

This identity implies that the value  $8(m-2)S_m(n) + (m-4)^2$  is a perfect square for any  $m$ -gonal number  $S_m(n)$ . So, in order to check if an arbitrary positive integer  $N$  is an  $m$ -gonal number, one can check if the value  $8(m-2)N + (m-4)^2$  is a perfect square. If it is not, the number  $N$  can not be  $m$ -gonal. If it is a perfect square  $S^2$ , then solving equation  $S = 2n(m-2) - m + 4$  for  $n$  gives  $n = \frac{S+m-4}{2(m-2)}$  (see [Dick05], [Weis11]).

For example, in order to check if the number 1540 is an 10-gonal number, let us consider the number  $8(m-2)N + (m-4)^2$  for  $m = 10$ . In fact, this number  $8 \cdot 8 \cdot 1540 + 6^2 = 98596$  is a perfect square:  $98596 = 314^2$ , i.e.,  $S = 314$ . The equation  $S = 2n(m-2) - m + 4$  with  $S = 314$  and  $m = 10$  obtains the form  $314 = 16n - 6$ , giving  $n = 20$ . Therefore, the number  $N = 1540$  is the 20-th 10-gonal number:  $1540 = S_{10}(20)$ .

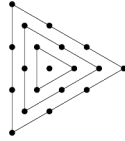
On the other hand, it is easy to show that the number 1540 is not an 8-gonal number. In fact, the number  $8(m-2)N + (m-4)^2$  for  $m = 8$  has the form  $8 \cdot 6 \cdot 1540 + 4^2 = 73936$  and is not a perfect square:  $73936 \neq a^2$ ,  $a \in \mathbb{Z}$ .

## 1.6 Centered polygonal numbers

Behind classical polygonal numbers, there are many other numbers, which can be constructed in the plane from points (or balls). The centered polygonal numbers form the next important class of such numbers.

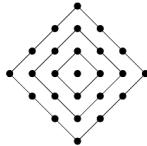
**1.6.1.** The **centered polygonal numbers** (or, sometimes, **polygonal numbers of the second order**) form a class of figurate numbers, in which layers of polygons are drawn centered about a point. Each centered polygonal number is formed by a central dot, surrounded by polygonal layers with a constant number of sides. Each side of a polygonal layer contains one dot more than any side of the previous layer, so starting from the second polygonal layer each layer of a **centered  $m$ -gonal number** contains  $m$  more points than the previous layer.

So, a **centered triangular number** represents a triangle with a dot in the center and all other dots surrounding the center in successive triangular layers. The following image shows the building of the centered triangular numbers using the associated figures: at each step the previous figure is surrounded by a triangle of new points.



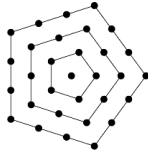
The first few centered triangular numbers are 1, 4, 10, 19, 31, 46, 64, 85, 109, 136, ... (Sloane's A005448).

A **centered square number** is consisting of a central dot with four dots around it, and then additional dots in the gaps between adjacent dots.



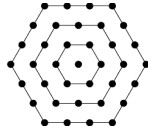
The first few centered square numbers are 1, 5, 13, 25, 41, 61, 85, 113, 145, 181, ... (Sloane's A001844).

A **centered pentagonal number** represents a pentagon with a dot in the center and all other dots surrounding the center in successive pentagonal layers.



The first few centered pentagonal numbers are 1, 6, 16, 31, 51, 76, 106, 141, 181, 226, ... (Sloane's A005891).

A **centered hexagonal number** represents a hexagon with a dot in the center and all other dots surrounding the center dot in a hexagonal lattice.



The first few centered hexagonal numbers are 1, 7, 19, 37, 61, 91, 127, 169, 217, 271, ... (Sloane's A003215). Centered hexagonal numbers are most known among centered polygonal numbers. Usually, they are called **hex numbers**.

Following this procedure, we can construct **centered heptagonal numbers** 1, 8, 22, 43, 71, 106, 148, 197, 253, 316, ... (Sloane's A069099), **centered octagonal numbers** 1, 9, 25, 49, 81, 121, 169, 225, 289, 361, ... (Sloane's A016754), **centered nonagonal numbers** 1, 10, 28, 55, 91, 136, 190, 253, 325, 406, ... (Sloane's A060544), **centered decagonal numbers** 1, 11, 31, 61, 101, 151, 211, 281, 361, 451, ... (Sloane's A062786), **centered hendecagonal numbers** 1, 12, 34, 67, 111, 166, 232, 309, 397, 496, ... (Sloane's A069125), **centered dodecagonal numbers** 1, 13, 37, 73, 121, 181, 253, 337, 433, 541, ... (Sloane's A003154), etc.

**1.6.2.** Algebraically,  $n$ -th centered  $m$ -gonal number  $CS_m(n)$  is obtained as the sum of the first  $n$  elements of the sequence  $1, m, 2m, 3m, \dots$ . So, by definition, it holds

$$CS_m(n) = 1 + m + 2m + \dots + (n-1)m.$$

In particular, we get

$$\begin{aligned} CS_3(n) &= 1 + 3 + 6 + 9 + \cdots + 3(n-1), \\ CS_4(n) &= 1 + 4 + 8 + 12 + \cdots + 4(n-1), \\ CS_5(n) &= 1 + 5 + 10 + 15 + \cdots + 5(n-1), \\ CS_6(n) &= 1 + 6 + 12 + 18 + \cdots + 6(n-1), \\ CS_7(n) &= 1 + 7 + 14 + 21 + \cdots + 7(n-1), \\ CS_8(n) &= 1 + 8 + 16 + 24 + \cdots + 8(n-1). \end{aligned}$$

The above formula implies the following recurrent formula for the centered  $m$ -gonal numbers:

$$CS_m(n+1) = CS_m(n) + nm, \quad CS_m(1) = 1.$$

In particular, we get

$$\begin{aligned} CS_3(n+1) &= CS_3(n) + 3n, & CS_4(n+1) \\ &= CS_4(n) + 4n, & CS_5(n+1) = CS_5(n) + 5n, \\ CS_6(n+1) &= CS_6(n) + 6n, & CS_7(n+1) \\ &= CS_7(n) + 7n, & CS_8(n+1) = CS_8(n) + 8n. \end{aligned}$$

Since  $m+2m+\cdots+(n-1)m = m(1+2+\cdots+(n-1)) = m\frac{(n-1)n}{2}$ , one obtains the following general formula for  $n$ -th centered  $m$ -gonal number:

$$CS_m(n) = 1 + m\frac{(n-1)n}{2} = \frac{mn^2 - mn + 2}{2}.$$

In particular, we have

$$\begin{aligned} CS_3(n) &= \frac{3n^2 - 3n + 2}{2}, & CS_4(n) &= 2n^2 - 2n + 1, \\ CS_5(n) &= \frac{(5n^2 - 5n + 2)}{2}, & CS_6(n) &= 3n^2 - 3n + 1, \\ CS_7(n) &= \frac{7n^2 - 7n + 2}{2}, & CS_8(n) &= 4n^2 - 4n + 1. \end{aligned}$$

These formulas for centered  $m$ -gonal numbers with  $3 \leq m \leq 30$ , as well as the first few elements of the corresponding sequences and the numbers of these sequences in the Sloane's *On-Line Encyclopedia*

of Integer Sequences (OEIS, [Sloa11]) classification, are given in the table below.

Name	Formula											Sloane
Cent. triangular	$\frac{1}{2}(3n^2 - 3n + 2)$	1	4	10	19	31	46	64	85	109	136	A005448
Cent. square	$2n^2 - 2n + 1$	1	5	13	25	41	61	85	113	145	181	A001844
Cent. pentagonal	$\frac{1}{2}(5n^2 - 5n + 2)$	1	6	16	31	51	76	106	141	181	226	A005891
Cent. hexagonal	$3n^2 - 3n + 1$	1	7	19	37	61	91	127	169	217	271	A003215
Cent. heptagonal	$\frac{1}{2}(7n^2 - 7n + 2)$	1	8	22	43	71	106	148	197	253	316	A069099
Cent. octagonal	$4n^2 - 4n + 1$	1	9	25	49	81	121	169	225	289	361	A016754
Cent. nonagonal	$\frac{1}{2}(9n^2 - 9n + 2)$	1	10	28	55	91	136	190	253	325	406	A060544
Cent. decagonal	$5n^2 - 5n + 1$	1	11	31	61	101	151	211	281	361	451	A062786
Cent. hendecagonal	$\frac{1}{2}(11n^2 - 11n + 2)$	1	12	34	67	111	166	232	309	397	496	A069125
Cent. dodecagonal	$6n^2 - 6n + 1$	1	13	37	73	121	181	253	337	433	541	A003154
Cent. tridecagonal	$\frac{1}{2}(13n^2 - 13n + 2)$	1	14	40	79	131	196	274	365	469	586	A069126
Cent. tetradecagonal	$7n^2 - 7n + 1$	1	15	43	85	141	211	295	393	505	631	A069127
Cent. pentadecagonal	$\frac{1}{2}(15n^2 - 15n + 2)$	1	16	46	91	151	226	316	421	541	676	A069128
Cent. hexadecagonal	$8n^2 - 8n + 1$	1	17	49	97	161	241	337	449	577	721	A069129
Cent. heptadecagonal	$\frac{1}{2}(17n^2 - 17n + 2)$	1	18	52	103	171	256	358	477	613	766	A069130
Cent. octadecagonal	$9n^2 - 9n + 1$	1	19	55	109	181	271	379	505	649	811	A069131
Cent. nonadecagonal	$\frac{1}{2}(19n^2 - 19n + 2)$	1	20	58	115	191	286	400	533	685	856	A069132
Cent. icosagonal	$10n^2 - 10n + 1$	1	21	61	121	201	301	421	561	721	901	A069133
Cent. icosihenagonal	$\frac{1}{2}(21n^2 - 21n + 2)$	1	22	64	127	211	316	442	589	757	946	A069178
Cent. icosidigonal	$11n^2 - 11n + 1$	1	23	67	133	221	331	463	617	793	991	A069173
Cent. icositrigonal	$\frac{1}{2}(23n^2 - 23n + 2)$	1	24	70	139	231	346	484	645	829	1036	A069174
Cent. icositetragonal	$12n^2 - 12n + 1$	1	25	73	145	241	361	505	673	865	1081	A069190
Cent. icosipentagonal	$\frac{1}{2}(25n^2 - 25n + 2)$	1	26	76	151	251	376	526	701	901	1126	
Cent. icosihexagonal	$\frac{1}{3}n^2 - 13n + 1$	1	27	79	157	261	391	546	728	936	1170	
Cent. icosiheptagonal	$\frac{1}{2}(27n^2 - 27n + 2)$	1	28	82	163	271	406	568	757	972	1216	
Cent. icosioctagonal	$14n^2 - 14n + 1$	1	29	84	169	281	421	589	785	1009	1261	
Cent. icosinonagonal	$\frac{1}{2}(29n^2 - 29n + 2)$	1	30	88	175	291	436	610	813	1045	1306	
Cent. triacontagonal	$15n^2 - 15n + 1$	1	31	91	181	301	451	631	841	1081	1351	

**1.6.3.** The generating function for the sequence  $CS_m(1), CS_m(2), \dots, CS_m(n), \dots$  of the centered  $m$ -gonal numbers has the form  $f(x) = \frac{x(1+(m-2)x+x^2)}{(1-x)^3}$  (see [Sloa11]), i.e., it holds

$$\frac{x(1+(m-2)x+x^2)}{(1-x)^3} = CS_m(1)x + CS_m(2)x^2 + CS_m(3)x^3 + \dots + CS_m(n)x^n + \dots, \quad |x| < 1.$$

In particular, one gets

$$\frac{x(x^2+x+1)}{(1-x)^3} = x + 4x^2 + 10x^3 + \dots + CS_3(n)x^n + \dots, \quad |x| < 1;$$

$$\frac{x(x+1)^2}{(1-x)^3} = x + 5x^2 + 13x^3 + \dots + CS_4(n)x^n + \dots, \quad |x| < 1;$$

$$\frac{x(x^2+3x+1)}{(x-1)^3} = x + 6x^2 + 16x^3 + \dots + CS_5(n)x^n + \dots, \quad |x| < 1;$$

$$\frac{x(x^2+4x+1)}{(1-x)^3} = x + 7x^2 + 19x^3 + \dots + CS_6(n)x^n + \dots, \quad |x| < 1.$$

In fact, let us consider the recurrent equation  $CS_m(n+1) = CS_m(n) + nm$ . Going from  $n$  to  $n+1$ , one obtains  $CS_m(n+2) = CS_m(n+1) + (n+1)m$ . Subtracting first equality from second one, we get  $CS_m(n+2) - CS_m(n+1) = CS_m(n+1) - CS_m(n) + m$ , i.e.,  $CS_m(n+2) = 2CS_m(n+1) - CS_m(n) + m$ . Similarly, one obtains  $CS_m(n+3) = 2CS_m(n+2) - CS_m(n+1) + m$ , and

$$CS_m(n+3) - CS_m(n+2) = 2CS_m(n+2) - 2CS_m(n+1) - CS_m(n+1) + CS_m(n),$$

i.e.,  $CS_m(n+3) = 3CS_m(n+2) - 3CS_m(n+1) + CS_m(n)$ . Hence, we get for the sequence of the centered  $m$ -gonal numbers the following linear recurrent equation:

$$CS_m(n+3) - 3CS_m(n+2) + 3CS_m(n+1) - CS_m(n) = 0.$$

It is a linear recurrent equation of 3-rd order with coefficients  $b_0 = 1, b_1 = -3, b_2 = 3, b_3 = -1$ . Its initial values are  $CS_m(1) = 1, CS_m(2) = m+1, CS_m(3) = 3m+1$ . Denoting  $CS_m(n+1)$  by  $c_n$ , one can rewrite above equation as

$$c_{n+3} - 3c_{n+2} + 3c_{n+1} - c_n = 0, \quad c_0 = 1, \quad c_1 = m+1, \quad c_2 = 3m+1.$$

Therefore, the generating function for the sequence of the centered  $m$ -gonal numbers has the form

$$\frac{f(x)}{g(x)} = \frac{a_0 + a_1x + a_2x^2}{b_0 + b_1x + b_2x^2 + b_3x^3},$$

where  $b_0 = 1, b_1 = -3, b_2 = 3, b_3 = -1$ , and  $a_0 = b_0c_0 = 1, a_1 = b_0c_1 + b_1c_0 = 1 \cdot (m+1) + (-3) \cdot 1 = m-2, a_2 = b_0c_2 + b_1c_1 + b_2c_0 = 1 \cdot (3m+1) + (-3)(m+1) + 3 \cdot 1 = 1$ . Since  $g(x) = 1 - 3x + 3x^2 - x^3 = (1-x)^3$  has three coinciding roots  $x_1 = x_2 = x_3 = 1$ , the generating function for the sequence of the centered  $m$ -gonal numbers has the form

$$\frac{1 + (m-2)x + x^2}{(1-x)^3} = CS_m(1) + CS_m(2)x + CS_m(3)x^2 + \dots + CS_m(n)x^{n-1} + \dots, \quad |x| < 1.$$

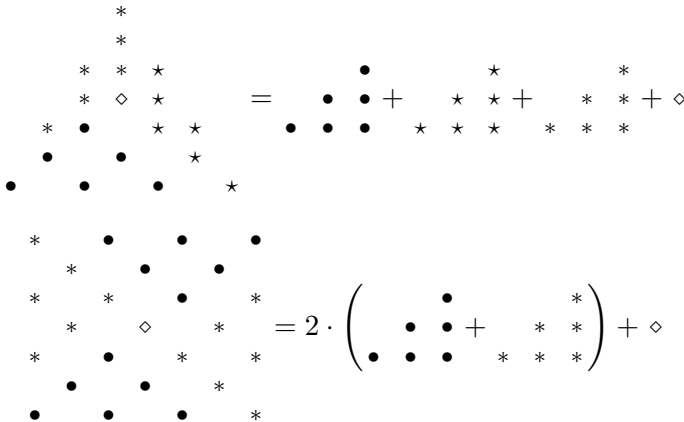
**1.6.4.** Now we are going to consider some interesting properties of centered polygonal numbers. (see [Weis11], [Wiki11], [CoGu96], [SIP195], [Gard88]).

**I.** Obviously,  $n$ -th centered  $m$ -gonal number  $CS_m(n)$  can be made up of a central point and  $m$  copies of the  $(n-1)$ -th triangular number  $S_3(n-1)$  surrounding the central dot:

$$CS_m(n) = 1 + mS_3(n-1).$$

In fact, this property follows from the formula  $CS_m(n) = 1 + m\frac{n(n-1)}{2}$  for the centered polygonal numbers, using that  $\frac{n(n-1)}{2} = S_3(n-1)$ .

The geometrical interpretation of this property is very natural. On the pictures below it is given, with  $n = 4$ , for centered triangular and centered square numbers.

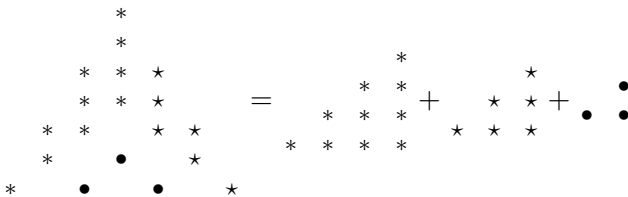


**II.** It is easy to see that each centered triangular number from 10 onwards is the sum of three consecutive ordinary triangular numbers:

$$CS_3(n) = S_3(n) + S_3(n-1) + S_3(n-2), \quad n \geq 3.$$

In fact,  $\frac{1}{2}((n-2)(n-1) + (n-1)n + n(n+1)) = \frac{1}{2}(3n^2 - 3n + 2)$ .

The geometrical illustration of this property is given below for  $n = 4$ .

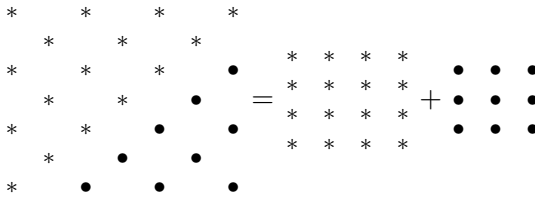


**III.** Similarly, *any centered square number is the sum of two consecutive square numbers:*

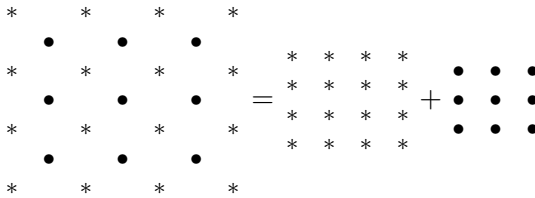
$$CS_4(n) = S_4(n) + S_4(n - 1).$$

In fact, the formula  $CS_4(n) = 2n^2 - 2n + 1$  for  $n$ -th centered square number can be rewritten as  $CS_4(n) = n^2 + (n^2 - 2n + 1) = n^2 + (n - 1)^2$ .

The geometrical illustration of this property can be easily obtained, if we remind, that  $n$ -th square number is the sum of the first  $n$  odd positive integers. In the picture below it is given for  $n = 4$ .



However, one can find both squares  $S_4(n)$  and  $S_4(n - 1)$  properly, as shown on the following picture:



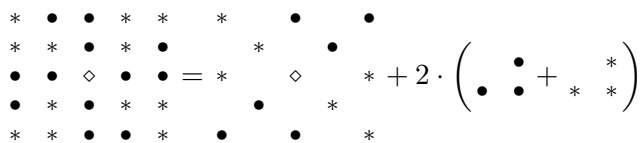
**IV.** Centered square numbers can be obtained using square and triangular numbers together:

$$CS_4(n) = S_4(2n - 1) - 4S_3(n - 1).$$

In fact,  $S_4(2n - 1) = (2n - 1)^2 = 4n^2 - 4n + 1 = (2n^2 - 2n + 1) + (2n^2 - 2n) = (2n^2 - 2n + 1) + 4\frac{n(n-1)}{2} = CS_4(n) + 4S_3(n - 1)$ .

The geometrical illustration of this fact, rewritten in the form  $S_4(2n - 1) = CS_4(n) + 4S_3(n - 1)$ , is implied by the fact that any centered square number is made up by four copies of a given triangular number and one central point. On the picture below, constructed for  $n = 3$ , it is shown, as 5-th square number can be constructed from eight copies of  $S_3(2)$  plus one point, giving the 3-rd centered

square number (four copies of  $S_3(2)$  plus one point) and four “free” copies of  $S_3(2)$ .

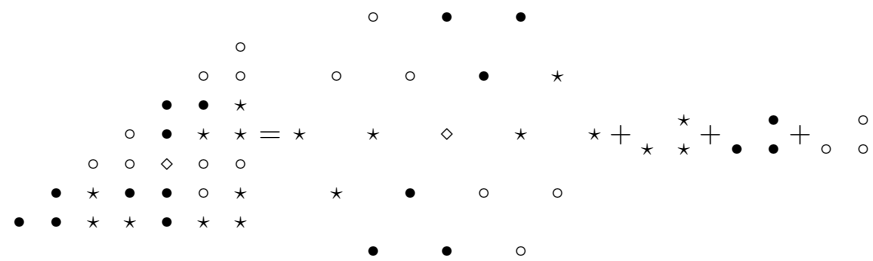


V. The similar connection exists between hex and triangular numbers:

$$CS_6(n) = S_3(3n - 2) - 3S_3(n - 1).$$

In fact,  $S_3(3n - 2) = \frac{(3n-2)(3n-1)}{2} = \frac{1}{2}(9n^2 - 9n + 2) = \frac{6n^2-6n+2}{2} + 3\frac{n(n-1)}{2} = CS_6(n) + 3S_3(n - 1)$ .

The geometrical illustration of this fact, rewritten in the form  $S_3(3n - 2) = CS_6(n) + 3S_3(n - 1)$ , is implied by the fact that any centered hexagonal number is made up by six copies of a given triangular number and one central point. On the picture below, constructed for  $n = 3$ , it is shown, as 7-th triangular number can be constructed from the nine copies of  $S_3(2)$  plus one point, giving 3-rd centered hexagonal number (six copies of  $S_3(2)$  plus one point) and three “free” copies of  $S_3(2)$ .



VI. Moreover, any centered hexagonal number is the difference of two consecutive perfect cubes:

$$CS_6(n) = n^3 - (n - 1)^3.$$

In fact, the formula  $CS_6(n) = 3n^2 - 3n + 1$  for  $n$ -th centered hexagonal number can be seen as  $CS_6(n) = n^3 - (n^3 - 3n^2 + 3n - 1) = n^3 - (n - 1)^3$ .

**VII.** Viewed from the opposite perspective, above property yields that *the sum of the first  $n$  centered hexagonal numbers is a perfect cube*:

$$CS_6(1) + CS_6(2) + \dots + CS_6(n) = n^3.$$

It follows immediately from the equation  $CS_6(n) = n^3 - (n - 1)^3$ , using the *telescoping summation*:

$$\begin{aligned} CS_6(1) + CS_6(2) + CS_6(3) + \dots + CS_6(n) \\ = (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \dots + (n^3 - (n - 1)^3) \\ = n^3 - 0^3 = n^3. \end{aligned}$$

**VIII.** Moreover, the centered hexagonal numbers satisfy the recurrence equation

$$CS_6(n) = 2CS_6(n - 1) - CS_6(n - 2) + 6.$$

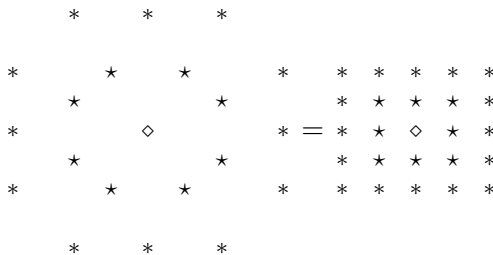
In fact,  $2CS_6(n - 1) - CS_6(n - 2) + 6 = 2(3(n - 1)^2 - 3(n - 1) + 2) - (3(n - 2)^2 + 3(n - 2) - 1) = 3n^2 - 3n + 1 = CS_6(n)$ .

**IX.** It is easy to check that *any centered octagonal number is equal to a square number with odd index*:

$$CS_8(n) = S_4(2n - 1).$$

In fact, the formula  $4n^2 - 4n + 1$  for  $n$ -th centered octagonal number  $CS_8(n)$  can be seen as  $(2n - 1)^2$ .

The geometrical illustration below, given for  $n = 3$ , shows, that centered octagonal number  $CS_8(n)$  and the squared number  $S_4(2n - 1)$  are numerically equal, consisting from one central point, and several equal sets of the points, belonging to the concentring circles, surrounding this central point, but these “circles” are differently arranged on the plane.



**X.** Similarly, every centered nonagonal number is a triangular number:

$$CS_9(n) = S_3(3n - 2).$$

In fact,  $CS_9(n) = \frac{9n^2 - 9n + 2}{2} = \frac{(3n-2)(3n-1)}{2} = S_3(3n - 2)$ . So, the sequence  $CS_9(1), CS_9(2), \dots, CS_9(n), \dots$  produces every third triangular number, starting with 1.

**XI.** The  $n$ -th centered dodecagonal number  $CS_{12}(n) = 1 + 12S_3(n)$  can be made up of a central point and 12 copies of the  $(n - 1)$ -th triangular number. It corresponds to the number of cells of generated *Chinese checker's board* (or *centered hexagram*), which is called **star number** and denoted by  $S(n)$ . So, any centered dodecagonal number coincides with the corresponding star number:

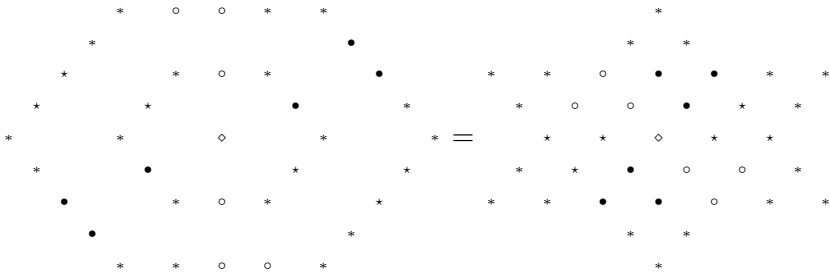
$$CS_{12}(n) = S(n).$$

In fact, by definition,  $n$ -th star number  $S(n)$  is constructed as  $n$ -th centered hexagonal number  $CS_6(n) = 1 + 6S_3(n - 1)$  with six copies of the  $(n - 1)$ -th triangular number  $S_3(n - 1)$  appended to each side. The classical Chinese checker's board, shown on the picture below, has a total of  $121 = S(5)$  holes.



So, we have  $S(n) = CS_6(n) + 6S_3(n - 1) = (1 + 6S_3(n - 1)) + 6S_3(n - 1) = 1 + 12S_3(n - 1)$ . Hence,  $n$ -th star number is numerically equal to  $n$ -th centered dodecagonal number, but is differently arranged.

The geometrical illustration of this fact for  $n = 3$  is given below.



The first few star numbers are 1, 13, 37, 73, 121, 181, 253, 337, 433, 541, ... (Sloane's A003154).

**XII.** The centered dodecagonal numbers (i.e., star numbers) satisfy the linear recurrent equation

$$CS_{12}(n) = CS_{12}(n - 1) + 12(n - 1).$$

In fact, it holds  $CS_{12}(n - 1) + 12(n - 1) = (1 + 12\frac{(n-1)(n-2)}{2}) + 12(n - 1) = 1 + 6(n - 1)(n - 2) + 12(n - 1)1 + 6(n - 1)(n - 2 + 2) = 1 + 6(n - 1)n = CS_{12}(n)$ .

**XIII.** Since  $CS_6(n) = 3n^2 - 3n + 1$  and  $CS_{12}(n) = 6n^2 - 6n + 1$ , we obtain that  $CS_6(n) = \frac{CS_{12}(n)+1}{2}$ . So, one gets  $CS_6(n)CS_{12}(n) = \frac{CS_{12}(n)(CS_{12}(n)+1)}{2}$ . In other words, *the product of  $n$ -th centered hexagonal and  $n$ -th centered dodecagonal number is always a triangular number*.

$$CS_6(n)CS_{12}(n) = S_3(CS_{12}(n)).$$

**1.6.5.** Let us consider now some centered polygonal numbers, which in the same time belong to other classes of figurate numbers.

**I.** At first, we list the centered  $m$ -gonal numbers, which are the classical  $m$ -gonal numbers. Just as in the case with classical polygonal numbers, the first centered  $m$ -gonal number is 1. Thus, for any  $m$ , the number 1 is both,  $m$ -gonal and centered  $m$ -gonal. The next number, which is both  $m$ -gonal and centered  $m$ -gonal, can be found using the formula  $\frac{m^3 - m^2 + 2}{2}$ , which tells us that 10 is both triangular and centered triangular, 25 is both square and centered square, 51 is both pentagonal and centered pentagonal, and so on (see [Wiki11]).

In general, this formula shows, that *the  $m$ -gonal number with index  $m + 1$  coincides with the centered  $m$ -gonal number with index  $m$ :*

$$S_m(m + 1) = CS_m(m).$$

In fact,  $n$ -th  $m$ -gonal number has the form  $S_m(n) = \frac{n(n(m-2)-m+4)}{2}$ , and the  $k$ -th centered  $m$ -gonal number has the form  $CS_m(k) = 1 + m\frac{(k-1)k}{2}$ . For  $n = m + 1$  and  $k = m$  one obtains:

$$\begin{aligned} \frac{(m + 1)((m + 1)(m - 2) - m + 4)}{2} &= \frac{(m + 1)(m^2 - 2m + 2)}{2} \\ &= \frac{m^3 - m^2 + 2}{2}, \quad \text{and} \\ 1 + m\frac{m(m - 1)}{2} &= \frac{2 + m^2(m - 1)}{2} = \frac{m^3 - m^2 + 2}{2}. \end{aligned}$$

**II. A triangular centered triangular number** is a number, which is simultaneously triangular and centered triangular. The first few such numbers are (Sloane's A128862)

$$1, 10, 136, 1891, 26335, \dots$$

In fact,  $S_3(u) = CS_3(v)$  for  $u = 1, 4, 16, 61, 229, \dots$  (Sloane's A133161), and  $v = 1, 3, 10, 36, 133, \dots$  (Sloanes A102871). These indices are found by solving the Diophantine equation

$$\frac{u^2 + u}{2} = \frac{3v^2 - 3v + 2}{2}.$$

**III. A square centered triangular number** is a number, which is simultaneously square and centered triangular. The first few such numbers are

$$1, 4, 64, 361, 6241, \dots$$

In fact,  $S_4(u) = CS_3(v)$  for  $u = 1, 2, 8, 19, 79, \dots$  (Sloane's A129445), and  $v = 1, 2, 7, 16, 65, \dots$  (see Sloane's A129444). These indices are found by solving the Diophantine equation

$$u^2 = \frac{3v^2 - 3v + 2}{2}.$$

**IV. A triangular centered hexagonal number (or triangular hex number)** is a number, which is simultaneously triangular

and centered hexagonal. The first few such numbers are (Sloane's A006244)

$$1, 91, 8911, 873181, 85562821, \dots$$

In fact,  $S_3(u) = CS_6(v+1)$  for  $u = 1, 13, 133, 1321, 13081, \dots$  (Sloane's A031138), and  $v = 0, 5, 54, 539, 5340$  (Sloane's A087125). These indices are found by solving the Diophantine equation

$$\frac{1}{2}u(u+1) = 3v^2 + 3v + 1.$$

**V. A square centered hexagonal number (or square hex number)** is a number, which is simultaneously square and centered hexagonal. The first few such numbers are (Sloane's A006051)

$$1, 169, 32761, 6355441, 1232922769, \dots$$

In fact,  $S_4(u) = CS_6(v+1)$  for  $u = 1, 13, 181, 2521, 35113, \dots$  (Sloane's A001570), and  $v = 0, 7, 104, 1455, 20272, \dots$  (Sloane's A001921). These indices are found by solving the Diophantine equation

$$u^2 = 3v^2 + 3v + 1.$$

The only centered hexagonal number that is both square and triangular is 1 ([Weis11]).

**VI. A triangular star number** is a number, which is simultaneously triangular and centered dodecagonal. The first few such numbers are (Sloane's A006060)

$$1, 253, 49141, 9533161, 1849384153, \dots$$

In fact,  $S_3(u) = CS_6(v)$  for  $u = 1, 22, 313, 4366, 60817, \dots$  (Sloane's A068774) and  $v = 1, 7, 91, 1261, 17557, \dots$  (Sloane's A068775). These indices are found by solving the Diophantine equation

$$\frac{u(u+1)}{2} = 6v^2 - 6v + 1.$$

**VII. A square star number** is a number, which is simultaneously square and centered dodecagonal. The first few such numbers are (Sloane's A006061)

$$1, 121, 11881, 1164241, 114083761, \dots$$

In fact,  $S_4(u) = CS_6(v)$  for  $u = 1, 11, 109, 1079, 10681, \dots$  (Sloane's A054320), and  $v = 1, 5, 45, 441, 4361, \dots$  (Sloane's A068778). These indices are found by solving the Diophantine equation

$$u^2 = 6v^2 - 6v + 1.$$

**VIII. A hex star number (or centered hexagonal star number)** is number, which is simultaneously centered hexagonal and centered dodecagonal. The first few such numbers are

(Sloane's A006062)

$$1, 37, 1261, 42841, 1455337, \dots$$

In fact,  $CS_6(u) = CS_{12}(v)$  for  $u = 1, 4, 21, 120, 697, \dots$  (see Sloane's A046090), and  $v = 1, 3, 15, 85, 493, \dots$  (see Sloane's A011900). These indices are found by solving the Diophantine equation

$$3u^2 - 3u + 1 = 6v^2 - 6v + 1.$$

## 1.7 Other plane figurate numbers

**1.7.1.** The **pronic number** (or **heteromecic number**) is a positive integer, which can be represented as a product of two consecutive integers. So, by definition,  $n$ -th pronic number  $P(n)$  has the following form:

$$P(n) = n(n + 1).$$

These numbers are sometimes called **oblong numbers** because they are represented by a rectangle with the sides  $n$  and  $n + 1$ , i.e., are figurate in this way:



The first few pronic numbers are 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, ... (Sloane's A002378).

Obviously,  $n$ -th pronic number  $P(n)$  is twice  $n$ -th triangular number:

$$P(n) = 2S_3(n).$$

The geometrical illustration of this fact for  $n = 5$  is given below.



Therefore,  $n$ -th pronic number is the sum of the first  $n$  even integers:

$$P(n) = 2 + 4 + \dots + 2n.$$

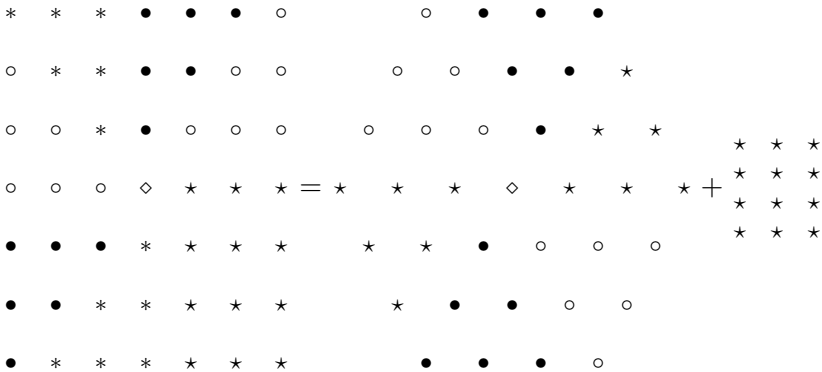
This fact implies the following recurrent formula for the pronic numbers:

$$P(n + 1) = P(n) + 2(n + 1), \quad P(1) = 2.$$

It is easy to see that *n*-th pronic number is the difference between (2*n* + 1)-th square number and (*n* + 1)-th centered hexagonal number:

$$P(n) = S_4(2n + 1) - CS_6(n + 1).$$

In fact,  $S_4(2n + 1) - CS_6(n + 1) = (2n + 1)^2 - (3(n + 1)^2 - 3(n + 1) + 1) = n^2 + n = n(n + 1) = P(n)$ . The geometrical illustration of this fact, rewritten in the form  $S_4(2n - 1) = CS_6(n) + P(n - 1)$ , is implied by the fact that any centered hexagonal number is made up by six copies of a given triangular number and one central point. On the picture below, constructed for  $n = 4$ , it is shown, as 7-th square number can be constructed from 3-rd pronic number and six copies of 3-rd triangular number plus one point, which, in turn, form the 4-th centered hexagonal number.



The generating function of the sequence of the pronic numbers is  $f(x) = \frac{2x}{(1-x)^3}$  (see [Weis11]), i.e., it holds

$$f(x) = \frac{2x}{(1-x)^3} = P(1)x + P(2)x^2 + P(3)x^3 + \dots + P(n)x^n + \dots, \quad |x| < 1.$$

The simplest way to derive this formula is just to multiply by two the generating function  $f(x) = \frac{x}{(1-x)^3} = S_3(1) \cdot x + S_3(2)x^2 + S_3(3)x^3 + \dots + S_3(n)x^n + \dots$  for triangular numbers. However, it is easy to obtain this result, following the standard procedure.

**1.7.2.** Using similar arguments, we can consider all composite numbers as **rectangular numbers**, since any composite number  $n = a \cdot b$ ,  $1 < a \leq b < n$ , can be represented as a non-trivial rectangle with the sides  $a$  and  $b$ . So, the sequence of the rectangular numbers starts with the entries 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, ... (Sloane's A002808).

In general, a composite number can be represented as a rectangle in several ways. The number of such non-trivial different rectangles is equal to the number of the above representations of  $n$  and can be obtained by the formula  $\frac{\tau(n)-2}{2}$  for non-square numbers  $n$ . For the squares the formula has the form  $\frac{\tau(n)-3}{2} + 1$ .

It is interesting to show that *for two different rectangular representations  $a \cdot b$  and  $c \cdot d$  of a given composite number, four squares  $a^2$ ,  $b^2$ ,  $c^2$  and  $d^2$ , constructed on the sides of the above rectangles, can be used to construct a new non-trivial rectangle.* For example, the number 12 has two non-trivial rectangular representations  $2 \cdot 6$  and  $3 \cdot 4$ . Then the sum of four squares  $2^2, 6^2, 3^2$  and  $4^2$  is equal to the composite number 65 and, hence, has a rectangular representation, for example,  $5 \cdot 12$ .

Speaking formally, one can check that *if  $N = ab = cd$ , then the number  $a^2 + b^2 + c^2 + d^2$  is composite.*

In fact, if  $ab = cd$ , then  $c$  divides  $ab$ . Let  $c = mn$ , where  $m$  divides  $a$ , and  $n$  divides  $b$ . Then there are  $p$  and  $q$  such that  $a = mp$ , and  $b = nq$ . Solving  $(mp)(nq) = (mn)d$  for  $d$  gives  $d = pq$ . It then follows that  $a^2 + b^2 + c^2 + d^2 = (mp)^2 + (nq)^2 + (mn)^2 + (pq)^2 = (m^2 + q^2)(n^2 + p^2)$ , and, therefore the number  $a^2 + b^2 + c^2 + d^2$  is always composite. The more general result that  $a^k + b^k + c^k + d^k$  is never prime for  $k \geq 0$  an integer also holds ([Hons91]).

**1.7.3.** The **trapezoidal numbers** are positive integers, which can be represented as an isosceles trapezoid. For example, a trapezoidal representation of 18 is given on the picture below.



The first trapezoidal numbers are 5, 7, 9, 11, 12, 13, 14, 15, 17, 18, ... (Sloane's A165513).

By definition, any trapezoidal number is a sum of two or more consecutive positive integers, greater than 1:

$$Tr(n, k) = n + (n + 1) + (n + 2) + \dots + (n + (k - 1)), \quad n \neq 1, \quad k \neq 1.$$

So, one obtains, that *any trapezoidal number is a difference of two non-consecutive triangular numbers*:

$$\begin{aligned} Tr(n, k) &= n + (n + 1) + (n + 2) + \dots + (n + (k - 1)) \\ &= (1 + 2 + \dots + (n + (k - 1))) - (1 + 2 + \dots + (n - 1)) \\ &= S_3(n + k - 1) - S_3(n - 1). \end{aligned}$$

Moreover, the above definition implies that *any trapezoidal number is a particular case of polite number* ([Smit97], [JoLo99]).

In fact, a **polite number** is a positive integer that can be represented as the sum of two or more consecutive positive integers. So, if such *polite representation* starts with 1, we obtain a triangular number, otherwise one gets a trapezoidal number. The first polite numbers are 1, 3, 5, 6, 7, 9, 10, 11, 12, 13, ... (Sloane's A138591).

Obviously, a given positive integer can have several polite representations. In the case of 18, one more such representation is given below.

$$\begin{array}{cccccc} & & * & & * & & * & & * & & * \\ & & & * & & * & & * & & * & & * \\ * & & * & & * & & * & & * & & * \\ & & & * & & * & & * & & * & & * \end{array}$$

The *politeness* of a positive integer number is the number of ways it can be expressed as a sum of consecutive positive integers. It is easy to show that *for any positive integer x, the politeness of x is equal to the number of odd divisors of x that are greater than one*.

In fact, suppose a number  $x$  has an odd divisor  $y > 1$ . Then  $y$  consecutive integers, centered on  $\frac{x}{y}$  (so that their average value is  $\frac{x}{y}$ ), have  $x$  as their sum:

$$x = \left(\frac{x}{y} - \frac{y - 1}{2}\right) + \dots + \frac{x}{y} + \dots + \left(\frac{x}{y} + \frac{y - 1}{2}\right).$$

Some of the terms in this sum may be zero or negative. However, if a term is zero, it can be omitted, and any negative terms may be used to cancel positive ones, leading to a polite representation for  $x$ . The requirement that  $y > 1$  corresponds to the requirement that a polite representation have more than one term. For instance, the

polite number  $x = 18$  has two non-trivial odd divisors, 3 and 9. It is therefore the sum of 3 consecutive numbers centered at  $\frac{18}{3} = 6$ :  $18 = 5 + 6 + 7$ . On the other hand, it is the sum of 9 consecutive integers centered at  $\frac{18}{9} = 2$ :  $18 = (-3) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6$ , or  $18 = 3 + 4 + 5 + 6$ .

Conversely, every polite representation of  $x$  can be formed by this construction. If a representation has an odd number  $y$  of terms, then the middle term can be written as  $\frac{x}{y}$ , and  $y > 1$  is a non-trivial odd divisor of  $x$ . If a representation has an even number  $2l$  of terms and its minimum value is  $m$ , it may be extended, in a unique way, to a longer sequence with the same sum and an odd number  $y = 2(m + l) - 1$  of terms, by including  $2m - 1$  numbers  $-(m - 1), -(m - 2), \dots, -1, 0, 1, \dots, m - 2, m - 1$ . After this extension the middle term of the new sequence can be written as  $\frac{x}{y}$ , and  $y > 1$  is a non-trivial odd divisor of  $x$ . By this construction, the polite representations of a number and its odd divisors greater than one may be placed into an one-to-one correspondence, that completes the proof (see [SyFr82]).

From this consideration it follows, that the **inpolite numbers**, i.e., positive integers which are not polite, are exactly the powers of two. So, the first few inpolite numbers are 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ... (Sloane's A000079), and the politeness of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ... is 0, 0, 1, 0, 1, 1, 1, 0, 2, 1, ... (Sloane's A069283).

Therefore, the only polite numbers that may be non-trapezoidal are the triangular numbers with only one non-trivial odd divisor, because for those numbers, according to the bijection described above, the odd divisor corresponds to the triangular representation and there can be no other polite representation. Thus, polite non-trapezoidal numbers must have the form of a power of two multiplied by a prime number. It is easy to show (see Chapter 5 for details), that there are exactly two types of triangular numbers with this form:

- \* the even perfect numbers  $2^{k-1}(2^k - 1)$  formed by the product of a Mersenne prime  $2^k - 1$  with half the nearest power of two;
- \* the products  $2^{k-1}(2^k + 1)$  of a Fermat prime  $2^k + 1 = 2^{2^n} + 1$  with half the nearest power of two.

**1.7.4.** The **gnomonic numbers** are closely associated with the square numbers. A gnomonic number is a figurate number of the *L*-shape, representing the area of the square gnomon obtained by removing a square of side  $n - 1$  from the square of side  $n$ . Since  $n^2 - (n - 1)^2 = 2n - 1$ , the gnomonic numbers are equivalent to the odd numbers  $2n - 1$ ,  $n \in \mathbb{N}$ , and the first few are 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, ... (Sloane's A005408). On the picture below an geometrical illustration for  $n = 1, 2, 3, 4, 5$  is given.



So, by definition,  $n$ -th gnomonic number  $Gn(n)$  has the form

$$Gn(n) = 2n - 1$$

and can be written as the difference of two consecutive square numbers:

$$Gn(n) = S_4(n) - S_4(n - 1).$$

Clearly, the gnomonic numbers can be obtained by the following recurrent formula:

$$Gn(n + 1) = Gn(n) + 2, \quad Gn(1) = 1.$$

The generating function for the gnomonic numbers is  $f(x) = \frac{x(1+x)}{(1-x)^2}$ , i.e., it holds

$$\begin{aligned} f(x) &= \frac{x(1+x)}{(1-x)^2} = Gn(1)x + Gn(2)x^2 \\ &\quad + Gn(3)x^3 + \dots + Gn(n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

The simplest way to get this result is to consider the decomposition

$$\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots, \quad |x| < 1,$$

which can be seen as the sum of the infinite geometric progression  $1, x, x^2, \dots, x^n, \dots$  with common ratio  $x$ . It is convergent for  $|x| < 1$

and represents the generating function for the sequence 1, 1, 1, ..., 1, ... . The direct multiplication gives

$$\begin{aligned} \frac{1}{(1-x)^2} &= (1+x+x^2+\cdots+x^n+\cdots) \\ &\quad \cdot (1+x+x^2+\cdots+x^n+\cdots) \\ &= 1 + (1+1)x + (1+1+1)x^2 + \cdots \\ &= 1 + 2x + 3x^2 + \cdots + (n+1) \cdot x^n + \cdots, \end{aligned}$$

i.e., we obtain the following generating function for the sequence 1, 2, 3, ...,  $n$ , ...:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + (n+1) \cdot x^n + \cdots, \quad |x| < 1.$$

Multiplication by two gives the generating function for the sequence 2, 4, 6, ...,  $2n$ , ...:

$$\frac{2}{(1-x)^2} = 2 + 4x + 6x^2 + \cdots + 2(n+1) \cdot x^n + \cdots, \quad |x| < 1.$$

Finally, the subtraction

$$\begin{aligned} \frac{2}{(1-x)^2} - \frac{1}{1-x} &= (2 + 4x + 6x^2 + \cdots + 2(n+1) \cdot x^n + \cdots) \\ &\quad - (1 + x + x^2 + \cdots + x^n + \cdots) \\ &= 1 + 3x + 5x^2 + \cdots + (2n+1)x^n + \cdots, \quad |x| < 1 \end{aligned}$$

gives the equality

$$\frac{1+x}{(1-x)^2} = x + 3x^2 + 5x^3 + \cdots + (2n+1)x^n + \cdots, \quad |x| < 1.$$

Of course, one can obtain the proof of this fact by the standard procedure, leading to the the linear recurrent equation  $Gn(n+2) - 2Gn(n+1) + Gn(n) = 0$  with initial values  $Gn(1) = 1$  and  $Gn(2) = 3$ .

**1.7.5.** Consider now the notion of **truncated plane figurate numbers**, which can be obtained by cutting figurate numbers of smaller size at each vertex of a given plane figurate number.

**I.** In the case of **truncated polygonal numbers**  $TS_m(n)$  we are going to consider only triangular and square numbers due to their symmetry.

The **truncated triangular numbers** are obtained by cutting triangular numbers of smaller size at each vertex of a given triangular number. More exactly,  $n$ -th truncated triangular number  $TS_3(n)$  is obtained after cutting from the triangular number  $S_3(3n - 2)$  three triangular numbers  $S_3(n - 1)$ , one of each vertex of the triangle of the size  $3n - 2$ , and therefore, it is given by the formula

$$TS_3(n) = S_3(3n - 2) - 3S_3(n - 1).$$

The first values of the sequence of truncated triangular numbers are 1, 7, 19, 37, 61, 91, 127, 169, 217, 271, ... (Sloane's A003215).

One can see that we get the sequence of hex numbers. This fact can be easily explained if we remind the formula  $CS_6(n) = S_3(3n - 2) - 3S_3(n - 1)$ , which was proven before. So,  $n$ -th truncated triangular number is numerically equal to  $n$ -th hex number:  $TS_3(n) = CS_6(n)$ .

Therefore, the general formula for the truncated triangular numbers is

$$TS_3(n) = 3n^2 - 3n + 1,$$

the recurrent equation is

$$TS_3(n + 1) = TS_3(n) + 6n, \quad TS_3(1) = 1,$$

while the generating function for this sequence is  $f(x) = \frac{x(x^2 + 4x + 1)}{(1 - x)^3}$ :

$$\begin{aligned} \frac{x(x^2 + 4x + 1)}{(1 - x)^3} &= TS_3(1)x + TS_3(2)x^2 + TS_3(3)x^3 \\ &+ \cdots + TS_3(n)x^n + \cdots, \quad |x| < 1. \end{aligned}$$

But we got already the formula  $S_3(2n - 1) = S_3(n) + 3S_3(n - 1)$ . So, we have the identity

$$S_3(n) = S_3(2n - 1) - 3S_3(n - 1),$$

and one can consider  $n$ -th triangular number as  $n$ -th truncated triangular number, which is obtained after cutting of the triangular number  $S_3(2n - 1)$  three triangular numbers  $S_3(n - 1)$ , one of each vertex of the triangle of the size  $2n - 1$ .

The **truncated square numbers** are obtained by cutting triangular numbers of smaller size at each vertex of a given square number. More exactly,  $n$ -th truncated square number  $TS_4(n)$  is obtained

after cutting of the square number  $S_4(3n - 2)$  four triangular numbers  $S_3(n - 1)$ , one of each its vertex, and therefore is given by the formula

$$TS_4(n) = S_4(3n - 2) - 4S_3(n - 1).$$

The first values of the sequence of truncated square numbers are 1, 12, 37, 76, 129, 196, 277, 372, 481, 604, ... (Sloane's A005892). It is easy to check that the general formula for  $n$ -th truncated square number has the form

$$TS_4(n) = 7n^2 - 10n + 4,$$

the recurrent equation is

$$TS_4(n + 1) = TS_4(n) + (14n - 3), \quad TS_4(1) = 1,$$

while the generating function is  $f(x) = \frac{x(1+9x+4x^2)}{(1-x)^3}$ , i.e.,

$$\begin{aligned} \frac{x(1 + 9x + 4x^2)}{(1 - x)^3} &= TS_4(1)x + TS_4(2)x^2 + TS_4(3)x^3 \\ &+ \dots + TS_4(n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

The last result can be obtained by the standard procedure, leading to the linear recurrent equation  $TS_4(n + 2) - 2TS_4(n + 1) + TS_4(n) = 0$  with initial conditions  $TS_4(1) = 1$ , and  $TS_4(2) = 12$ .

But we got already the formula  $S_4(2n - 1) = CS_4(n) + 4S_3(n - 1)$ , and, therefore, we have the following identity:

$$CS_4(n) = S_4(2n - 1) - 4S_3(n - 1).$$

It allows to consider  $n$ -th centered square number as  $n$ -th truncated square number, which is obtained after cutting of the square number  $S_4(2n - 1)$  four triangular numbers  $S_3(n - 1)$ , one of each its vertex.

**II. The truncated pronic numbers** are obtained by cutting triangular numbers of smaller size at each vertex of a given pronic number. More exactly,  $n$ -th truncated pronic number  $TP(n)$  is obtained after cutting of the pronic number  $P(3n - 2)$  four triangular numbers  $S_3(n - 1)$ , one of each vertex of  $(3n - 2) \times (3n - 1)$  rectangle, and therefore is given by the formula

$$TP(n) = P(3n - 2) - 4S_3(n - 1).$$

The first values of the sequence of truncated pronic numbers are 2, 16, 44, 86, 142, 208, 292, 390, 502, 628, ... It is easy to check that

the general formula for  $n$ -th truncated pronic number is

$$TP(n) = 7n^2 - 7n + 2,$$

the recurrent equation is

$$TP(n+1) = TP(n) + 14n, \quad TP(1) = 2,$$

while the generating function is  $f(x) = \frac{x(2+10x+2x^2)}{(1-x)^3}$ :

$$\begin{aligned} \frac{x(2+10x+2x^2)}{(1-x)^3} &= TP(1)x + TP(2)x^2 + TP(3)x^3 \\ &+ \cdots + TP(n)x^n + \cdots, \quad |x| < 1. \end{aligned}$$

The last result can be obtained by the standard procedure, leading to the linear recurrent equation  $TP(n+2) - 2TP(n+1) + TP(n) = 0$  with initial conditions  $TP(1) = 1$ , and  $TP(2) = 16$ .

In general, in order to obtain truncated pronic numbers we should delete four triangular numbers, relating to the four angles of the corresponding rectangles.

So, for  $P(2k) = 2k(2k+1)$ , we can construct truncated pronic numbers  $P(2k) - 4S_3(t)$  for any  $t = 1, 2, \dots, k$ . For the biggest,  $t = k$ , we get  $2k(2k+1) - 2k(k+1) = 2k^2 = 2S_4(k)$ , i.e., such truncated numbers give the doubled squares. For  $P(2k+1) = (2k+1)(2k+2)$  we can construct truncated pronic numbers  $P(2k+1) - 4S_3(t)$  for any  $t = 1, 2, \dots, k$ . For the biggest,  $t = k$ , we get  $(2k+1)(2k+2) - 2k(k+1) = 2k^2 + 4k + 2 = 2(k^2 + 2k + 1) = 2(k+1)^2 = 2S_4(k+1)$ , i.e., such truncated numbers also give the doubled squares.

**III.** To obtain **truncated centered polygonal numbers**, one should cut squares of smaller size, corresponding to each angle of a given  $m$ -gon, since, in the geometrical settings, the deleted small triangles have  $1 = 1^2$ ,  $1 + 3 = 2^2$ ,  $1 + 3 + 5 = 9^2$ ,  $\dots$  points each.

So, the **truncated centered  $m$ -gonal numbers** are obtained by cutting square numbers of smaller size at each vertex of a given centered  $m$ -gonal number. More exactly,  $n$ -th truncated centered  $m$ -gonal number  $TCS_m(n)$  is obtained by cutting of the centered  $m$ -gonal number  $CS_m(3n-2)$   $m$  square numbers  $S_4(n-1)$ , one of each

its vertex, and so, it is given by the formula

$$TCS_m(n) = CS_m(3n - 2) - mS_4(n - 1).$$

It is easy to check that the general formula for  $n$ -th truncated centered  $m$ -gonal number is

$$TCS_m(n) = 1 + \frac{m}{2}(7n^2 - 11n + 4),$$

the recurrent equation is

$$TCS_m(n + 1) = TCS_m(n) + m(7n - 2), \quad TCS_m(1) = 1,$$

while the generating function is  $f(x) = \frac{x(1+(5m-2)x+(1+2m)x^2)}{(1-x)^3}$ :

$$\begin{aligned} & \frac{x(1 + (5m - 2)x + (1 + 2m)x^2)}{(1 - x)^3} \\ &= x + (1 + 5m)x + (1 + 17m)x^2 \\ & \quad + (1 + 36m)x^3 + (1 + 62m)x^4 + \dots \\ &= TCS_m(1)x + TCS_m(2)x^2 + TCS_m(3)x^3 \\ & \quad + \dots + TCS_m(n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

The last result can be obtained by the standard procedure, leading to the linear recurrent equation  $TCS_m(n + 2) - 2TCS_m(n + 1) + TCS_m(n) = 0$  with initial conditions  $TCS_m(1) = 1$ , and  $TCS_m(2) = 1 + 5m$ .

In particular, the general formula for  $n$ -th **truncated centered triangular number** is

$$TCS_3(n) = \frac{21n^2 - 33n}{2} + 7,$$

giving the first values 1, 16, 52, 109, 187, 286, 406, 547, 709, 892, ...

The recurrent equation is

$$TCS_3(n + 1) = TCS_3(n) + (21n - 6), \quad TCS_3(1) = 1,$$

while the generating function is  $f(x) = \frac{x(1+13x+7x^2)}{(1-x)^3}$ :

$$\begin{aligned} & \frac{x(1 + 13x + 7x^2)}{(1 - x)^3} = TCS_3(1)x + TCS_3(2)x^2 + TCS_3(3)x^3 \\ & \quad + \dots + TCS_3(n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

The general formula for  $n$ -th **truncated centered square number** is

$$TCS_4(n) = 14n^2 - 22n + 9,$$

giving the first values 1, 21, 69, 145, 249, 381, 541, 729, 945, 1189, . . . . The recurrent equation is

$$TCS_4(n+1) = TCS_4(n) + (28n - 8), \quad TCS_4(1) = 1,$$

while the generating function is  $f(x) = \frac{x(1+18x+9x^2)}{(1-x)^3}$ :

$$\begin{aligned} \frac{x(1+18x+9x^2)}{(1-x)^3} &= TCS_4(1)x + TCS_4(2)x^2 + TCS_4(3)x^3 \\ &+ \cdots + TCS_4(n)x^n + \cdots, \quad |x| < 1. \end{aligned}$$

The general formula for  $n$ -th **truncated centered pentagonal number** is

$$TCS_5(n) = \frac{35n^2 - 55n}{2} + 3,$$

giving the first values 1, 26, 86, 181, 303, 468, 668, 903, 1173, 1478, . . . . The recurrent equation is

$$TCS_5(n+1) = TCS_5(n) + (35n - 10), \quad TCS_5(1) = 1,$$

while the generating function is  $f(x) = \frac{x(1+8x+11x^2)}{(1-x)^3}$ :

$$\begin{aligned} \frac{x(1+8x+11x^2)}{(1-x)^3} &= TCS_5(1)x + TCS_5(2)x^2 + TCS_5(3)x^3 \\ &+ \cdots + TCS_5(n)x^n + \cdots, \quad |x| < 1. \end{aligned}$$

The general formula for  $n$ -th **truncated centered hexagonal number** (or **truncated hex number**) is

$$TCS_6(n) = 21n^2 - 33n + 13,$$

giving the first values 1, 31, 103, 217, 373, 571, 811, 1093, 1417, 1783, . . . . The recurrent equation is

$$TCS_6(n+1) = TCS_6(n) + (42n - 12), \quad TCS_6(1) = 1,$$

while the generating function is  $f(x) = \frac{x(1+28x+13x^2)}{(1-x)^3}$ :

$$\begin{aligned} \frac{x(1+28x+13x^2)}{(1-x)^3} &= TCS_6(1)x + TCS_6(2)x^2 + TCS_6(3)x^3 \\ &+ \cdots + TCS_6(n)x^n + \cdots, \quad |x| < 1. \end{aligned}$$

Using the standard procedure to construct truncated objects, more general concept of truncated centered polygonal numbers can

be obtained. In fact, truncated centered  $m$ -gonal numbers come by the formulas  $CS_m(2n) - mk^2$ ,  $k = 1, 2, \dots, n$ , and  $CS_m(2n+1) - mk^2$ ,  $k = 1, 2, \dots, n$ . So, the truncated centered triangular numbers can be obtained by the formulas  $CS_3(2n) - 3k^2$ ,  $k = 1, 2, \dots, n$ , and  $CS_3(2n+1) - 3k^2$ ,  $k = 1, 2, \dots, n$ . The truncated centered square numbers come by the formulas  $CS_4(2n) - 4k^2$ ,  $k = 1, 2, \dots, n$ , and  $CS_4(2n+1) - 4k^2$ ,  $k = 1, 2, \dots, n$ . The truncated hex numbers come by the formulas  $CS_6(2n) - 6k^2$ ,  $k = 1, 2, \dots, n$ , and  $CS_6(2n+1) - 6k^2$ ,  $k = 1, 2, \dots, n$ .

**1.7.6.** The **polygram numbers** (or **centered star polygonal numbers**) correspond to the *polygrams* (or *star polygons*) and can be obtained as corresponding centered polygonal numbers with attached triangular numbers of corresponding size to each side of a given polygon. In order to obtain the  **$m$ -gram numbers**, we use such procedure for each term of the sequence of the centered  $m$ -gonal numbers. The most known figurate numbers of such kind, the star numbers, corresponding to the centered hexagram, and, hence, to the hex numbers, were considered before.

Formally,  $n$ -th  **$m$ -gram number**  $P_m(n)$ ,  $m \geq 3$ , is defined as  $n$ -th centered  $m$ -gonal number  $CS_m(n)$  with  $m$  copies of the  $(n-1)$ -th triangular number  $S_3(n-1)$  appended to each side:

$$P_m(n) = CS_m(n) + mS_3(n-1) = mn^2 - mn + 1.$$

As  $CS_m(n) = 1 + m \cdot S_3(n-1)$ , we get

$$\begin{aligned} P_m(n) &= CS_m(n) + m \cdot S_3(n-1) \\ &= (1 + m \cdot S_3(n-1)) + m \cdot S_3(n-1) \\ &= 1 + (2m) \cdot S_3(n-1) = CS_{2m}(n). \end{aligned}$$

Hence,  $n$ -th  $m$ -gram number  $P_m(n) = mn^2 - mn + 1$  is numerically equal to  $n$ -th centered  $(2m)$ -gonal number, but is differently arranged.

For  $m = 3$ , and 4 this procedure is trivial: attaching to the centered triangular number  $CS_3(n)$  three triangular numbers  $S_3(n-1)$  just gives the shape of  $n$ -th centered hexagonal number; attaching to the centered square number  $CS_4(n)$  four triangular numbers  $S_3(n-1)$  gives the shape of the  $(2n+1)$ -th square number, which can be seen as  $n$ -th octagonal number.

For  $m \geq 5$ , the polygram numbers correspond to the classical *regular polygrams* (or *star polygons*), where a *regular*  $\{p/q\}$  *polygram*, with positive integers  $p, q$ , is a figure formed by connecting every  $q$ -th point out of  $p$  regularly spaced points lying on a circumference. In fact,  $m$ -gram numbers correspond to the *regular*  $\{m/2\}$  *polygram*.

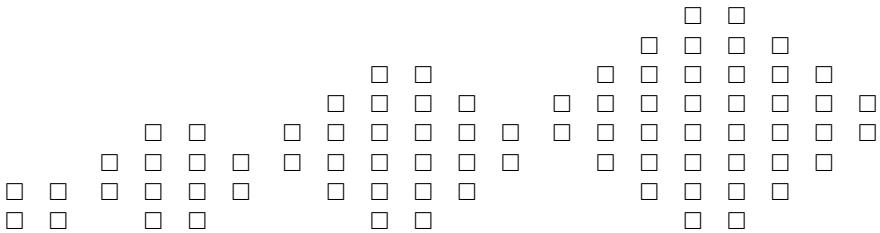
In particular, the **pentagram numbers** are connected with classical pentagram. By definition,  $n$ -th pentagram number  $P_5(n)$  is constructed as  $n$ -th centered pentagonal number  $CS_5(n)$  with five copies of the  $(n - 1)$ -th triangular number  $S_3(n - 1)$  appended to each side. So, it holds

$$P_5(n) = CS_5(n) + 5S_3(n - 1).$$

Hence,  $n$ -th pentagram number  $P_5(n) = 5n^2 - 5n + 1$  is numerically equal to  $n$ -th centered dodecagonal number  $CS_{10}(n)$ , but is differently arranged.

The star numbers, corresponding to centered hexagrams, can be considered as **hexagram numbers**.

**1.7.7.** The **Aztec diamond numbers** arise by *Aztec diamond construction*. The *Aztec diamond of  $n$ -th order* is the region obtained from four staircase shapes of height  $n$  by gluing them together along the straight edges. On the picture below the geometrical illustration of this notion is given for  $n = 1, 2, 3$  and 4.



The Aztec diamond of  $n$ -th order can be considered also as the union of unit squares in the plane whose edges lie on the lines of a square grid and whose centers  $(x, y)$  satisfy to  $|x - 0.5| + |y - 0.5| \leq n$ .

The  $n$ -th **Aztec diamond number**  $Az(n)$ , i.e., the number of squares in the Aztec diamond of  $n$ -th order, has the form

$$Az(n) = 2n(n + 1),$$

giving the values 4, 12, 24, 40, 60, 84, 112, 144, 180, 220, ... (Sloane's A046092).

So, for  $n$ -th Aztec diamond number  $Az(n)$  it holds

$$Az(n) = 4S_3(n), \quad \text{and} \quad Az(n) = 2P(n),$$

i.e.,  $n$ -th Aztec diamond number is four times  $n$ -th triangular number and doubled  $n$ -th pronic number.

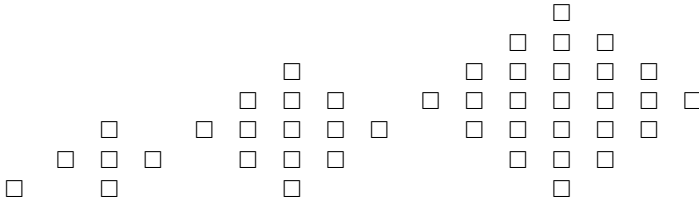
It is easy to see that the recurrent formula for the Aztec diamond numbers has the form

$$Az(n + 1) = Az(n) + 4(n + 1), \quad Az(1) = 4,$$

while the generating function is  $f(x) = \frac{4x}{(1-x)^3}$ , i.e.,

$$\frac{4x}{(1-x)^3} = Az(1)x + Az(2)x^2 + Az(3)x^3 + \dots + Az(n)x^n + \dots, \quad |x| < 1.$$

On the other hand, the classical *diamond of  $n$ -th order* can be considered as the set of squares whose centers satisfy the inequality  $|x| + |y| \leq n$ . On the picture below the geometrical illustration for  $n = 1, 2, 3, 4$  is given.



The  $n$ -th **diamond number**, giving the number of union squares in  $n$ -th order diamond, is  $2n(n - 1) + 1$ , that is precisely  $n$ -th centered square number  $CS_4(n)$ , giving values 1, 5, 13, 25, 41, 61, 85, 113, 145, 181, ... (Sloane's A001844). So,  $n$ -th diamond number is numerically equal to  $n$ -th centered square number, and can be seen geometrically as the corresponding square, rotated on  $45^\circ$ .

**1.7.8.** The **cross numbers** are defined as numbers of the form  $4n - 3$ ,  $n = 1, 2, 3, \dots$ , which can be represented geometrically as a cross on the plane. So,  $n$ -th cross number  $Cr(n)$  has the form

$$Cr(n) = 4n - 3,$$

and the sequence starts from 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, ... (Sloane's A016813). On the figure below we construct  $Cr(n)$

for  $n = 1, 2, 3, 4$ .



The recurrent equation for these numbers is

$$Cr(n + 1) = Cr(n) + 4, \quad Cr(1) = 1,$$

and the generating function has the form  $f(x) = \frac{x(1+3x)}{(1-x)^2}$ , i.e.,

$$\frac{x(1 + 3x)}{(1 - x)^2} = Cr(1)x + Cr(2)x^2 + Cr(3)x^3 + \dots + Cr(n)x^n + \dots, \quad |x| < 1.$$

The last result can be obtained via the standard procedure, giving the linear recurrent equation  $Cr(n + 2) - 2Cr(n + 1) + Cr(n) = 0$  with initial values  $Cr(1) = 1$  and  $Cr(2) = 5$ . On the other hand, the function  $f(x)$  can be calculated using the sum of the generating function  $\frac{4x}{(1-x)^2} = 4x + 8x^2 + 12x^3 + \dots$  for the numbers of the form  $4n$ , and the generating function  $\frac{1}{1-x} = 1 + x + x^2 + \dots$  for the sequence  $1, 1, 1, \dots$ .

The positive integers of such form have many interesting properties. For example, *any prime cross number can be represented as a sum of two squares, and a cross number can be represented as a sum of two squares if and only if it has a prime divisors of the form  $4n + 3$  only in even powers* (see [Sier64]).

## 1.8 Generalized plane figurate numbers

The **generalized plane figurate numbers** are defined as all values of the standard formulas for a given class of the plane figurate numbers, taken for any integer value of the index.

**1.8.1.** The **generalized polygonal numbers** or, specifically, the **generalized  $m$ -gonal numbers**, are defined as all values of the

formula

$$S_m(n) = \frac{n((m-2)n - m + 4)}{2} = \frac{(m-2)}{2}(n^2 - n) + n$$

for  $n$ -th  $m$ -gonal number, taken for any integer value of  $n$ .

For positive integers  $n$ , we get ordinary polygonal numbers. For  $n = 0$ , one has  $S_m(0) = 0$ . Finally,  $S_m(-n) = \frac{n((m-2)n+m-4)}{2} = \frac{m-2}{2}(n^2 + n) - n$  for negative values of the argument, i.e., for any positive integer  $n$ .

Let us consider the properties of the above numbers  $S_m(-n)$ ,  $n \in \mathbb{N}$ . For convenience, denote  $S_m(-n)$  by  ${}^-S_m(n)$ , i.e., by definition,

$$\begin{aligned} {}^-S_m(n) &= S_m(-n) = \frac{n((m-2)n+m-4)}{2} \\ &= \frac{m-2}{2}(n^2 + n) - n, \quad n \in \mathbb{N}. \end{aligned}$$

In particular, one has

$$\begin{aligned} {}^-S_3(n) &= \frac{n(n-1)}{2}, \quad {}^-S_4(n) = n^2, \quad {}^-S_5(n) = \frac{n(3n+1)}{2}, \\ {}^-S_6(n) &= n(2n+1), \quad {}^-S_7(n) = \frac{n(5n+3)}{2}, \quad {}^-S_8(n) = n(4n+2). \end{aligned}$$

So, the **generalized triangular numbers** with negative indices coincide with ordinary triangular numbers of smaller index:

$${}^-S_3(n) = S_3(-n) = S_3(n-1) = \frac{(n-1)n}{2}.$$

They have values 0, 1, 3, 6, 10, ... So, the generalized triangular numbers are

$$\dots, 10, 6, 3, 1, 0, 0, 1, 3, 6, 10, 15, \dots,$$

forming sequence 0, 1, 0, 3, 1, 6, 3, 10, 6, ... for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$  (see Sloane's A008795).

Obviously, the **generalized square numbers** with negative indices coincide with ordinary square numbers:

$${}^-S_4(n) = S_4(-n) = S_4(n) = n^2.$$

They have values 1, 4, 9, 16, 25, ... So, the generalized square numbers are

$$\dots, 25, 16, 9, 4, 1, 0, 1, 4, 9, 16, 25, \dots$$

forming sequence 0, 1, 1, 4, 4, 9, 9, 16, 16, ... for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$  (see Sloane's A008794).

The **generated pentagonal numbers** with negative indices have the form

$${}^{-}S_5(n) = S_5(-n) = \frac{3n^2 + n}{2}.$$

They have values 2, 7, 15, 26, 40, ... So, the generalized pentagonal numbers are

$$\dots, 40, 26, 15, 7, 2, 0, 1, 5, 12, 22, 35, \dots$$

forming sequence 0, 1, 2, 5, 7, 12, 15, 22, 26, ... for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$  (Sloane's A001318).

The generalized pentagonal numbers are the most known class of the generalized figurate numbers; for example, they play an important role in the theory of the unrestricted partitions (see Chapter 4).

The **generalized hexagonal numbers** with negative indices have the form

$${}^{-}S_6(n) = S_6(-n) = 2n^2 + n.$$

They have values 3, 10, 21, 36, 55, ... In other words, the numbers  ${}^{-}S_6(n)$  coincide with the triangular numbers of even indices:  ${}^{-}S_6(n) = 2n^2 + n = \frac{2n(2n+1)}{2} = S_3(2n)$ . On the other hand, as it was shown before, the ordinary hexagonal numbers are triangular numbers with odd indices:  $S_6(n) = S_3(2n - 1)$ . So, the set of the generalized hexagonal numbers

$$\dots, 55, 36, 21, 10, 3, 0, 1, 6, 15, 28, 45, \dots$$

contains only triangular numbers, forming the standard sequence 0, 1, 3, 6, 10, 15, 21, 28, 36, ... (Sloane's A000217) of all ordinary triangular numbers for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$

The recurrent formula for the sequence  ${}^{-}S_m(1), {}^{-}S_m(2), {}^{-}S_m(3), \dots$  of generalized  $m$ -gonal numbers with negative indices has the form

$${}^{-}S_m(n+1) = {}^{-}S_m(n) + ((m-2)(n+1) - 1), {}^{-}S_m(1) = m - 3.$$

In fact, one obtains from the recurrent formula  $S_m(n+1) = S_m(n) + (1 + (m-2)n)$  for the ordinary  $m$ -gonal numbers, that  $S_m(n) = S_m(n+1) - (m-2)n - 1$ , or  $S_m(n-1) = S_m(n) + (m-2)(1-n) - 1$ . So,  $S_m(0) = 0$ , and, for negative integers, one has  $S_m(-(n+1)) = S_m(-n) + (m-2)(1+n) - 1$ ,  $n \in \mathbb{N}$ .

The generating function for the sequence  ${}^{-}S_m(1), {}^{-}S_m(2), {}^{-}S_m(3), \dots$  of the generalized  $m$ -gonal numbers with negative indices, i.e., for the sequence  $S_m(-1), S_m(-2), S_m(-3), \dots$ , has the form  $f(x) = \frac{x(x+(m-3))}{(1-x)^3}$ ; so, it holds

$$\begin{aligned} \frac{x(x+(m-3))}{(1-x)^3} &= S_m(-1)x + S_m(-2)x^2 + S_m(-3)x^3 \\ &+ \dots + S_m(-n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

In particular, we have

$$\begin{aligned} \frac{x^2}{(1-x)^3} &= S_3(-1)x + S_3(-2)x^2 + S_3(-3)x^3 + \dots + S_3(-n)x^n + \dots \\ &= S_3(0)x + S_3(1)x^2 + S_3(2)x^3 + \dots \\ &+ S_3(n-1)x^n + \dots, \quad |x| < 1; \end{aligned}$$

$$\begin{aligned} \frac{x(x+1)}{(1-x)^3} &= S_4(-1)x + S_4(-2)x^2 + S_4(-3)x^3 + \dots + S_4(-n)x^n + \dots \\ &= S_4(1)x + S_4(2)x^2 + S_4(3)x^3 + \dots \\ &+ S_4(n)x^n + \dots, \quad |x| < 1; \end{aligned}$$

$$\begin{aligned} \frac{x(x+2)}{(1-x)^3} &= S_5(-1)x + S_5(-2)x^2 \\ &+ S_5(-3)x^3 + \dots + S_5(-n)x^n + \dots, \quad |x| < 1; \end{aligned}$$

$$\begin{aligned} \frac{x(x+3)}{(1-x)^3} &= S_6(-1)x + S_6(-2)x^2 \\ &+ S_6(-3)x^3 + \dots + S_6(-n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

This fact can be obtained via the standard procedure, leading to the recurrent linear equation  ${}^{-}S_m(n+3) - 3{}^{-}S_m(n+2) + 3{}^{-}S_m(n+1) - {}^{-}S_m(n) = 0$  with initial conditions  ${}^{-}S_m(1) = m - 3$ ,  ${}^{-}S_m(2) = 3m - 8$ , and  ${}^{-}S_m(3) = 6m - 15$ .

However, for  $m = 3$  and  $m = 4$  the situation is much more simple. In fact, the generating function for the sequence of the ordinary triangular numbers is  $f(x) = \frac{x}{(1-x)^3}$ , i.e.,

$$\frac{x}{(1-x)^3} = S_3(1)x + S_3(2)x^2 + S_3(3)x^3 + \dots + S_3(n)x^n + \dots, \quad |x| < 1.$$

Since  $S_3(n) = -S_3(n + 1)$ ,  $n \geq 1$ , and  $-S_3(1) = 0$ , we get

$$\begin{aligned} \frac{x}{(1-x)^3} &= -S_3(2)x + -S_3(3)x^2 + \dots + -S_3(n+1)x^n + \dots \\ &= -S_3(1) + -S_3(2)x + -S_3(2)x^2 + \dots \\ &\quad + -S_3(n+1)x^n + \dots, \quad |x| < 1, \quad \text{and} \\ \frac{x^2}{(1-x)^3} &= -S_3(1)x + -S_3(2)x^2 + -S_3(3)x^3 + \dots \\ &\quad + -S_3(n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

As for  $m = 4$ , the generating function  $f(x) = \frac{x(1+x)}{(1-x)^3}$  for the sequence of the generated square numbers with negative indices coincides, clearly, with the generating function for the sequence of the ordinary square numbers.

Now we can get the generating function for the sequence  $S_m(0), S_m(1), S_m(-1), S_m(2), S_m(-2), \dots$  of all generalized  $m$ -gonal numbers, written for  $n = 0, \pm 1, \pm 2, \dots$ . It can be obtained from the generating functions  $f_1(x) = \frac{1+(m-3)x}{(1-x)^3} = S_m(1) + S_m(2)x + \dots + S_m(n+1)x^n + \dots$  and  $f_2(x) = \frac{(m-3)+x}{(1-x)^3} = S_m(-1) + S_m(-2)x + \dots + S_m(-(n+1))x^n + \dots$  using the formula  $x^2 f_1(x^2) + x^3 f_2(x^2)$ . The direct computation gives  $f(x) = \frac{x^2(1+(m-3)x+(m-3)x^2+x^3)}{(1-x^2)^3} = \frac{x^2(1+x)(1+(m-4)x+x^2)}{(1-x^2)^3} = \frac{x^2(1+(m-4)x+x^2)}{(1-x)(1-x^2)^2}$ . In other words, it holds

$$\begin{aligned} \frac{x^2(1+(m-4)x+x^2)}{(1-x)(1-x^2)^2} &= S_m(0)x + S_m(1)x^2 + S_m(-1)x^3 \\ &\quad + S_m(2)x^4 + S_m(-2)x^5 + \dots, \quad |x| < 1. \end{aligned}$$

In particular, we have

$$\begin{aligned} \frac{x^2(1-x+x^2)}{(1-x)(1-x^2)^2} &= S_3(0)x + S_3(1)x^2 + S_3(-1)x^3 + S_3(2)x^4 \\ &\quad + S_3(-2)x^5 + \dots, \quad |x| < 1; \\ \frac{x^2(1+x^2)}{(1-x)(1-x^2)^2} &= S_4(0)x + S_4(1)x^2 + S_4(-1)x^3 + S_4(2)x^4 \\ &\quad + S_4(-2)x^5 + \dots, \quad |x| < 1; \end{aligned}$$

$$\begin{aligned} \frac{x^2(1+x+x^2)}{(1-x)(1-x^2)^2} &= S_5(0)x + S_5(1)x^2 + S_5(-1)x^3 + S_5(2)x^4 \\ &\quad + S_5(-2)x^5 + \dots, \quad |x| < 1; \\ \frac{x^2(1+2x+x^2)}{(1-x)(1-x^2)^2} &= S_6(0)x + S_6(1)x^2 + S_6(-1)x^3 + S_6(2)x^4 \\ &\quad + S_6(-2)x^5 + \dots, \quad |x| < 1. \end{aligned}$$

We can consider some properties of the generated polygonal numbers, similar to the properties of ordinary polygonal numbers.

For example, using the fact  $S_4(n) = S_3(n) + S_3(n-1)$  and the identities  ${}^-S_4(n) = S_4(n)$ ,  ${}^-S_3(n) = S_3(n-1)$ , we obtain that *any generated square number is a sum of two generalized triangular numbers with opposite indices*:

$$S_4(n) = {}^-S_4(n) = S_3(n) + {}^-S_3(n).$$

Furthermore, *any generalized square number is a sum of two consecutive generalized triangular numbers*:

$$S_4(n) = {}^-S_4(n) = S_3(n-1) + S_3(n) = {}^-S_3(n) + {}^-S_3(n+1).$$

On the same way, we can rewrite all properties, connecting triangular and square numbers. For example, the property  $S_3(2n) = 3S_3(n) + S_3(n-1)$  gets the form

$${}^-S_3(2n+1) = 3{}^-S_3(n+1) + {}^-S_3(n).$$

Since  $\frac{3n^2+n}{2} = n^2 + \frac{n(n+1)}{2}$ , we get the property

$${}^-S_5(n) = {}^-S_4(n) + {}^-S_3(n+1),$$

which is similar to the property  $S_5(n) = S_4(n) + S_3(n-1)$ .

The reader can easily obtain many new identities, connecting generalized polygonal numbers.

**1.8.2. The generalized centered polygonal numbers**, or, specifically, the **generalized centered  $m$ -gonal numbers** are defined as all values of the formula  $CS_m(n) = \frac{mn^2 - mn + 2}{2}$  for any integer value of the index  $n$ .

For positive integers  $n$ , we get ordinary centered  $m$ -gonal numbers. For  $n = 0$ , it holds  $CS_m(0) = 1$ . Finally, for negative integers we get  $CS_m(-n) = \frac{mn^2 + mn + 2}{2}$ ,  $n \in \mathbb{N}$ .

So, denoting  $CS_m(-n)$  by  ${}^-CS_m(n)$ , we get the formula

$${}^-CS_m(n) = CS_m(-n) = \frac{mn^2 + mn + 2}{2}, \quad n \in \mathbb{N}.$$

Hence, the **generalized centered triangular numbers** with negative indices have the form

$${}^-CS_3(n) = CS_3(-n) = \frac{3n^2 + 3n + 2}{2},$$

giving values 4, 10, 19, 31, 46, ... So, the generalized centered triangular numbers are

$$\dots, 46, 31, 19, 10, 4, 1, 1, 4, 10, 19, 31, \dots$$

forming sequence 1, 1, 4, 4, 10, 10, 19, 19, 31, ... for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$  (see Sloane's A005448).

The **generalized centered square numbers** with negative indices have the form

$${}^-CS_4(n) = CS_4(-n) = 2n^2 + 2n + 1,$$

giving values 5, 13, 25, 41, 61, ... So, the generalized centered square numbers are

$$\dots, 61, 41, 25, 13, 5, 1, 1, 5, 13, 25, 41, \dots$$

forming sequence 1, 1, 5, 5, 13, 13, 41, 41, 61, ... for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$  (see Sloane's A001844).

The **generated centered pentagonal numbers** with negative indices have the form

$${}^-CS_5(n) = CS_5(-n) = \frac{5n^2 + 5n + 2}{2},$$

giving values 6, 16, 31, 51, 76, ... So, the generalized centered pentagonal numbers are

$$\dots, 76, 51, 31, 16, 6, 1, 1, 6, 16, 31, 51, \dots$$

forming sequence 1, 1, 6, 6, 16, 16, 31, 31, 51, ... for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$  (see Sloane's A005891).

The **generalized centered hexagonal numbers** (or **generalized hex numbers**) with negative indices have the form

$${}^{-}S_6(n) = S_6(-n) = 3n^2 + 3n + 1,$$

giving values 7, 19, 37, 61, 91, ... So, the generalized hex numbers are

$$\dots, 91, 61, 37, 19, 7, 1, 1, 7, 19, 37, 61, \dots,$$

forming the sequence 1, 1, 7, 7, 19, 19, 37, 37, ... for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$  (see Sloane's A003215).

It is easy to see that in this case the situation is much more simple than in the case of generalized polygonal numbers, since *all generalized centered  $m$ -gonal numbers are ordinary centered  $m$ -gonal numbers*:

$${}^{-}CS_m(n) = CS_m(-n) = CS_m(n+1), \quad n \in \mathbb{N}.$$

In fact, it holds  ${}^{-}CS_m(n) = CS_m(-n) = \frac{mn^2+mn+2}{2} = \frac{(mn^2+2mn+1)-(mn+m)+2}{2} = \frac{m(n+1)^2-m(n+1)+2}{2} = CS_m(n+1)$ .

So, the sequence of the generalized centered  $m$ -gonal numbers is

$$\dots, 1 + 15m, 1 + 10m, 1 + 6m, 1 + 3m, 1 + m, \\ 1, 1, 1 + m, 1 + 3m, 1 + 6m, 1 + 10m, \dots,$$

forming the sequence 1, 1,  $1 + m$ ,  $1 + m$ ,  $1 + 3m$ ,  $1 + 3m$ ,  $1 + 6m$ ,  $1 + 6m$ ,  $1 + 10m$ , ... for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$

One can check that the recurrent formula for the sequence of the generalized centered  $m$ -gonal numbers with negative indices has the form

$${}^{-}CS_m(n+1) = {}^{-}CS_m(n) + (n+1)m, \quad {}^{-}CS_m(1) = 1 + m.$$

The generating function for the sequence  ${}^{-}CS_m(1), {}^{-}CS_m(2), {}^{-}CS_m(3), \dots$  of generalized centered  $m$ -gonal numbers with negative indices, i.e., for the sequence  $CS_m(-1), CS_m(-2), CS_m(-3), \dots$ , has

the form  $f(x) = \frac{x(x^2-2x+(m+1))}{(1-x)^3}$ , i.e., it holds

$$\frac{x(x^2 - 2x + (m + 1))}{(1 - x)^3} = CS_m(-1)x + CS_m(-2)x^2 + CS_m(-3)x^3 + \dots + CS_m(-n)x^n + \dots, \quad |x| < 1.$$

In fact, using the identity  ${}^-CS_m(n) = CS_m(n + 1)$  and the decomposition

$$\frac{1 + (m - 2)x + x^2}{(1 - x)^3} = CS_m(1) + CS_m(2)x + \dots + CS_m(n + 1)x^n + \dots, \quad |x| < 1,$$

one has

$$\begin{aligned} &\frac{1 + (m - 2)x + x^2}{(1 - x)^3} - CS_m(1) \\ &= CS_m(2)x + \dots + CS_m(n + 1)x^n + \dots \\ &= {}^-CS_m(1)x + {}^-CS_m(2)x^2 + \dots \\ &+ {}^-CS_m(n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

Since  $CS_m(1) = 1$ , and  $\frac{1+(m-2)x+x^2}{(1-x)^3} - 1 = \frac{x(x^2-2x+(m+1))}{(1-x)^3}$ , it holds

$$\frac{x(x^2 - 2x + (m + 1))}{(1 - x)^3} = {}^-CS_m(1)x + {}^-CS_m(2)x^2 + \dots + {}^-CS_m(n)x^n + \dots, \quad |x| < 1.$$

The generating function for the sequence  $CS_m(0), CS_m(1), CS_m(-1), CS_m(2), CS_m(-2), \dots$  of all generalized  $m$ -gonal numbers, written for  $n = 0, \pm 1, \pm 2, \dots$ , can be obtained from the generating functions  $f_1(x) = \frac{1+(m-2)x+x^2}{(1-x)^3} = CS_m(1) + CS_m(2)x + \dots + CS_m(n+1)x^n + \dots$  and  $f_2(x) = \frac{(m+1)-2x+x^2}{(1-x)^3} = CS_m(-1) + CS_m(-2)x + \dots + CS_m(-(n+1))x^n + \dots$  using the formula  $x^2 f_1(x^2) + x^3 f_2(x^2)$ . The direct computation gives  $f(x) = \frac{x^2(1+(m+1)x+(m-2)x^2-2x^3+x^4+x^5)}{(1-x^2)^3}$ .

In other words, it holds

$$\begin{aligned} & \frac{x^2(1 + (m + 1)x + (m - 2)x^2 - 2x^3 + x^4 + x^5)}{(1 - x^2)^3} \\ &= CS_m(0)x + CS_m(1)x^2 + CS_m(-1)x^3 + CS_m(2)x^4 \\ & \quad + CS_m(-2)x^5 + \cdots, \quad |x| < 1. \end{aligned}$$

For example, in the case of the generalized hex numbers, we have

$$\begin{aligned} & \frac{x^2(1 + 7x + 4x^2 - 2x^3 + x^4 + x^5)}{(1 - x^2)^3} = \frac{x^2(1 + 6x - 2x^2 + x^4)}{(1 - x)(1 - x^2)^2} \\ &= CS_6(0)x + CS_6(1)x^2 + CS_6(-1)x^3 \\ & \quad + CS_m(2)x^4 + CS_m(-2)x^5 + \cdots, \quad |x| < 1. \end{aligned}$$

One can obtain many properties of generalized centered polygonal numbers, using known properties for ordinary centered polygonal numbers and the above identity  ${}^-CS_m(n) = CS_m(n + 1)$ .

For example, the identity  $CS_m(n + 1) = 1 + mS_3(n)$  implies that *any generalized centered  $m$ -gonal number can be constructed as  $m$  copies of generalized triangular numbers, places around a given central point:*

$${}^-CS_m(n) = 1 + mS_3(n) = 1 + m({}^-S_3(n + 1)).$$

**1.8.3.** Following the similar procedure, one can consider other generalized plane figurate numbers.

For example, the **generalized pronic numbers** are defined as all values of the formula  $P(n) = n(n + 1)$  for any integer value of the index  $n$ .

For positive integer  $n$ , we get ordinary pronic numbers. For  $n = 0$ , one has  $P(0) = 0$ . Finally, for negative values of the index one has

$$P(-n) = (n - 1)n, \quad n \in \mathbb{N}.$$

So, *any generalized pronic number coincides with the ordinary pronic number of smaller size:*

$${}^-P(n) = P(-n) = P(n - 1), \quad n \in \mathbb{N}.$$

Hence, the generalized pronic numbers  ${}^{-}P(n) = P(-n)$  with negative integers are 0, 2, 6, 12, 20, ..., while the generalized pronic numbers are

$$\dots, 20, 12, 6, 2, 0, 0, 2, 6, 12, 20, 30, \dots,$$

forming the sequence 0, 2, 0, 6, 2, 12, 6, 20, 12, ... for  $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \dots$

The recurrent formula for the generalized pronic numbers with negative indices is

$${}^{-}P(n + 1) = {}^{-}P(n) + 2n, {}^{-}P(1) = 0,$$

while the generating function for this sequence has the form  $f(x) = \frac{2x^2}{(1-x)^3}$ , i.e.,

$$\begin{aligned} \frac{2x^2}{(1-x)^3} &= P(-1)x + P(-2)x^2 + P(-3)x^3 + \dots \\ &+ P(-n)x^n + \dots, \quad |x| < 1. \end{aligned}$$

The recurrent formula for the sequence  $P(0), P(1), P(-1), P(2), P(-2), \dots$  can be obtained from the generating functions  $f_1(x) = \frac{2}{(1-x)^3} = P(1) + P(2)x + \dots + P(n+1)x^n + \dots$  and  $f_2(x) = \frac{2x}{(1-x)^3} = P(-1) + P(-2)x + \dots + P(-(n+1))x^n + \dots$  using the formula  $x^2 f_1(x^2) + x^3 f_2(x^2)$ . The direct computation gives  $f(x) = \frac{2+2x^3}{(1-x^2)^3}$ , i.e., it holds

$$\begin{aligned} \frac{x(2+2x^3)}{(1-x)^3} &= P(0)x + P(1)x^2 + P(-1)x^3 \\ &+ P(2)x^4 + P(-2)x^5 \dots, \quad |x| < 1. \end{aligned}$$