

Part One

Fundamental Principles of Toothed Bodies in Mesh

Before we can understand the future, we must learn about the past

—Anonymous

1

Introduction to the Kinematics of Gearing

1.1 Introduction

A brief history of gearing and some established gear concepts are presented in this chapter as an introduction to the development of a generalized kinematic theory for the design and manufacture of gears. The primary objective is to familiarize the kinematician with gear terminology in a format that is familiar to them (compatible with established kinematic theory) as well as to introduce the gear specialist to some of the relevant kinematic concepts that are used in developing a generalized methodology for the concurrent design and manufacture of gear pairs. This approach includes the synthesis and analysis of the gear elements concurrently with the design of the corresponding cutter elements used for their fabrication. These introductory concepts will be built upon throughout this book to develop a generalized methodology based on kinematic geometry for the integrated design and manufacturing of appropriate toothed body to transmit a specified speed and load between generally oriented axes and the constraints that may restrict implementation.

1.2 An Overview

An introduction to the complexities involved in the design and manufacture of toothed bodies in mesh can be achieved by first examining the kinematic structure of conjugate motion between parallel axes. One purpose of this chapter is to introduce the concept of toothed wheels and demonstrate the basic kinematic geometry of toothed wheels in mesh as well as their fabrication. This extended introduction is intended to establish a foundation that will be used as a corollary to exemplify the intricacies of spatial gearing (namely, worm and hypoid gearing). A similar introductory treatment on gears is presented in existing textbooks on kinematics and machine design (e.g., Spotts, 1964; Martin, 1969; Shigley and Uicker, 1980; Erdman and Sandor, 1997; Budynas and Nisbett, 2011). The elementary treatment provided in these textbooks on kinematics and machine design is essentially based on the books by

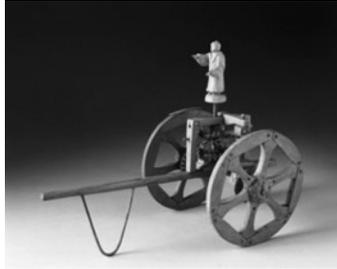


Figure 1.1 South pointing chariot (reproduced by permission of Science Museum London/Science and Society Picture Library)

Buckingham 1949 and Merritt (1971). Because of its practical importance, the design and manufacture of toothed bodies continues to attract the attention of researchers in a variety of fields (e.g., geometry, lubrication, dynamics, elasticity, material science, and computer science). Dudley (1969) provides a brief account on the history of gears, and additional information regarding the history of gears is provided by Cromwell (1884) and Grant (1899). An overview on the design and manufacture of gears is presented by Dudley (1984) and Drago (1988). Specialists in the gear industry have contributed to the second edition of *Dudley's Gear Handbook* edited by Townsend (1991). A more extensive and up-to-date analysis for the design and manufacture of gears is provided by the following organizations:

- American Gear Manufacturers Association (AGMA)
- International Standards Organization (ISO)
- Deutsches Institute für Normung (DIN)
- Japanese Gear Manufacturers Association (JGMA)
- American National Standards Institute (ANSI)
- British Gear Association (BGA)

One of the earliest documented geared devices is the *South Pointing Chariot*. A model of a South Pointing Chariot is depicted in Figure 1.1. The function of this device is to serve as a mechanical compass in crossing the Gobi desert. The statue atop of the wheeled cart maintains a constant direction of pointing independent of the cart track. Various claims to the date of the device range from 2700 BC to 300 AD. *Heron of Alexandria* devised many mechanical systems involving mechanisms (some geared). Example systems include special temple gates, mechanized plays, coin-operated water dispensers, and the aeolipile. Leonardo da Vinci is one of the most celebrated designers of all times. Da Vinci is credited with the various sketching of gears in Figure 1.2.

Norton (2001) credits James Watt as the “first” kinematician for documenting the coupler motion of a four-link mechanism. This documentation was part of his effort to achieve long strokes on his steam engine. More noted is Euler (father of involute gearing) and his analytical treatment of mechanisms. Yet, Reuleaux is considered the “father” of modern kinematics for his text *Theoretical Kinematics*. Reuleaux defined six basic mechanical components (namely, a link, wheel, cam, screw, ratchet, and belt). A gear can be considered a manifestation of the wheel, cam, and screw.

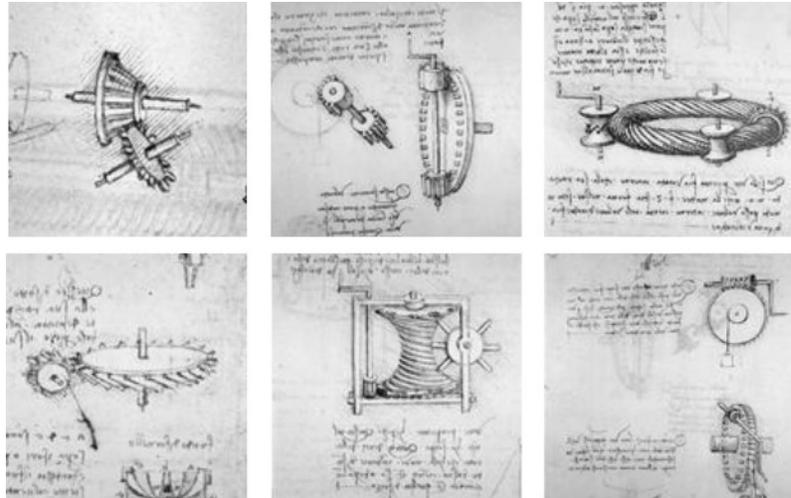


Figure 1.2 Gear sketches by da Vinci (reproduced by permission of Biblioteca Nacional)

Geared devices remain vital components in many machine systems today. As a result, the field of gearing endures an extensive pedigree and can require a devoted apprenticeship to master the subject. Due to the nature of the evolution of gearing, current research and practice, have for the most part, built on concepts charted by nineteenth century geometers. These contributions include modern concepts in kinematic synthesis and analysis, methods of manufacture, analysis of vibrations and noise, the development and integration of tribological behavior into the field of gearing, and the widespread availability of digital computers. Improvements in the field of gearing can be achieved by directing new energies toward these areas. *In order to give the field of gearing a new genesis, gears (special toothed bodies) are classified in general as elements of a mechanism that are used to control an input/output relationship between two axes via surfaces in direct contact.* As this manuscript evolves the discrepancies, limitations, inconsistencies, different design philosophies, and the need for new technology within the gear community will become more apparent and the concept of a gear will take on a new identity. The primary goal of this manuscript is to provide the gear designer with new technology and simultaneously provide the gear designer with a practical and unified approach to design and manufacture general toothed bodies. This unified approach provides the analytical foundation to better establish a correlation between theory and practice for generalized gear design and manufacture. It is written with the assumption that the reader has access to the numerous texts which illustrate traditional methods of gear design and manufacture.

1.3 Nomenclature and Terminology

An essential and important aspect of gear design and manufacture is to identify a nomenclature that distinguishes different phenomena with as few symbols as possible. Currently, each of the

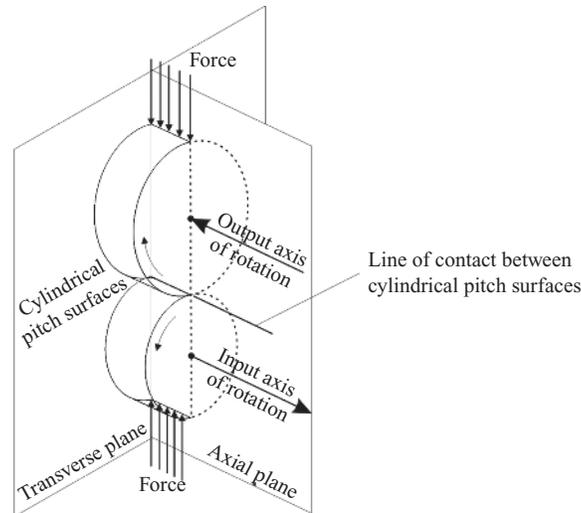


Figure 1.3 Two cylindrical wheels (friction wheels) in line contact. An applied force F exists between the two wheels in order to facilitate motion transmission

different gear types (planar, bevel, hypoid,¹ worm, and worm gears) utilize a nomenclature applicable to the particular gear type. The vernacular of a gear specialist can be misleading and confusing for the novice and may require clarification among gear specialists. Also, due to the interdisciplinary nature of gear, design and manufacture some of the established nomenclature within each discipline becomes nebulous. An attempt is made here to adhere to standard “gearing” nomenclature whenever possible.

The purpose of toothed wheels is to transmit uniform motion from one axis to another independent of the coefficient of friction that exists between the teeth in mesh. Grant was one of the first to document a treatise on toothed wheels in mesh (1899). He reveals that at the close of the nineteenth century the design and manufacture of toothed bodies was becoming more analytical, and less of a craft. As the design and manufacture of toothed wheels became more analytical the nomenclature and terminology attained more significance. The following are some of the common terms presently used in the gear community, and additional nomenclature and terminology will be established throughout this book as the analysis of toothed bodies in mesh increases.

Pitch radius: When two cylindrical wheels (input and output wheel) are in line contact as shown in Figure 1.3, the radii of the input and output cylinders are referred to as the pitch radii u_{pi} and u_{po} , respectively. Two cylinders are in line contact when the two axes of rotation are parallel. As the two cylinders rotate, there is no slippage at the line of contact. Motion transmission via two

¹The phrase *Hypoid Gear Drive* is a trademark of Gleason Works. Other forms of similar gear drives not fabricated by Gleason are referred to as simply hypoid gears or skew axis gears. The term *hyperboloidal* gears is used frequently in this manuscript when referring the theoretical development of spatial gearing where current methods for the design and fabrication of hypoid gears are not applicable.

cylinders (friction wheels) in contact is limited by the applied radial force F and the coefficient of friction that exists between the two cylinders.

Number of teeth: In order to maintain a desired speed ratio between two axes of rotation, an integer number of teeth N must exist on each wheel. The combination of the number of teeth on each wheel and the size of the two cylinders determine the load-carrying capacity of the toothed wheels in mesh.

Transverse surface: For motion transmission between parallel axes, a transverse surface of any plane is perpendicular to the axis of rotation. The transverse surface is used to parameterize toothed wheels.

Pitch circle: The pitch circle is the intersection between a cylindrical wheel and a transverse surface. The pitch circle is used as a reference for which many calculations are based. The radius of the input pitch circle is u_{pi} , and the radius of the output pitch circle is u_{po} .

Diametral pitch: The diametral pitch P_d is a rational expression for the number of teeth N divided by twice the pitch radius u : $P_d = N/2u$. The purpose for introducing such an immeasurable quantity is to specify tooth sizes using integer values. It is customary for SI designated standards to use the module m instead of the diametral pitch P_d to specify gear tooth sizes, where $P_d = 1/m$. The diametral pitch is always the same for two gears in mesh. Accordingly, $P_d = N_i/2u_{pi} = N_o/2u_{po} = (N_i + N_o)/2E$, where the center distance $E = u_{pi} + u_{po}$. The possibility of specifying an irrational I/O relationship is alleviated by defining the pitch radii in terms of the diametral pitch. $P_d < 20$ is considered coarse pitch; afterward fine pitch ($P_d \leq 20$).

Transverse pitch: The transverse or circular pitch p_t is an irrational expression for the circumferential distance along the pitch circle between adjacent teeth: $p_{ti} = 2\pi u_{pi}/N_i = p_{to} = 2\pi u_{po}/N_o = \pi/P_d$.

Addendum circle: The addendum circle is a hypothetical circle in the transverse surface whose radius is the outermost element of any tooth. The addendum is the region between the pitch circle and the addendum circle. The amount by which the radius of the addendum circle exceeds the radius of the pitch circle is expressed in terms of an addendum constant a : $u_a = u_p + a/P_d$. The active region of the gear tooth that lies in the addendum is referred to as the gear face.

Dedendum circle: The dedendum circle is a hypothetical circle in the transverse surface whose radius is the innermost element between adjacent teeth. The dedendum is the region between the pitch circle and the dedendum circle.

Center line: The two points in the transverse plane where the two axes of rotation for the input and the output wheel intersect, the transverse plane are instant centers. The line connecting these two instant centers is the center line. When the two axes of rotation are skew, the center line is the single line perpendicular to the two axes of rotation.

Center distance: The distance along the center line between the two axes of rotation is the center distance. This length is sometimes referred to as the *interaxial distance*.

Line of action: The line that passes through the point that is coincident with the two teeth in mesh and also perpendicular to the two teeth is the line of action.

Pitch point: The pitch point is the intersection between the center line and the line of action.

Clearance: The distance along the center line between the dedendum of one gear and the addendum of its mating gear is the clearance. Like the dedendum and addendum, the clearance is defined in terms of the clearance constant c and the diametral pitch P_d .

Tooth width: The distance along the pitch circle between adjacent profiles of a single tooth is the tooth width t_t .

Tooth space: The distance along the pitch circle between two adjacent teeth is the tooth space t_s . The sum of the tooth width t_t plus the tooth space t_s must be equal to the transverse pitch p_t (i.e., $p_t = t_t + t_s$).

Backlash: The amount the tooth space of one gear exceeds the tooth width of its mating gear. AGMA recommends that the face width b be proportional to tooth size. This is accomplished via the following AGMA recommendation:

$$\frac{9}{P_d} \leq b \leq \frac{14}{P_d}$$

Pressure angle: The included angle between the common tangent between the two pitch circles and the line of action.

IPS: A US customary system of measurements based on length, force, and time whose units are inches, pounds, and seconds, respectively.

CGS: A SI system of measurements based on length, mass, and time whose units are centimeters, grams, and seconds, respectively.

1.4 Reference Systems

Three distinct coordinate systems are used to parameterize the geometry of a gear pair. The three distinct Cartesian coordinate systems are

1. (X, Y, Z) fixed to the ground,
2. input (X_i, Y_i, Z_i) attached to the driving or input wheel, and
3. output (X_o, Y_o, Z_o) attached to the driven or output wheel.

Each reference frame is a conventional right-handed Cartesian coordinate system as depicted in Figure 1.4. The z_i -axis of the input reference frame (axis of rotation for the input body) is collinear with the Z -axis of the fixed reference frame. The distance E between the two axes of rotation is a fixed distance directed along the positive X -axis of the stationary reference frame. The z_o -axis of the output reference frame (axis of rotation for the output body) is perpendicular to the X - Y plane of the fixed reference frame. Associated with each of the two Cartesian coordinate systems (x_i, y_i, z_i) and (x_o, y_o, z_o) are, respectively, two systems of

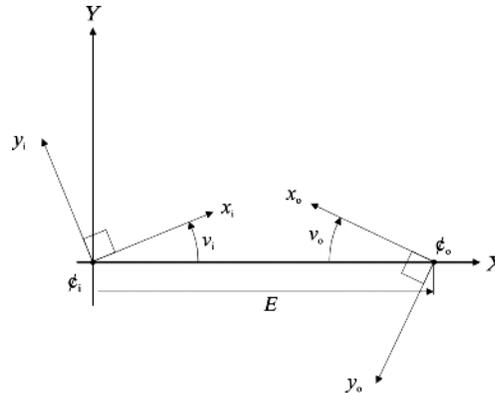


Figure 1.4 Three Cartesian coordinates systems (X, Y, Z) , (x_i, y_i, z_i) , and (x_o, y_o, z_o) are used to parameterize toothed wheels in mesh

curvilinear coordinates (u_i, v_i, w_i) and (u_o, v_o, w_o) . The curvilinear coordinates (u, v, w) are introduced to facilitate the parameterization of gear pairs and are indistinguishable from the cylindrical coordinates (r, θ, z) for motion transmission between parallel axes. The special curvilinear coordinates (u, v, w) will be introduced in Chapter 3 where a single system of curvilinear coordinates can be used to analyze the general case of toothed bodies in mesh.

1.5 The Input/Output Relationship

The relationship between the fixed coordinate system (X, Y, Z) and the input coordinate system (x_i, y_i, z_i) is defined by the net angular position v_i about the input Z -axis as measured from the fixed X -axis (see Figure 1.5). Similarly, the coordinate system (x_o, y_o, z_o) is defined by the net angular position v_o about a line parallel to the Z -axis and located at a distance E along the X -axis. The I/O relationship between the angular position v_i of the input body to that of the position (angular or linear) v_o of the output body is defined as the *transmission function*. The *instantaneous gear ratio* g is the ratio between the instantaneous angular displacement dv_o of the output and the corresponding instantaneous angular displacement dv_i of the input; thus,

$$g \equiv \frac{dv_o}{dv_i} = \frac{\text{Instantaneous angular displacement of the output body}}{\text{Instantaneous angular displacement of the input body}}. \quad (1.1)$$

Here, the differential displacements dv_i and dv_o refer to an instantaneous change in angular positions v_i and v_o , respectively. The displacements dv_i and dv_o are angular displacements about the z_i and z_o axes, respectively. The angular speeds ω_i and ω_o are, respectively, the angular displacements dv_i and dv_o per unit time dt . For uniform motion transmission between fixed axes, the transmission function is linear and its slope is a constant equal to the gear ratio. When this occurs the gear ratio is also defined by the ratio N_i/N_o of gear teeth. This ratio is defined to accommodate non-circular gears and is the reciprocal of the gear ratio used by AGMA.

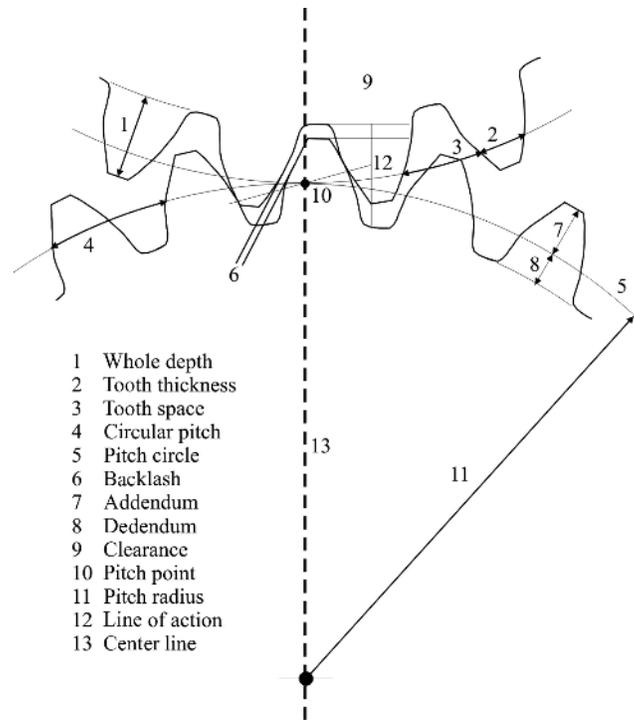


Figure 1.5 Basic terminology for toothed wheels in mesh

The z_i -axis of the input moving reference frame and the z_o -axis of the output moving frame are parallel for two external gears in mesh. The I/O relationship g is negative in this case for two external gears in mesh. Although the majority of gears are external gears, it is convenient to plot the I/O relationship g as positive for both two external gears and internal–external gears in mesh with clarification on the gear type (namely, external–external or external–internal). The elements of a gear pair are usually identified as either the *gear* or *pinion*, where the pinion is the smaller of the two gears.² It is possible in special circumstances regarding a hypoid gear pair that the pinion is physically larger than the gear and yet have fewer teeth! The reason for this phenomenon will be presented in Chapter 5. Use of “gear” and “pinion” to identify two gears in mesh does not explicitly indicate if the gear pair is used for speed increasing or speed decreasing. As a result, trailing subscripts “i” and “o” are added to identify the input and output

² In Spanish, “pinion” translates to *piñón* and “gear” to *Catalina* or Catherine (literally, gear is engrènement when simply referring to a generic “gear” element). Interestingly, St. Catherine of Alexandria has become emblematic for wheelwrights, machinist, and mechanical engineers. St. Catherine was condemned in 305 AD by the pagan emperor Maximian (305–313) for her confessed faith in Christianity. Accounts of this event vary, but one version is that a special machine consisting of wheels and axles was devised to shred Catherine. Another version is that a rack- and pinion-related device was used to stretch and torture Catherine. Both versions involve wheels and torture to discourage the spread of Christianity. St. Catherine was invested by the Catholic Church and celebrated on November 25.

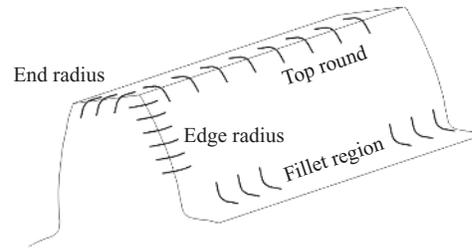


Figure 1.6 Edge radius, end radius, and top round reduce nicks and burrs encountered in shipping and handling prior to assembly

respectively. Neither subscript is used in certain situations where a notation is applicable to both the input and the output gears.

The simplest scenario of toothed wheels in mesh is motion transmission between parallel axes. Depicted in Figures 1.5 and 1.6 is some terminology used to describe toothed wheels. In general, gear designers parameterize gear teeth in a plane. This same planar parameterization is also applied to analyze bolts and nuts, presses, rotary compressors, and planar four-bar linkages. Since motion transmission between parallel axes can be adequately illustrated in a plane perpendicular to the axis of rotation it is commonly referred to as planar motion. The ease of visualizing planar motion attributes to its usage.

1.6 Rigid Body Assumption

Initially, when analyzing the kinematic geometry of toothed bodies in mesh, it is assumed that the bodies in mesh are rigid although they will inevitably deform depending on the transmitted load. These deformations are accounted for by the compliance of the housing used to support the bearings, the deflections in the bearing supports, the bending and torsional displacements in the gear blanks and shafts, and the deflection of the teeth relative to the gear blank. The assumption of rigid bodies not only simplifies analysis but also necessary in order to initially determine the geometry of the toothed bodies in mesh. The elastic deformations are subsequently calculated and compensated for by profile modifications such as profile relief and crowning of the teeth. Due to errors encountered in manufacturing, assembly, and operation of a gear pair, the amount of profile modification varies for each gear type and is generally based on experience. If the proper modifications are not incorporated then the smooth transmission of motion from one axis to another can no longer be expected to occur, and the gear teeth will be subjected to impulsive loading producing higher stresses and noise.

1.7 Mobility

Earlier in this chapter gears were described as elements of a mechanism. Reuleaux (1876) defines a mechanism as a closed kinematic chain where one of its links is held stationary. The stationary link or ground is usually indicated by feathered marks as shown in Figure 1.7a. The mobility \mathfrak{M} or the degree of freedom (dof) of a mechanism refers to the number of independent parameters that must be specified to uniquely determine the configuration or

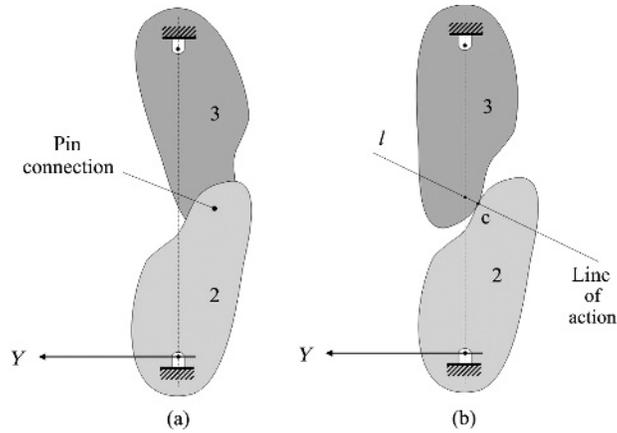


Figure 1.7 Two mechanisms where (a) has mobility one and (b) has zero mobility

arrangement of the remaining links within the mechanism. One task of a kinematician is to specify the configuration of a mechanism given the independent parameters. The mechanism shown in Figure 1.7a has no mobility (over constrained), whereas the special mechanism shown in Figure 1.7b has mobility one. The difference between the two mechanisms is that the links 2 and 3 in Figure 1.7a are connected by a pin, whereas the two links 2 and 3 in Figure 1.7b are tangent to one another at point c. A mechanism with zero or less mobility is a structure or truss. The concept of mobility is important and far reaching when considering toothed bodies in mesh. First, a brief discussion regarding planar three link 1-dof mechanisms is discussed; then, later in Chapter 5 the more general case of a five link 1-dof mechanism will be discussed. More insight on mobility is found in many textbooks on mechanisms and kinematics (e.g., Hunt, 1978; Shigley and Uicker, 1980; Edman and Sandor, 1997).

The analysis of mechanisms involves identification of the types of motion that may exist between two objects. The displacement or change in position of a point relative to a fixed coordinate system is defined as *absolute displacement*. The displacement or change in position and orientation of an object relative to a fixed coordinate system is defined as *vehicular displacement*. The displacement of a point relative to another moving coordinate system is defined as *relative displacement*.

Depicted in Figure 1.8 is a movable lamina 2 (planar body) relative to the fixed coordinate system (X, Y, Z). Three independent parameters $X_2, Y_2,$ and θ_{z2} are used to specify the position

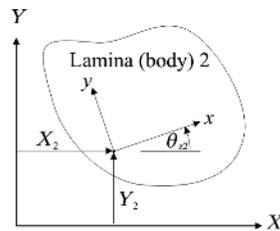


Figure 1.8 Fixed coordinate moving coordinate systems

and orientation of this movable lamina with respect to the fixed coordinate system (X, Y, Z) . The mobility m or freedom of the lamina 2 relative to the fixed coordinate system (X, Y, Z) is three:

1. A translation ΔX_2 in the X -direction
2. A translation ΔY_2 in the Y -direction
3. A rotation $\Delta\theta_{z2}$ about the Z -axis

Planar displacement can be parameterized by a linear combination of the above three displacement and that at any instant the displacement of lamina 2 can be reduced to a rotation about a fixed line parallel to the Z -axis (Theorem of Chasles). The point where this axis of rotation intersects the X - Y plane is the instant center of rotation for the moving lamina 2. By restricting the movable lamina 2 depicted in Figure 1.7b to only rotations about the Z -axis, the mobility of lamina 2 relative to (X, Y, Z) reduces to one (i.e., a rotation without translation). Similarly, restricting the movable lamina 3 depicted in Figure 1.7b to only rotations about a line parallel to the Z -axis (and located a distance E along the positive X -axis) also restricts the mobility of lamina 3 relative to (X, Y, Z) to one. Rosenauer and Willis (1953) define the connection between two bodies according to its mobility \mathfrak{M} . If the mobility between two bodies is one then it is a lower pair, and if the mobility is greater than one then it is a higher pair. Thus, the connection between body 2 and ground as well as the connection between body 3 and ground both comprise lower pairs.

In order to assess the mobility of the three link mechanism shown in Figure 1.7b, it is necessary to determine the freedom or mobility that exists between bodies 2 and 3. There cannot exist any relative motion along the line of action l at the point of contact if contact is maintained between bodies 2 and 3. The mobility between bodies 2 and 3 increases to two (a higher pair) by restricting the relative displacement between bodies 2 and 3 to rotations about a point of the line of action l . The relative mobility between the links of a planar mechanism is given by the planar mobility criterion (Hunt, 1978)³:

$$\mathfrak{M} = 3(n - k - 1) + \sum_{j=1}^k f_j, \quad (1.2)$$

where m is the mobility, n is the number of bodies, k is the number of joints, and f_j is the freedom at each joint. The above mobility criterion is frequently referred to as Grübler's mobility criterion and is a special form of a more general mobility criterion to be discussed in Chapter 5. Applying the above mobility criterion to the three link mechanism depicted in Figure 1.7, the mobility becomes

$$\mathfrak{M} = 3(3 - 3 - 1) + (1 + 1 + 2) = 1.$$

In this case, there are three elements or bodies: two gear elements and a fixed housing element. There are also three joints: one between each of the gear elements and the fixed housing thus comprising a total of two joints, and a third one at the point of contact between the two gear elements. The latter joint has 2 dof. Thus, the three link mechanism is a 1-dof mechanism. In other words, as one of the gears rotate, the other gear must rotate according to Equation (1.1).

³The number of independent parameters to determine rigid body motion in d -dimensional space is $d(d + 1)/2$. In 3D space, $3(4)/2$ or 6 independent parameters are necessary to uniquely define position and orientation.

Caution should be exercised when using the above mobility relation. Misleading or wrong results can occur for special geometries and overconstraints. The above relation treats all joints as active and does not consider idle dof or redundant constraints.

1.8 Arhbold-Kennedy Instant Center Theorem

A point that is common to two planar bodies in motion that has the same absolute velocity is referred to as an *instant center of rotation*. A transverse section of a three link mechanism is shown in Figure 1.9. The intersections between the two axes of rotation z_i and z_o for the two bodies shown in Figure 1.9 and a transverse surface (the X - Y plane) are referred to as the instant centers of rotation ϕ_i and ϕ_o , respectively. Using the special notation “ ϕ_i ” and “ ϕ_o ” to represent axes of rotation by points is valid for motion transmission between parallel axes. Here, the absolute velocities between the fixed coordinate system (X, Y, Z) and the centers of rotation ϕ_i and ϕ_o corresponding to the two moving coordinate systems) and (x_o, y_o, z_o) , respectively, are zero, thus instant centers. Since the z_i -axis of the input Cartesian coordinate system (x_i, y_i, z_i) is coaxial with the Z -axis of the fixed Cartesian coordinate system (X, Y, Z) , the point coordinates of the instant center ϕ_i relative to the fixed coordinate system are determined by $(X, Y, Z) = (0, 0, 0)$. The transverse surface is defined by the X - Y plane of the fixed Cartesian coordinate system (X, Y, Z) . The z_o -axis of the output Cartesian coordinate system (x_o, y_o, z_o) is perpendicular to the X - Y plane and intersects the positive X -axis at a distance E from the origin; thus, the coordinates of the instant center ϕ_o relative to the fixed coordinate system are determined by $(X, Y, Z) = (E, 0, 0)$.

Uniform motion transmission between two parallel axes is possible only if the line of action passes through a fixed point ϕ_{irp} known as the *pitch point*. The subscript “irp” signifies that ϕ_{irp} is the *instantaneous rotation pole*. The locus of pitch points (relative to the input coordinate system) for each angular position v_i of the input determines the input’s *pitch curve*

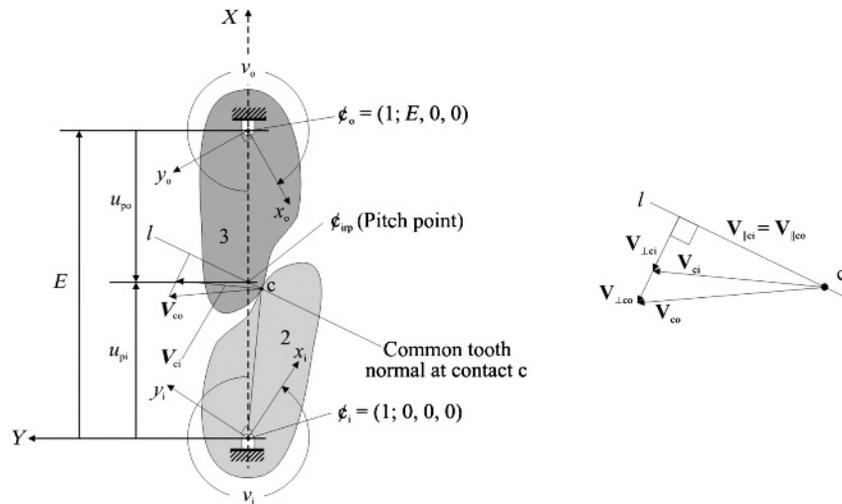


Figure 1.9 Two bodies in direct contact for motion transmission between parallel axes

or *centrode*. Likewise, the locus of pitch points (relative to the output coordinate system) for each angular position v_o of the output determines the output's *pitch curve* or *centrode*. For uniform motion, transmission the pitch curves become circles whose radii u_{pi} and u_{po} depends on the magnitudes of the I/O relationship g and the center distance E .

Illustrated in Figure 1.9 are the input and output bodies, the line of action l , the point of contact c , and the pitch point for the two bodies in contact. In the majority of gearing applications, the position ϕ_{irp} of the pitch point p remains invariant for each angular position v_i ; nonetheless, circumstances can exist (see Appendix C) where the position of the pitch point p varies for the different input positions v_i . Two “planar” curves in direct contact are *conjugate* if the line of action l passes through the *desired* pitch point for each angular position v_i of the input. In general, the line of action does not have to pass through the desired pitch point ϕ_{irp} as will be explained in Chapter 5.

Two toothed wheels in mesh are in effect two links of a three link 1-dof mechanism. Bodies 2 and 3 depicted in Figure 1.9 are in mesh. The input body is one link, the output body is another link, and ground or the hypothetical link connecting the two axes of rotation is the third link. Considering the toothed wheels in mesh as a three link 1-dof kinematic chain, any one of the three links may be held stationary. The process of holding stationary different links of a kinematic chain is known as *inversion*. Although the absolute motion of the three link 1-dof kinematic chain is different depending on which link is held stationary, the relative motion between the three links remains unaltered. Knowledge of the relative displacements between two elements of a mechanism is necessary for its design (both kinematic and structural). It is important to understand that the relative motion (planar in this case) between the three links necessary to define a gear pair can be obtained from the special vector loop equation:

$$dv_{12}\phi_{12} + dv_{23}\phi_{23} + dv_{31}\phi_{31} = 0, \quad (1.3)$$

where $dv_{12}\phi_{12}$ is the relative angular displacement of body 2 with respect to body 1 (i.e., an angular displacement dv_{12} about point ϕ_{12}), $dv_{23}\phi_{23}$ is the relative angular displacement of body 3 with respect to body 2, and $dv_{31}\phi_{31}$ is the relative angular displacement of body 1 with respect to body 3. Body 1 represents ground or the fixed reference system, body 2 is the input body, and body 3 is the output body. By holding stationary link 1 then its displacement is always zero and can also be specified using the vector loop Equation (1.3). That is, the relative displacement of body 2 with respect to body 1 plus the relative displacement of body 3 with respect to body 2 plus the relative displacement of body 1 with respect to body 3 must always sum to zero for the closed three link 1-dof kinematic chain.

The displacement $dv_{12}\phi_{12}$ of the input body with respect to ground is denoted $dv_i\phi_i$, the displacement $dv_{13}\phi_{13}$ (where $dv_{13}\phi_{13} = -dv_{31}\phi_{31}$) of the output body with respect to ground is denoted $dv_o\phi_o$, and the displacement $dv_{23}\phi_{23}$, denoted $dv_{irp}\phi_{irp}$, is the relative displacement of the output body with respect to the input body. The subscript *irp* is used to indicate ϕ_{irp} is the instantaneous rotation pole between bodies 2 and 3. The instantaneous angular speeds ω_i and ω_o are extracted from the angular displacements dv_i and dv_o , respectively, by dividing Equation (1.3) through by the incremental change in time dt , where $\omega_i = dv_i/dt$ and $\omega_o = dv_o/dt$. Hunt (1978), Bottema and Roth (1979), and Phillips (1984) each present a more general treatment on the closure of general kinematic chains.

In order for two pitch circles to rotate without slippage at the point of contact (for two circles in mesh the point of contact and the pitch point are coincident), the absolute velocity of

the point of contact on bodies 2 and 3 must be the same relative to the fixed coordinate system (X, Y, Z) . An important theorem from planar kinematics is the Arhnhold-Kennedy instant center theorem: for three rigid bodies 1, 2, and 3 in mesh, the instant centers ϕ_{12} , ϕ_{23} , and ϕ_{31} between bodies 1 and 2, bodies 2 and 3, and bodies 3 and 1 all lie on a straight line. Applying the Arhnhold-Kennedy instant center theorem to toothed wheels in mesh reveals that the pitch point (instant center ϕ_{irp}) must always lie on the line connecting the two wheel's center of rotation. In a mathematical sense, the linear combination of the two points ϕ_i and ϕ_o must be a third point ϕ_{irp} on the line connecting the two points ϕ_i and ϕ_o .

Before the coordinates dv_{irp} ϕ_{irp} of the instant center can be determined using Equation (1.3), it is beneficial to first parameterize the instant centers ϕ_i and ϕ_o in terms of the special point coordinates $\phi_i = (W_i; 0, 0, 0)$ and $\phi_o = (W_o; E, 0, 0)$. The special coordinates for the instant centers ϕ_i and ϕ_o are supplemented by introducing the additional reference parameters W_i and W_o where $W_i = W_o = 1$. Introducing the additional reference parameters W_i and W_o to describe the instant centers ϕ_i and ϕ_o , respectively, enables ϕ_{irp} to be obtained by simply summing the special point coordinates ϕ_i and ϕ_o . The special coordinates used to uniquely define the position of a point are known as *homogeneous point coordinates*. This is demonstrated diagrammatically in Figure 1.10. The sum of the two points ϕ_1 and ϕ_2 results in a third point ϕ_3 . Without the additional reference parameters W_1 and W_2 , the sum of the two points (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) yields a third point $(X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2)$ on the line l_3 . By introducing the reference parameters W_1 and W_2 , the sum of the two points ϕ_1 and ϕ_2 becomes a third point on the line l_3 as well as the line l_{12} . Depending on the ratio

$$\sqrt{W_1^2 + X_1^2 + Y_1^2 + Z_1^2} : \sqrt{W_2^2 + X_2^2 + Y_2^2 + Z_2^2},$$

the sum $\phi_1 + \phi_2$ is always a third point ϕ_3 on the line l_{12} connecting the two points ϕ_1 and ϕ_2 . Initially, the use of the special point coordinates ϕ_i and ϕ_o to determine the instant center or pitch point ϕ_{irp} might appear unnecessary. The advantage for implementing such an approach

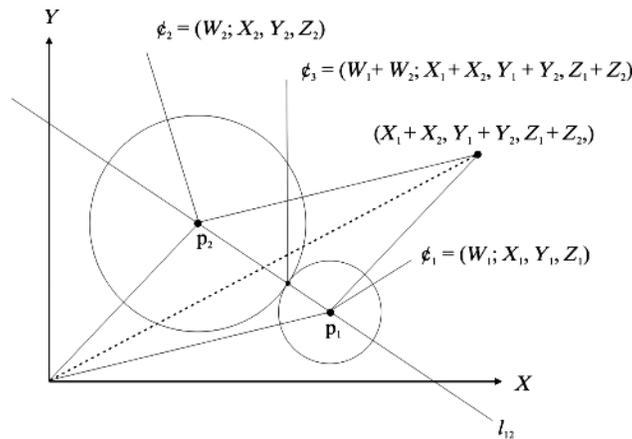


Figure 1.10 The sum of the two points p_1 and p_2 using homogeneous coordinates is a third point ϕ_3 on the line l_{12}

to determine the instant center is that an analogous approach is used to determine the spatial equivalence of the pitch point as will be demonstrated in Chapter 3.

Application of the Arhnhold-Kennedy instant center theorem indicates that the pitch point or instant center $dv_{irp}\phi_{irp}$ must lie on line which passes through the two centers ϕ_i and ϕ_o of rotation (this is the fixed X -axis). The scaling or weighting factors dv_i and dv_o determine the location of ϕ_{irp} on this line. The radius u_{pi} or the distance along the fixed X -axis from the origin of the fixed coordinate system (X, Y, Z) is the magnitude of $dv_{irp}\phi_{irp}$. Solving the vector loop Equation (1.3) for the pitch point $dv_{irp}\phi_{irp}$ yields (where g is negative for two external gears)

$$dv_{irp}\phi_{irp} = -dv_i(\phi_i - g\phi_o) = dv_i(1 - g; gE, 0, 0). \quad (1.4)$$

The magnitude of $dv_{irp}\phi_{irp}$ from the origin of the fixed coordinate system (X, Y, Z) is the radius u_{pi} of the input centrode for each angular position v_i where

$$u_{pi} = E \frac{g}{g - 1}, \quad (1.5a)$$

and the radius u_{po} of the output centrode for each angular position v_i is

$$u_{po} = E - u_{pi} = \frac{-E}{g - 1}. \quad (1.5b)$$

Recognize that when the I/O relationship $g = 1$ that the denominators of Equations (1.5a) and (1.5b) both vanish and the pitch point for the gear pair is infinitely located.

The fact that the instant center must lie on the line connecting the two wheels' center of rotation will be demonstrated synthetically. As shown in Figure 1.9, the point of contact c is a general point in the transverse plane. The absolute velocity \mathbf{V}_{ci} of the point c coincident with the input body must be perpendicular to the line connecting c and the instant center ϕ_i . Similarly, the absolute velocity \mathbf{V}_{co} of c coincident with the output body must be perpendicular to the line connecting c and the instant center ϕ_o . Thus, the absolute velocities are not collinear. Unless the point of contact c lies on the line connecting the two instant centers ϕ_i and ϕ_o , it cannot be a pitch point. In order for two conjugate surfaces to remain in mesh, the component of the absolute velocities along the line of action l must be identical, otherwise rigid bodies 2 and 3 become separated.

The velocity of the point on the input body coincident with the point of contact c can be resolved into two components: one component $\mathbf{V}_{\perp ci}$ is perpendicular to the line of action l and another component $\mathbf{V}_{\parallel ci}$ parallel to l . Similarly, the velocity of the point on the output body coincident with the point of contact c can also be resolved into two components: one component $\mathbf{V}_{\perp co}$ is perpendicular to the line of action l and another component $\mathbf{V}_{\parallel co}$ parallel to l . In order for the two teeth to remain in contact the two components $\mathbf{V}_{\parallel ci}$ and $\mathbf{V}_{\parallel co}$ must be equal. The difference in the two perpendicular components $\mathbf{V}_{\perp ci} - \mathbf{V}_{\perp co}$ is the relative sliding \mathbf{V}_s between the two teeth. The cyclic behavior of the relative sliding is a source of vibrations and noise. The presence of friction at the point of contact c causes the line of action of the net force $\$w$ ⁴ (the net force $\$w$ is the sum of the force along the common normal between the teeth in contact plus the frictional force) between the input and output to no longer pass through the desired pitch point p . During the engagement of gear teeth, lubricant is rapidly displaced

⁴The dollar symbol "\$" is used to indicate a line, whereas the cents symbol "¢" refers to a point. It is a series of points that determine a line analogous to the convention that an amount of "cents" define a "dollar."

within the contact zone. Due to the friction forces at the contact, the product of speed and torque put into the system does not equal the product of speed and torque extracted from the gear system (i.e., $T_o \neq T_i/g$). The difference $T_i\omega_i - T_o\omega_o$ due to friction at the contact is what is generally known as the *mesh losses*.

1.9 Euler-Savary Equation for Envelopes

Another important theorem from planar kinematics that can be useful in analyzing toothed bodies in mesh is the Euler-Savary equation. In planar kinematics, there exist several forms of the Euler-Savary Equation.⁵ The Euler-Savary Equation for envelopes reveals a limitation on the relative gear tooth curvature. Motion transmission between two axes by gear elements is produced by two surfaces in direct contact. It was discussed in Section 1.9 regarding the Arhbold-Kennedy instant center theorem that conjugate motion exists when the line of action l intersects the desired pitch point p between the input and output and the instantaneous gear ratio $g = \omega_o/\omega_i$ bodies is $g = u_{pi}/u_{po}$. This result is used to establish the planar Euler-Savary equation for envelopes. Additional information of the spatial analog to the planar Euler-Savary equation is presented in Appendix J.

Determination of the relative gear tooth curvature $\Delta\kappa$ between two planar involute gear teeth is demonstrated prior to presenting the relative gear tooth curvature between two generalized gear teeth. Depicted in Figure 1.11 are two involute gear teeth in mesh. The radius of the input pitch circle is R_i , whereas R_o is the radius of the output pitch circle. ρ_i and ρ_o are the radii of curvature for the input and output gear teeth, respectively. Projecting the pitch radii R_i and R_o onto the contact normal yields

$$\rho_i = R_i \sin \phi \quad (1.6a)$$

$$\rho_o = R_o \sin \phi, \quad (1.6b)$$

where ϕ is the angle between the pitch circle tangency and the tooth contact normal. For planar curves, the curvature κ and radius of curvature ρ are reciprocals (i.e., $\kappa = 1/\rho$); thus, relative gear tooth curvature $\Delta\kappa$ can be expressed as follows:

$$\Delta\kappa = \left(\frac{1}{\rho_i} + \frac{1}{\rho_o} \right) \quad (1.7a)$$

or

$$\Delta\kappa = \left(\frac{1}{R_i} + \frac{1}{R_o} \right) \frac{1}{\sin \phi}, \quad (1.7b)$$

where

- R_i radius of input pitch circle,
- R_o radius of output pitch circle, and
- ϕ pressure angle.

⁵ Felix Savary (1797–1841) was a student and professor at the Ecole Polytechnique. His interests included astronomy and geodesy. Savary worked with Andre Marie Ampere and is credited with an ability to focus intensely and this skill seems to have been influential on Ampere and his research. *Kinematics* as a separate science was defined by Ampere.

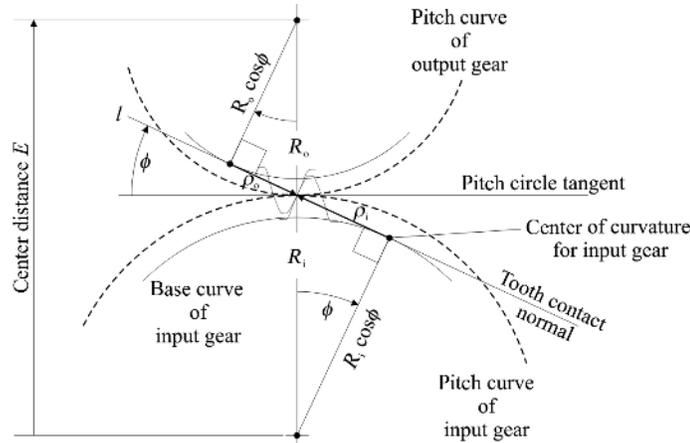


Figure 1.11 Two pitch circles in contact

Equating the above expression establishes a unique relation among the pressure angle ϕ , the pitch radii R_i and R_o , and the relative gear tooth curvature $\Delta\kappa$. This relation is known as the Euler-Savary equation for envelopes. Regardless of the radii of tooth curvature ρ_i and ρ_o , the relative gear tooth curvature $\Delta\kappa$ depends solely on pitch radii R_i and R_o and pressure angle ϕ . The above relation for relative gear tooth curvature is for cylindrical gears with spur type gear teeth. Further, this relation is valid only for contact at the pitch point.

1.10 Conjugate Motion Transmission

Initially toothed bodies began as pegged wheels depicted in Figure 1.12. As loads and speeds increased, the idea of pegged wheels no longer sufficed. The speed fluctuations due to the non-conjugate behavior were sources of dynamics which eventually led to tooth failure. Craftsmen or wheelwrights modified these pegged wheels to reduce the dynamics, and these modifications eventually evolved into the modern tooth profile. The Arnhold-Kennedy instant center theorem (Section 1.8) was presented to demonstrate that uniform motion transmission between two parallel axes exists if the line of action between the two bodies in mesh passes through

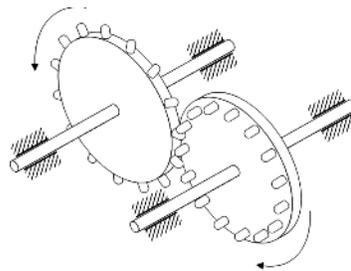


Figure 1.12 Primitive gears originated as pegged wheels

a fixed point; the pitch point. The mechanization of the eighteenth century required gears to operate at higher speeds and sustain higher loads. Thus the pegged wheel gave way to the cast iron gear that cannot tolerate nonconjugate motion. Included in the advancing gear technology was also the concept of interchangeability between different gear pairs.

1.10.1 *Spur Gears*

Analytical treatment of toothed wheels produced many forms of conjugate tooth profiles. Initially, an arbitrary profile was chosen for one body and the mating or conjugate profile for the other body was determined by satisfying the Arhbold-Kennedy instant center theorem (Equation (1.4)) (Reuleaux, 1876). This type of analysis led to many impractical profile forms which were eventually abandoned. Reuleaux (1876) provides an excellent treatment regarding the kinematics of early tooth profiles, their methods of synthesis, and how they evolved. Three forms of conjugate tooth profiles that have received universal recognition are

1. the cycloidal tooth profiles,
2. the involute tooth profiles, and more recently
3. the circular-arc tooth profiles.

Currently, the procedures used to obtain coordinates for each of the above three forms of candidate tooth profiles are different. Tooth forms where the profile remains parallel to the generators of the pitch surface are defined as *spur gears*. It is necessary to investigate multiple transverse surfaces to determine if a gear has spur-type teeth.

So far a single transverse section has been used to illustrate conjugate motion. However, in order to physically transmit power from one axis to another, the toothed wheels must be of finite thickness (i.e., an infinite number of transverse sections). What is the tooth shape for each transverse section that will ensure conjugate motion? The simplest and most immediate solution is that the tooth shape remains invariant for each transverse surface. By introducing a family of tooth profiles each parallel to one another, the pitch point is no longer a point but instead becomes a line comprised many pitch points. At each instant, the locus of the pitch points comprise a line parallel to the axes of rotation z_i and z_o . Conjugate motion continues provided the line of action intersects the locus of pitch points. Spur gear can also be determined by recognizing that the line of action is perpendicular to the locus of pitch points. The pitch curves are no longer curves in this case but are instead pitch cylinders as shown in Figure 1.13.

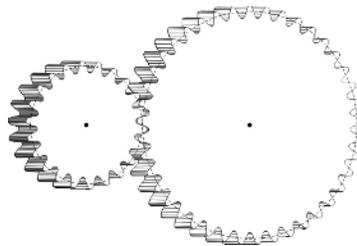


Figure 1.13 Two cylindrical wheels in mesh

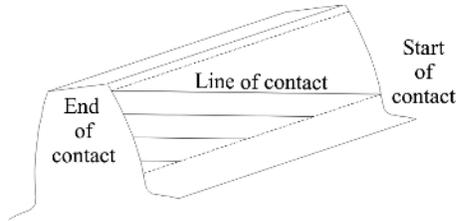


Figure 1.14 Lines of contact with a spiral tooth on a cylindrical wheel

1.10.2 Helical and Crossed Axis Gears

One feature of spur-type gears is that the initial contact between two teeth is independent of the axial position along the axis of rotation. The initial engagement of two teeth is a line segment parallel to both axes of rotation. As the two gears mesh, this line of contact remains parallel to the locus of pitch points. The length of this line segment of contact between two spur-type gears does not change in magnitude during the mesh cycle. A more favorable engagement between gear teeth would be for the teeth to gradually enter into line contact. This is achieved by inducing an angular offset between each transverse section. When this occurs toothed wheels are said to be *helical* or *spiral*. The term helical is usually associated with cylindrical gearing whereas spiral is associated with bevel and hypoid gearing. In pursuit of developing a unified approach, the term spiral is used to identify nonspur tooth forms. An example of a spiral tooth on a cylindrical pitch surface is shown in Figure 1.14. The line segment of contact is no longer parallel to the locus of pitch points. Contact between spiral gears begins as an infinitesimal line segment (point contact). As contact progresses the line segment increases in length to a certain limit, then propagates across the tooth surface, and finally decreases in length. This is illustrated by the various line segments shown in Figure 1.14.

The axial displacement dw_i (or instantaneous lead dL_i) of the line segment of contact must be the same for any radius u_i . The only way to achieve an instantaneous invariant lead dL_i is for the spiral angle ψ_i to change for each radius u_i . If the instantaneous lead dL_i is not constant for each transverse section then the axial displacement dw_i associated with each radius u_i will be different and the gear teeth would bind or become locked. Shown in Figure 1.15 is the

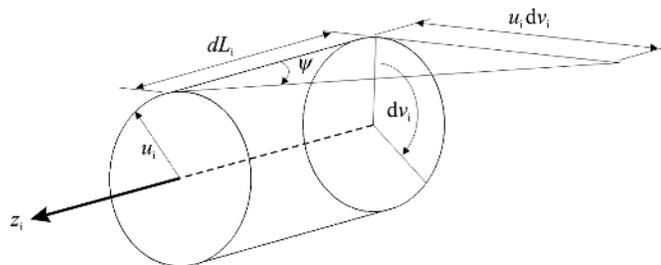


Figure 1.15 The relationship among the radius u_i , the angular displacement dv_i , the instantaneous lead dL_i , and the spiral angle ψ_i



Figure 1.16 Double spiral or herringbone gears

radius u_i , an angular offset dv_i , the spiral angle ψ_i , and the instantaneous lead dL'_i . In order to preserve an instantaneous invariant lead dL'_i , the spiral angle ψ_i for each radius u_i becomes

$$\psi_i = \tan^{-1} \frac{u_i}{dL'_i}. \quad (1.8)$$

The amount of angular displacement dv_i between successive transverse profiles depends on the axial displacement dw_i and the instantaneous lead dL'_i . Typically, the instantaneous lead dL'_i is defined in terms of the spiral angle ψ_{ρ_i} associated with the pitch radius u_{ρ_i} , where $10^\circ < |\psi_{\rho_i}| < 30^\circ$. For example, the angular offset Δv between the two axial position w_1 and w_2 is (ψ_{ρ_i} is constant)

$$\Delta v = \frac{(w_1 - w_2) \tan \psi_{\rho_i}}{u_{\rho_i}}. \quad (1.9)$$

If a pitch spiral angle $\psi_{\rho_i} = 20^\circ$ exists on the input gear then the pitch spiral angle ψ_{ρ_o} for the output gear must be $\psi_{\rho_o} = -20^\circ$. Otherwise the two axes of rotation z_i and z_o cannot be parallel. When the two spiral angles ψ_{ρ_i} and ψ_{ρ_o} are not equal and opposite, then the included angle between the input axis z_i and the output axis z_o becomes $\psi_{\rho_i} + \psi_{\rho_o}$. Two cylindrical gears in mesh are referred to as *crossed axis cylindrical gears* or *nonenveloping gears* when the shaft angle is nonzero (i.e., $\Sigma \neq 0$).

Spur gears are special spiral gears where the spiral angle ψ_{ρ_i} is zero. Usually, the spiral angle ψ_{ρ_i} is constant for cylindrical gears. The freedom to arbitrarily choose the spiral angle ψ_{ρ_i} and still satisfy conjugate motion does not exist for motion transmission between nonparallel axes as will be discussed later. One aspect of spiral gears is that an axial thrust is produced, and hence, an increase in the contact force that must exist between the two surfaces in mesh in order to transmit the same load. One method of balancing or eliminating the axial thrust produced by spiral gears is to use *herringbone gears* that incorporate two equal and opposite spiral angles on each gear. The sign of the spiral angle determines the hand. Looking along the gears axis of rotation, if the spiral angle is positive then the gear is said to have a left hand. Illustrated in Figure 1.16 are two herringbone gears or gears that incorporate equal and opposite hands. Each transverse surface of a pair of spiral cylindrical gears is equivalent to a transverse section of a pair of spur cylindrical gears.

1.11 Contact Ratio

At least one pair of teeth must come into contact before the adjacent pair of teeth in contact become separated in order to sustain conjugate motion for toothed wheels in mesh. It can be

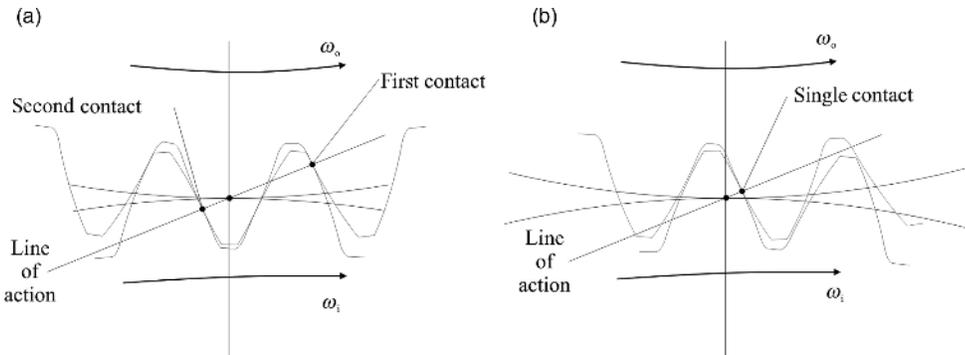


Figure 1.17 Gear pair with single contact and double contact

seen in Figure 1.17a that one pair of teeth is in contact and that in Figure 1.17b two pair of teeth are in contact. The number of teeth in contact at any given instant is referred to as the *engagement factor*. A measure of the average number of teeth in contact is the *total contact ratio* m . The total contact ratio is the sum of two components; the *transverse contact ratio* m_t and the *axial contact ratio* m_a . Figure 1.18 is used to illustrate the transverse contact ratio. For example, if the transverse contact ratio m_t is 1.3 then the average number of teeth in contact is between one and two. In this case one pair of teeth is in contact for 70% of the mesh cycle and two pairs of teeth are in mesh for the remaining 30%. Similarly, if the transverse contact ratio is 2.1 then three pairs of teeth are in contact for 10% of the mesh cycle and two pairs of teeth are in contact for the remaining 90%. If the transverse contact ratio $m_t = 0.75$ and the axial contact ratio m_a is 0.35 then the total contact ratio m is $0.75 + 0.35 = 1.1$; thus, conjugate motion can only be achieved through a combination of transverse and axial displacement of the line segment of contact.

Load sharing exists as gear teeth mesh. One common assumption is the ISO 6336 standard for metal spur gears. This assumption is that a pair of teeth support 1/3 of the total transmitted load at the instant of engagement and this distribution of load increases to 2/3 the total

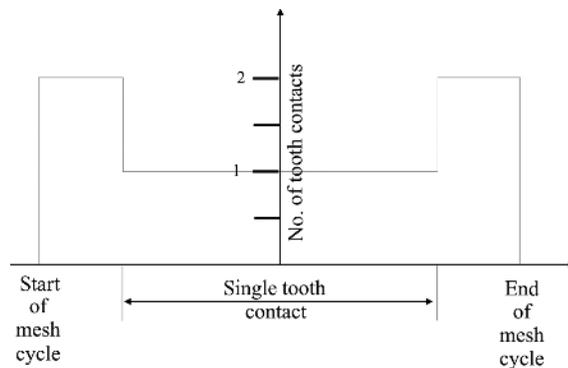


Figure 1.18 Contact ratio

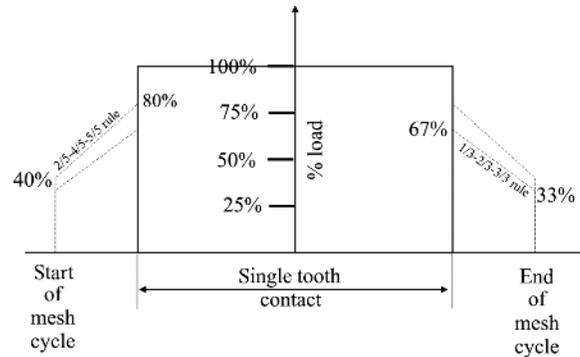


Figure 1.19 1/3-3/3-1/3 and 2/5-4/5-5/5 rule

transmitted load at the instant the adjacent set of teeth are disengaging. The load distribution is increased to 3/3 or 100% of the total when the gear teeth are the only set in mesh. This loading is known as the 1/3-2/3-3/3 rule (also exists the 2/5-3/5-5/5 rule). This phenomenon is illustrated in Figure 1.19. Gears with large difference in compliance typically do not follow this assumption (e.g., a plastic gear in mesh with a metal gear).

1.11.1 Transverse Contact Ratio

The transverse contact ratio is a dimensionless measure between the mesh cycle and the angle subtended by the transverse pitch. The angle subtended by the transverse pitch is defined as the total *arc of action*. The total arc of action is further decomposed into the *angle of approach* and the *angle of recess*.

The strength of the teeth depends on the tooth size. If the transmitted force is high, it is necessary to incorporate large teeth (small diametral pitch P_d) such that the teeth do not fracture. Also, as the diametral pitch P_d decreases the gear size must increase (i.e., an increase in pitch radii u_{pi} and u_{po}) to maintain $m_t \geq 1$. It is generally desirable to keep the pitch radii u_{pi} and u_{po} as small as possible in order to reduce the size and weight of the gears and thus reduce the tangential velocity of the pitch point or pitch line velocity. One of the adverse features of toothed wheels with small diametral pitch is the variation in compliance or change in bending stress (and hence deflection of the point of contact) as the point of contact traverses the mesh cycle. An increase in diametral pitch P_d reduces the variation in compliance as teeth mesh. For these reasons, gears with small diametral pitch gears are generally used to transmit large loads at low speeds where the change in compliance is not of primary importance. As the speed of operation for gears in mesh increases the change in compliance gives rise to higher vibrations and noise. High transverse contact ratios m_t are obtained by increasing the diametral pitch P_d , increasing the addendum constant, or decreasing the pressure angle ϕ .

Illustrated in Figure 1.18 is transverse contact ratio $m_t = 1.3$. The number of teeth in mesh is an integer and the change in load distribution among teeth in mesh is more significant for low transverse contact ratios. Near integer contact ratios require special attention due to the short duration of contact and affects the dynamic behavior of toothed bodies in mesh. The benefits of these phenomena necessitates exactly when teeth engage and disengage.

1.11.2 Axial Contact Ratio

In cylindrical gears with a nonzero spiral angle ψ_{pi} (i.e., helical gears), the length or face width F of a gear determines the angular offset between the two transverse surfaces that determine the end planes of the gear. It was discussed in Section 1.10.2 that the initial engagement between two teeth depends upon the transverse surface defined by the axial position w_i and that conjugate motion can be achieved by allowing the line segment of contact to propagate along an axis parallel to the locus of the pitch points. The transverse pitch p_{ti} is the distance along the pitch circle (within a transverse surface defined by a constant axial position w_i) between adjacent teeth. Similarly, the axial pitch p_{ai} is the distance between adjacent teeth in the direction of the axis of rotation z_i (within an axial surface define by a constant angular position v_i). Provided the spiral angle ψ_{pi} is constant, the axial pitch is expressed in terms of the transverse pitch:

$$p_{ai} = \frac{p_{ti}}{\tan \psi_{pi}}. \quad (1.10)$$

The axial contact ratio m_{ai} or face contact ratio is the face width divided by the axial pitch; hence,

$$m_{ai} = \frac{F}{p_{ai}}. \quad (1.11)$$

Provided $\psi_{pi} = -\psi_{po}$, then the axial contact ratio is the same for both the input and output. The axial contact ratio differs from the transverse contact ratio in that the transverse contact ratio depends upon both the input and output gear whereas the axial contact ratio does not. In general, two gears with different axial contact ratios cannot mesh with one another. An exception is for crossed axes cylindrical gearing (i.e., the included angle $\Sigma \neq 0$). Why this occurs will be explained in Chapter 5. Finally, the total contact ratio m_{tot} is the sum of the transverse contact ratio m_{ti} and the axial contact ratio m_{ai} :

$$m_{tot} = m_{ti} + m_{ai}. \quad (1.12)$$

If the total contact ratio is always greater than two then the gear pair is referred to as one with a *high contact ratio*. This is usually achieved by increasing the addendum. Anytime the transverse contact ratio m_{ti} is greater than zero then the contact between the two gears is a line segment.

1.12 Backlash

The amount the tooth spacing exceeds the tooth width is referred to as *backlash*, which is used to prevent the nondriving side of gear teeth from contact and is usually specified in terms of length. Backlash is also needed to accommodate tooth deflections, thermal expansion of the gear pair, foreign material in the lubricant, as well as errors in the manufacture, assembly, and operation of gears in mesh. Gears with zero backlash, or *antibacklash* gears, are sometimes incorporated in gear systems if the input or driver gear frequently changes direction of rotation.

Backlash may be measured along the pitch circle or measured perpendicular to the tooth surface using a feeler gauge. If the normal tooth width t_{ni} is determined in a manner similar to the normal pitch p_{ni} then the backlash δ_B becomes

$$\delta_B = p_{ni} - 2t. \quad (1.13)$$

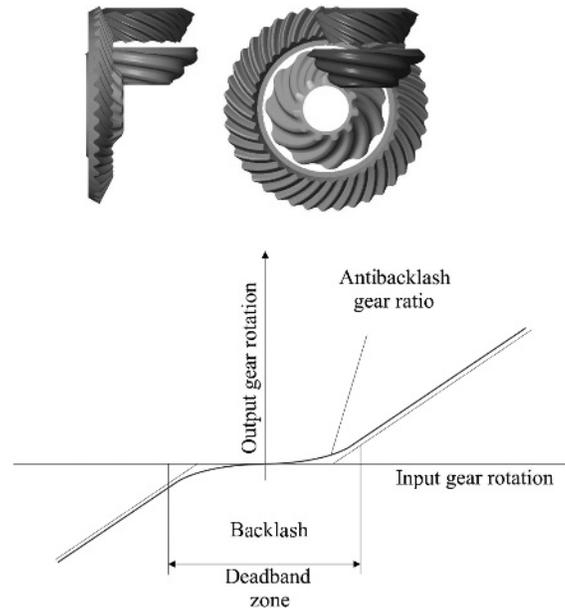


Figure 1.20 Antibacklash gears

Backlash is a measure of the amount of angular displacement Δv_i that must accompany a change in direction of rotation and depends upon the entire gear topology and not just a single transverse surface. A dimensionless backlash value of zero implies that the tooth thickness is equal to the tooth space. A value of one implies that the tooth width t_{ti} is zero. It is customary that backlash is achieved by equal reductions in tooth thickness between the input and the output. Backlash is also important in finishing operations such as shaving, honing, burnishing, and inspection.

With backlash, there is a small angle that the input gear can rotate without contacting or imparting a load to the output or mating gear element. This angular region is referred to as the *deadband zone* as depicted in Figure 1.20. The concept of *antibacklash* gears is when the input gear changes directions of rotation there is no deadband zone. This can be achieved by introducing two gear pairs for a single drive as depicted later. One gear pair is the *primary drive* and the other gear pair is the *secondary drive*. One output elements of the primary and secondary drives are rigidly connected, whereas the input elements of the primary and secondary drives are torsionally loaded via a spring element. The deflections of gear bodies, shafts, and housing all contribute to the change in compliance during a change in direction of rotation. The deadband zone is not eliminated but reduced with such antibacklash gears. Also, the depicted antibacklash gears experience a decrease in efficiency due to the primary and secondary meshes.

1.13 Special Toothed Bodies

Situations exists where the “pitch surfaces” and the “axodes” are not the same. An interesting form of this situation emerges when synthesizing toothed bodies for motion transmission

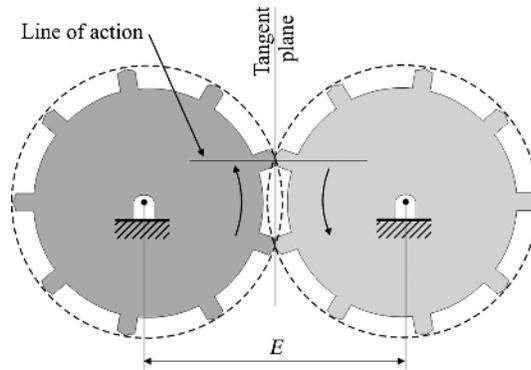


Figure 1.21 Special tooth bodies

where the I/O relationship $g = -1$. For this special case, the line of action and the plane containing the two axes of rotation are parallel and, hence, intersect at infinity. From Equation (1.6), the input pitch radius u_{pi} becomes infinite (i.e., $u_{pi} = Eg/(g - 1) = \infty$). The use of two external toothed bodies in mesh to produce the I/O relationship $g = 1$ was developed at the Bauman Institute in Moscow. Depicted in Figure 1.21 is a planar view of two such toothed bodies. The gears must be helical where the helix angles are both of the same hand. Conjugate motion between the two toothed bodies is sustained by axial displacement of the point of contact. The point of contact changes position (both transverse and axial) with a change in direction of rotation. One difference between a rack and pinion and the two external gears shown in the Figure 1.21 is that the pitch radius u_{po} for the rack is infinite and the pitch point is finitely located, whereas for the two external gears the pitch radius u_{po} for the output is finite and the pitch point is located at infinity. These gear forms experience high tooth loads and are not common for high torques and speeds.

Another special gear form is intermittent gearings. Such gears are used as counting mechanisms. The gear pair begins as a conventional gear pair. The input gear has at least one tooth as an ordinary gear tooth designed for continuous rotation. Shown in Figure 1.22 is a 1:1 gear pair with 20 teeth originally. The gears are modified to enable the output gear to rotate 1/10

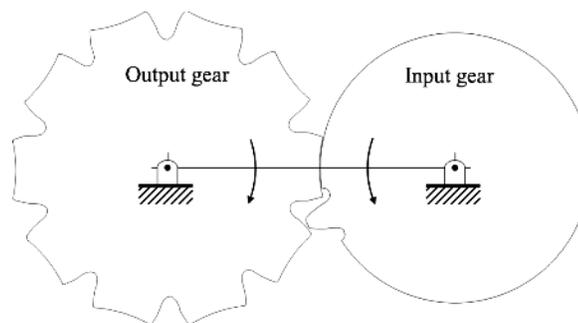


Figure 1.22 Intermittent gearing

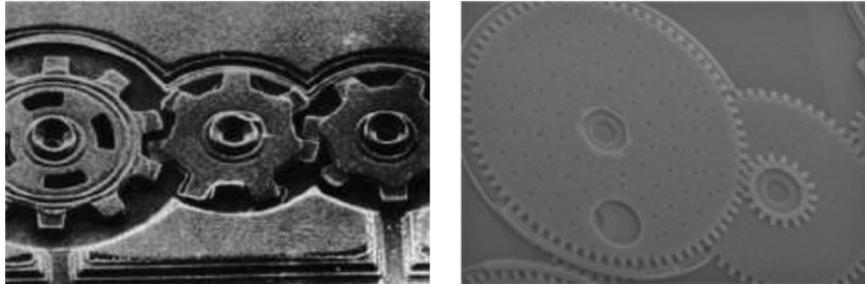


Figure 1.23 Microgears (reproduced by permission of Sandia National Laboratories)

of a revolution for 1/10 revolution of the input gear. The output gear is locked against rotation during the remainder of the input gear rotation as illustrated in Figure 1.22. The single tooth on the input gear meshes with each space on the output gear. Such gears result in a velocity jump at tooth engagement and disengagement between the input gear and the output gear.

1.13.1 Microgears

Microgears derive their name from micrometer (i.e., 10^{-6} m). Their applications are with Micro-Electro-Mechanical Systems (MEMS) in the United States or Micro-Systems Technology (MST) outside the United States. In general, MEMS range from millimeters up to a centimeter. Depicted in Figure 1.23 are microgears. MEMS typically operate at greater speeds than macromachines due to their reduced size and inertia. MEMS origins are with the electronics industry and the fabrication of Integrated Circuits. Their low cost result from the batch fabrication techniques developed. Silicon is currently the material most commonly used in MEMS. Metals, polymers, and ceramics can also be employed in MEMS.

One application involving MEMS is with accelerometers. MEMS accelerometers have replaced conventional accelerometers for crash air-bag deployment systems in automobiles. An important aspect of MEMS involves friction and wear. It is reported that wear is the dominate failure mode in MEMS devices (Ananthasuresh, 2004). This increased wear is attributed to size. Friction and wear are surface related. Area is related to length squared and volume is related to length cubed. Decreasing the dimension by a factor of 10, the surface area decreased by a factor of 100 and the volume decreases by a factor of 1,000.

1.13.2 Nanogears

Nanogears derive their name from nanometer (i.e., 10^{-9} m). The principle here differs from both macromachinery and micromachinery. These molecule-sized gears can be made from pipes of carbon atoms with benzene atoms attached to the outside of the pipe to form the teeth. The shape of the teeth are not critical as with macrogears. Depicted in Figure 1.24 is a nanogear. Based on simulations, nanogears (one-billionth of a meter in diameter) can rotate at 100 billion turns per second or six trillion RPM. Nanomachines based on gear are futuristic.

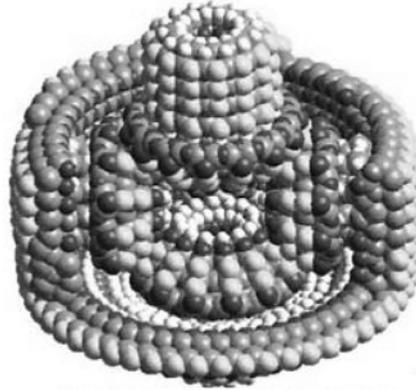


Figure 1.24 Nanogears (reproduced by permission of IMM)

1.14 Noncylindrical Gearing

Cylindrical gearing is a degenerate form of general spatial gearing. The purpose here is to discuss noncylindrical forms of toothed bodies for motion transmission. Due to the complexity of noncylindrical gears, the kinematic geometry of such gears is less developed. The highly successful implementation of hypoid, bevel, and worm gearing is attributed to much experience and the manner in which they are produced. Consequently, much of the information used for their design and manufacture are based on experience and is referred to as *gear art*. A distribution of the various gear types produced is presented in Figure 1.25.

1.14.1 Hypoid Gear Pairs

In the general theory of constant speed gearing, hypoid gearing is the most general gear type where worm, bevel, and planar gearing are special cases. Current methods of gear design and manufacture do not enable the special cases to be analyzed using general hypoid gear technology. There are numerous publications on this subject because of its practical importance. Wildhaber (1946a, 1946b, 1946c, 1946d, 1946e, 1946f) contributed to the foundation for bevel and hypoid gear design and manufacture in the first half of the twentieth century. Litvin and Fuentes (2004) offers a comprehensive account of the design and manufacture of hypoid gears. Minkof-Petrof (1983) and Shtipelman (1978) have contributed to the design of hypoid gears, but both have limited their work to uniform motion transmission between fixed axes.

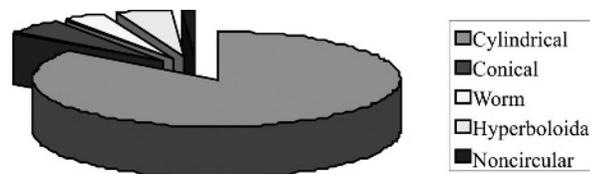


Figure 1.25 Distribution of gear types

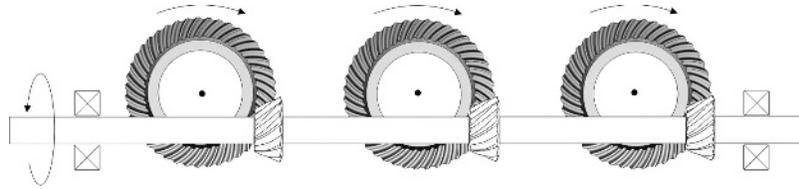


Figure 1.26 Hypoid gears enable a single input to drive multiple outputs

Dyson (1969) gives an in-depth approach to the topological analysis of surfaces in contact, but he never delves into the kinematics of motion or the manufacture of gears. Chen (1978) uses the “calculus of rotations” to investigate conjugate surfaces of gear pairs.

The widespread use of hypoid gears for power transmission originated with the automobile at the beginning of the twentieth century and are currently used in many other forms of transportation including tractors, earth moving equipment, and construction equipment. Hypoid gears are used as intermediate means to transmit power from the engine to the rear drive wheels. Initially, an overall lower center of mass for the automobile (and hence stability) was obtained by requiring that the pinion or input axis remain below the axis of the drive wheels. Another feature of hypoid gears is that their circumferential pitch can be adjusted for a given gear ratio by choice of the axial contact ratio. This is verified by recognizing that for hypoid and worm gears that the gear ratio g is not equal to the ratio of radii. The smaller of the two gears (usually the input or pinion) is subjected to the higher torque, thus a judicious choice for the axial contact ratio results in a larger input gear and hence decreases the stresses within the gear teeth. The adverse effect of increasing the size of the input gear is an increase in axial thrust. Other applications for hypoid gearing occur in industry where single input multioutput right angle drives are necessary where the input shaft cannot intersect the driven or output shafts (see Figure 1.26). The relative motion inherent to hypoid gearing can be beneficial, provided it enhances fluid film development in the mesh. This relative motion increases wear and tends to polish the contacting surfaces under certain conditions. This polishing effect can increase efficiency over time as well as reduce pitting, wear, and surface fatigue.

1.14.2 Worm Gears

The general case where both the center distance E and the shaft angle Σ are nonzero is referred to a hypoid gears. A case of hypoid gears has already been encountered in Section 1.7.1 for the special case of crossed cylindrical gears. Another special case of hypoid gears in mesh is for worm and worm gears. Like hypoid gears, the most common occurrence for worm gearing is when the included angle Σ is $\pi/2$ radians. The smaller of the two gears is referred to as the *worm* and the larger is referred to as the *worm wheel*. This popular and useful form of gearing is sometimes referred to as *globoidal gearing* and can be found in the following applications:

- Window regulating devices in both houses and automobiles
- Electric mixers, can openers, and food processors (namely, house appliances)
- The steering mechanism in automobiles and heavy equipment
- Large reducers used in industrial applications
- Nonback-drivable or self-locking positioning devices

High spiral angles necessary for the desired gear ratio are reflected in increased thrust loads and usually require special thrust bearings for support. The relative sliding encountered in the contact can produce high temperatures within the contact and impose additional demands on the lubricants.

Kinematically, the design and manufacture of worm gearing differs from cylindrical and hypoid gearing. As such, the nomenclature established for the synthesis and analysis of this special form of gearing also differ from cylindrical and hypoid gearing. Worm gearing is further classified as *nonenveloping*, *single enveloping*, or *double enveloping* according to the shape of the reference pitch surfaces. Nonenveloping gears are the simplest form of worm gearing and is the name given to crossed cylindrical gears in mesh. Nonenveloping gears have low power-to-weight ratio and are used mainly for motion transmission between skew axes. An increase in power-to-weight ratio is achieved by arbitrarily fabricating a cylindrical worm and using an identically shaped cutter to manufacture the mating worm wheel using a generation-type process. This common form of worm gearing is referred to as single enveloping worm gearing. An intrinsic characteristic of this type of gear fabrication is that conjugate action between the worm and worm wheel is guaranteed. A third form of worm gearing that approximates the methods for motion transmission between skew axes presented in this book is double-enveloping worm gears. Like single-enveloping gears, conjugate action is ensured as a result of the generation-type process.

A worm and worm gear are special hypoid gears, where the two reference pitch surfaces in mesh are symmetric about the throat. This is a special region due to the variation in the radius of the reference pitch surface. Illustrated in Figure 1.27 is a worm pitch surface double-enveloping gear drive. This worm pitch surface is a doubly ruled surface. That is, the pitch surface can be defined using two sets of generators: a primary and secondary generators. The shape of the pitch surface depends on these generators where the distance and angle between the generator and axis of rotation define the shape of the hyperboloid worm. Depicted in Figure 1.28 is the hyperboloidal worm and worm wheel mesh. Here, there are two hyperboloidal pitch surfaces tangent to each other along the primary generator: the worm and the worm wheel. This primary generator also defines our worm wheel pitch surface. Gear mesh occurs along this primary generator. In the special case of orthogonal worm and worm wheel (90° shaft angle), the two pitch surfaces (both hyperboloids) are tangent to each other along two separate lines: the primary generator and the secondary generator. Gear mesh can exist along both generators. As a result, additional restrictions exist for the number of starts, the lead, and the tooth shape to avoid interference along the secondary generator.

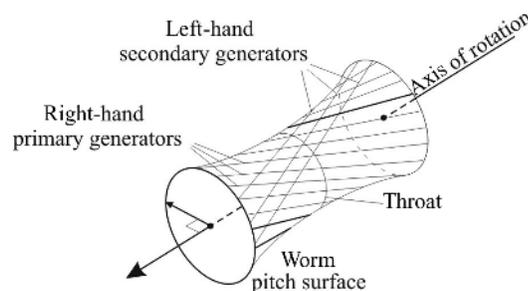


Figure 1.27 Doubly ruled worm pitch surface

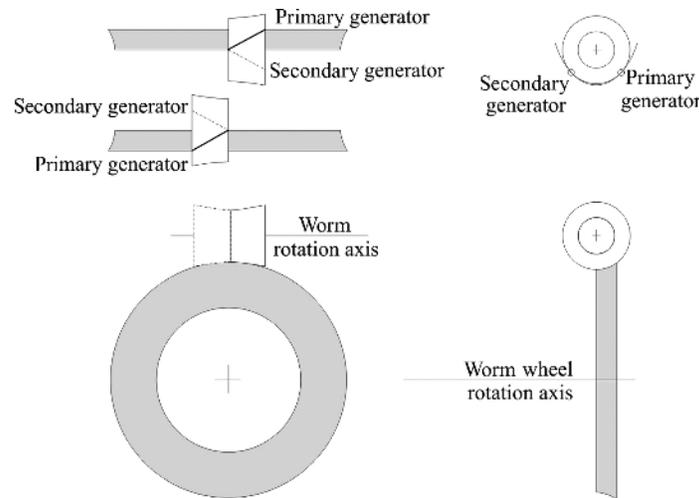


Figure 1.28 Worm gearing

1.14.3 Bevel Gears

The transmission of motion between two parallel axes is a special case of motion transmission between two generally disposed axes. Another special case of motion transmission between two axes is when the distance E is zero (i.e., intersecting axes). Motion transmission between two intersecting axes is commonly referred to as bevel gearing, where the most common occurrence is when the included angle Σ is $\pi/2$ radians. It has been demonstrated that the path of contact, contact ratio, start of active profile (SAP), end of active profile (EAP), and tooth curvature depend on the tooth profile in a transverse plane for cylindrical gear elements. In order to extend these principles to also parameterize bevel gears, it is necessary to introduce a conical pitch or base surface and a spherical transverse surface. An “octoid” or “8”-shaped path exists on a transverse spherical surface when rolling a crown rack on the base cone. The apex of the base cone and the crown rack are coincident with the sphere’s center. In order to utilize existing knowledge and understanding of planar gearing, equivalent planar gears are defined using *Tredgold’s approximation* (see Grant, 1899; Buckingham, 1949; Figliolini and Angeles, 2005) by projecting the gear teeth onto a “back cone” as illustrated in Figure 1.29. One limitation of employing such a procedure is encountered when designing bevel gears for a particular application (e.g., high I/O relationships $|g|$). Like planar gears, bevel gears can be either spur or spiral. The specification of tooth properties for bevel gears is based on the heel or outer radius.

Bevel gearing has much in common with hypoid gearing where the axial position w_i is much greater than the center distance E . When this occurs, the amount of axial displacement relative to the transverse displacement of the reference pitch surface is approximately zero and the reference pitch surfaces appear conical. Current methods as well as the machines used for their design and manufacture are essentially the same as those employed for hypoid gearing. Kinematically, the efficiency of conical gearing should exceed that of comparable hypoid gearing as a result of the reduction in axial sliding between gear teeth in mesh. As

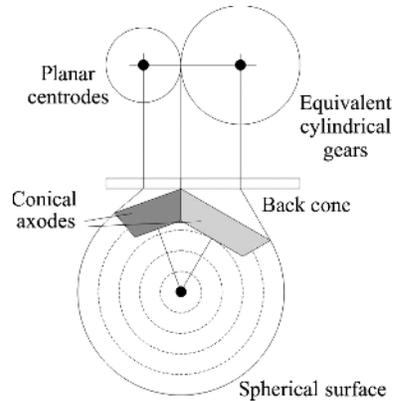


Figure 1.29 Tregold's method of transforming the analysis of motion transmission between intersecting axes to that of parallel axes

expected, the choice to use either bevel or hypoid gearing depends on the particular application and its corresponding restrictions. Currently, the use of bevel gearing (namely, spiral bevel gears) is increasing as a result of many automotive manufacturer's decision to switch from rear wheel drive (RWD) to front wheel drive (FWD) automobiles. Independent of an automotive manufacturer's choice of either RWD or FWD, each automobile is equipped with a differential to regulate the difference in wheel speeds during cornering where each differential consists of conical gear elements (usually spur gears or straight bevel gears). Another important application for bevel gearing is in the right angle drives for helicopters. Bevel gearing is usually preferred over hypoid gearing for motion transmission between nonparallel axes due to their added simplicity. Henceforth, they are more frequently employed than hypoid gears. Although the kinematic geometry of bevel gearing is better understood than hypoid gearing, bevel gear manufacture suffers many of the same difficulties currently encountered in the manufacture of hypoid gears.

One special case of bevel gears is *face gears*. Face gears are bevel gears where the cone angle is zero. This feature enables conventional cylindrical spur gears to mesh with a bevel crown gear or face gear. Face gears are often used as a "substitute" for bevel gears. One benefit of face gears is the decreased sensitivity to axial position of the pinion gear. One limitation being a decrease in load-carrying capacity.

1.15 Noncircular Gears

The origin of toothed wheels as function generators is not known. mG (miniGears) of Padua Italy produces a replica planet-tracking clock called the *Astrarium* that consists of elliptical gears (GearTechnology (2006)—*Addendum May/June*). The original *Astrarium* was built by Giovanni Dondi circa 1360. Ollson (1953) reports that Leonardo da Vinci illustrated the use noncircular gears to control the tension in crossbows. One of the first known publications concerning noncircular gears is by Holditch (1842). In his treatise on gearing, Grant (1899) recommended that the most practical method of noncircular gear manufacture was to

approximate the pitch curve with a circle (osculating circle) along the centerline of each tooth and proceed to cut the tooth as would be done for circular gears. This method of noncircular gear manufacture usually fails to ensure conjugate motion and is recommended for low-speed operation.

Noncircular gears have received limited attention because few designers have recognized their potential for use as elements of a mechanism and as a result very few gear manufacturers are capable of fabricating such toothed bodies. Also, the limited success of applying noncircular gears in practice can be attributed to the large amount of computations necessary for their accurate design and manufacture. The majority of references are limited to the analysis of noncircular gears between parallel axes. The only known published work regarding motion transmission between intersecting axes is by Ollson (1959). In 1931, noncircular gear were used to provide specialized motion in printing presses and other industrial machines (Gobler, 1939). These gears rarely exhibited complete rotatability. The most common type of noncircular gears are a pair of elliptical gears. Ollson (1953) uses the wealth of knowledge concerning elliptical gears to demonstrate the design and manufacture of noncircular gears. Here, Ollson divides the pitch curve into a number of segments, each of which are replaced by an equivalent segment of an ellipse, then proceeds to demonstrate the design and manufacture of general noncircular gears based upon the analysis of elliptical gears. Litvin (1956) also demonstrated the design and manufacture of noncircular gear. Cunningham (1957) was one of the first in the United States to publish a methodology to synthesize pitch curves for a general I/O relationship. Bloomfield (1960), Benford (1968), Horiuchi (1988) and more recently Quintero et al. (2007) have demonstrated the synthesis of pitch curves for noncircular gears. Al-Sabeeh (1991) combines segments of circular gears to achieve speed variations.

One application of noncircular gear pairs is the motion specification for the *loom slay* found in textile combing machines (Kowalczyk and Urbanek, 2003). A second application of noncircular gears is the motion modification of a conventional forging die to reduce the dwell time during forging. Doege et al. (2001) report a reduction in dwell time by 48% (from 75 ms to 39 ms) resulting in lower die temperatures. A third application involves noncircular gears combined with links to perform a polishing motion (Liu et al., 2006). Other applications involve continuous casting of steel using a nonsinusoidal motion (vs. conventional sinusoidal motion) improves the surface quality of the cast product (see Liu et al. (2002)) as well as variable pump flow (<http://www.ovalasia.com.sg/index.php?option=displaypage&Itemid=74&op=page&SubMenu>).

The I/O relationship g (Equation (1.1)) for circular gears is constant. For noncircular gears, the I/O relationship is no longer constant. An additional constraint for circular and noncircular gear pairs is that the integral

$$\int_0^{2\pi} g dv_i$$

must always be rational; otherwise, the output could not sustain an indefinite number of cycles with the desired functional relationship. This is discussed in detail by Freudenstein (1962). Similar to cam system design, the I/O relationship g for a pair of noncircular gears needs to be as “smooth” as possible to minimize dynamics. The actual form of the I/O relationship g is important when designing noncircular gears (more so for high angular speeds ω_i). The integral of the I/O relationship g is the angular position v_o of the output for a given angular position v_i

Table 1.1 Relation between kinematic and time-based motion properties

	Time	Kinematic
Velocity	$g\omega_i$	g
Acceleration	$g'\omega_i^2 + g\alpha_i$	g'
Jerk	$g''\omega_i^3 + 3g'\alpha_i\omega_i + g\dot{\alpha}_i$	g''
Snap	$g'''\omega_i^4 + 6g''\omega_i^2\alpha_i + g'(\alpha_i^2 + \omega_i\dot{\alpha}_i) + g\ddot{\alpha}_i$	g'''
Crackle	$g'''\omega_i^5 + 10g'''\omega_i^3\alpha_i + g''\omega_i(13\alpha_i^2 + 7\omega_i\dot{\alpha}_i) + g'(3\alpha_i\dot{\alpha}_i + 2\omega_i\ddot{\alpha}_i) + g\ddot{\alpha}_i$	g''''
Pop	—	g'''''

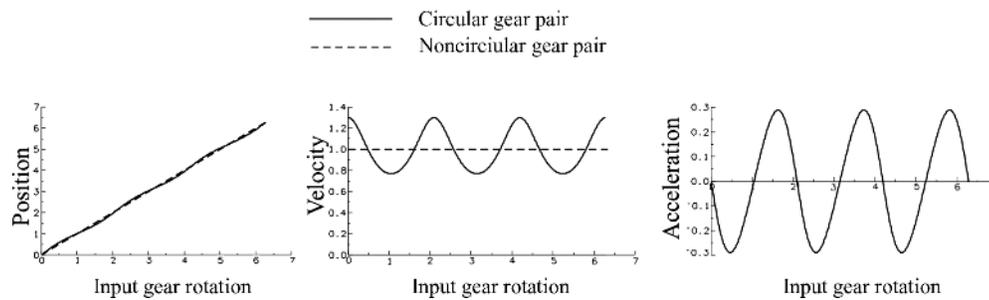


Figure 1.30 A plot of the angular position v_o , the velocity ratio g , and the kinematic acceleration g' as a function of the angular position v_i of the input

of the input. A discontinuity in v_o is paramount to specifying the impossibility that the output be in two different angular positions simultaneously.

As defined in Section 1.4, the I/O relationship g is the change dv_o in the angular position v_o of the output relative to the change dv_i in angular position v_i of the input; thus, g is also referred to as the *kinematic velocity*. The derivative g' of g with respect to the angular position v_i is denoted by the prime superscript and is referred to as the *kinematic acceleration*. The curvature of the two centrodes in mesh is directly related to the kinematic acceleration g' . If g' is too high then the centrodes become pointed and are difficult to manufacture. Also, if the acceleration g' is high then the output torque is high and thus high loads exist between the teeth in mesh. The second derivative g'' of the velocity ratio g is referred to as the *kinematic jerk*. Subsequent derivatives of the kinematic velocity are referred to as *kinematic snap* and *kinematic crackle* and are denoted by additional prime superscripts.⁶ Table 1.1 provides the relation between time-based motion properties and kinematic-based properties. Illustrated in Figure 1.30 are sample plots for the position v_o , the kinematic velocity v'_o or g , and the kinematic acceleration v''_o or g' for circular and noncircular gear pairs, and a sketch of each of the two gear pairs are shown in Figure 1.31. Although the functional relationship $v_o = f(v_i)$ that defines the position of the output gear element in terms of the input angular position, the kinematic velocity g

⁶The terms “snap” and “crackle” (in addition to “pop”) to identify higher derivatives are attributed to Harvey Crane [Tesar and Matthew, 1976] and the motion characteristics of disk cams.

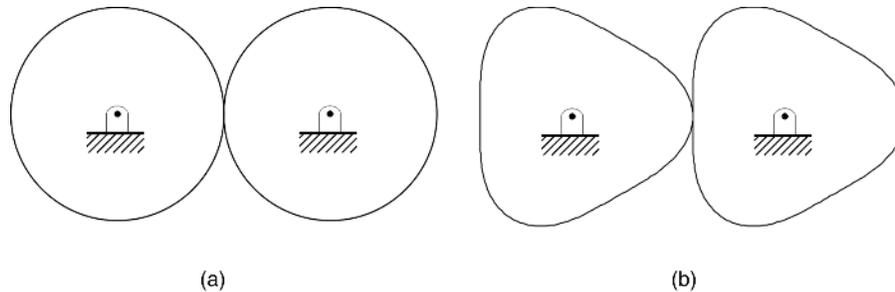


Figure 1.31 Gear pairs that satisfy the position, velocity, and acceleration curves depicted in Figure 1.30 are a circular gear pair (a) and a noncircular gear pair (b)

and the kinematic acceleration g' are necessary to specify pitch curves and tooth profiles for noncircular gear pairs.

Scroll gears are a special form of noncircular gear pairs where continuous rotation of the input is not essential. One such example is illustrated in Figure 1.32. A scroll-type gear where continuous rotation of both gear elements is presented in Figure 1.32. Additionally, certain applications of noncircular gearing the load variation as a function of input angular position can be significant. In such applications, a variable face width can be implemented to better match instantaneous face width with load. Also depicted in Figure 1.32 is an example of a variable face width noncircular gear pair.

Improvements in mechanism performance can be accomplished by extending the concepts and analysis of toothed bodies such that they are not limited to gears as in contemporary connotation. Utilizing toothed bodies to control displacement between two generally disposed axes can render a valuable resource to the machine designer. Generalized function generators incorporating toothed bodies can be synthesized to satisfy the optimal I/O relationship for a specified task. Examples include

- recreational equipment designed to maximize the use of human output (bicycles, rowing machines, compound bows, rehabilitation devices, etc.),
- coordinated steering for automobiles (use of noncircular gears to achieve coordinated or Ackerman steering), and
- manufacturing processes requiring sequencing or indexing with constant speed conveyer (stamping, bottling, inspection, etc.).

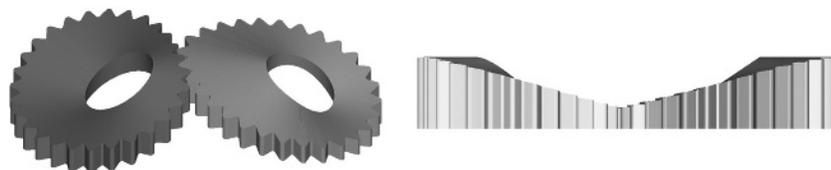


Figure 1.32 Scroll gears and variable face width gears

Also, a generalized function generator can be used to rid machines of unwanted vibrations. Here, the undesirable torque displacement versus angular displacement is used to synthesize devices for flywheels to achieve theoretically zero speed fluctuation, thus reducing backlash, windup, fatigue, and noise. Some efforts have focused on band mechanisms for torque balancing (Hain, 1961) and (McPhate, 1966) with consideration of cam systems. Much attention has been given to cam systems for function generators; however, in certain situations, there are advantages for selecting toothed bodies over cam drives. Chakraborty and Dhande (1977) discuss a wide variety of cam systems used for motion generation. In some applications, gear systems can provide benefits over cam systems for the following reasons:

- No gross separation or decoupling of members occur in gear systems.
- Gear systems can be generally designed with fewer parts and higher strength-to-weight ratios than cam systems.
- Cam systems with different orientation of input and output axes (disk, wedge, barrel, and face) are fundamentally different.
- Cam systems generally transform rotary motion into oscillatory motion.

In addition to cam systems, some consideration has been given to the use of belts and pulleys as well as chains and sprockets for function generation (e.g., Freudenstein and Chen (1988) used chains and sprockets as function generators). The primary difference between belts and pulleys and chains and sprockets is that belts and pulleys rely upon friction to transmit motion whereas chains and sprockets do not. One advantage of belts and pulleys over direct contact mechanisms is that the output can be insulated from the input. Reasons for selecting toothed bodies over chains and sprockets are as follows:

- Chains and sprockets are generally nonback-drivable.
- Chains and sprockets are unable to facilitate negative I/O relationships g .
- Sprocket profiles must remain convex.
- Chains are intermediate elements with additional design constraints.

Finally, linkages can also provide an invaluable means of function generation. Two commonly incurred problems with linkage synthesis are sequencing and branching. Some advantages for selecting toothed bodies over linkages are as follows:

- Motion specification for linkages is not general (I/O relationship can only be satisfied or optimized for discrete values of the input).
- Linkages are usually restricted to planar or spherical motion.
- Linkages are space inefficient.
- Linkages are difficult to simultaneously balance with regards to shaking forces and shaking moments.

In many circumstances, toothed bodies can be relied upon to provide the most direct, compact, and versatile means of power transmission and function generation. Examples of non-circular gears used for torque balancing and function generation are provided in Appendix C.

The difference between gear pairs and cam systems for motion transmission is not well defined in the literature. Both forms of motion transmission utilize bodies in direct contact to achieve motion transmission. Buckingham defines motion transmission via a body where pins or rollers are attached to a plate as *Lantern pinions*. Nugent (2001) extends the concept of Lantern pinions to include nonconstant motion transmission. The pressure angle for NC gears can be defined as the angle between the tooth surface normal and the pitch curve tangency or the angle between the tooth surface normal and the line connecting the axes of rotation. Here, the pressure angle is defined as the angle between the tooth surface normal and the pitch curve tangency. Using this definition enables conventional cutters (namely, hobs and shapers) to produce spiral NC gear elements.

1.15.1 Gear and Cam Nomenclature

Gear pairs can be identified as a direct contact mechanism the desired I/O relation is achieved via two surfaces or bodies in direct contact. Chains and sprockets along with belts and pulleys generate motion indirectly via an intermediate body (namely, the chain or belt). Two common examples of direct contact mechanisms include gear pairs and cam systems. Typically, cam systems provide oscillating behavior of the output body (translating or rotating follower) for continuous rotation of the input (cam), whereas gear pairs provide linear output rotation of the “gear” for an angular rotation of the input gear element or pinion. However, cam systems can be used to provide continuous rotation of the output (Gonzalez-Palacios and Angeles, 2000), and gear pairs can be used to achieve nonconstant motion as referenced in the preceding text.

The nomenclature for each field is well established. The term “gear pair” encompasses the transmission of motion between two teeth in direct contact. The shapes of these gear teeth are selected to achieve a sought motion (usually uniform motion). The gear teeth in mesh are said to be conjugate when they yield the desired motion. Cam systems differ from gear pairs where the geometry of one body (usually the follower) is selected a priori, and the geometry of the moving body (the cam) in direct contact is calculated such that the two bodies in direct contact achieve the sought motion. The nomenclature used throughout this work more closely follows the gear community while enabling general nonlinear output motion characteristics. This difference in terminology or nomenclature between cam system design and gear pair design is revealed for the special case of “harmonic” motion as depicted in Figure 1.33. Depicted in Figure 1.33 is a circular disk cam with a translating roller follower (zero offset) and an input gear element in mesh with an output gear element. Central to both fields are the following terms:

- Pitch curve
- Base curve
- Pressure angle
- Contact normal

In the considered example of harmonic motion, each term aforementioned embodies a different concept. The base curve or base circle in cam terminology defines the cam coordinates or contact loci between the follower and the cam. The base curve or the base circle in the gearing community is the reference curve or circle for the generation of the commonly used

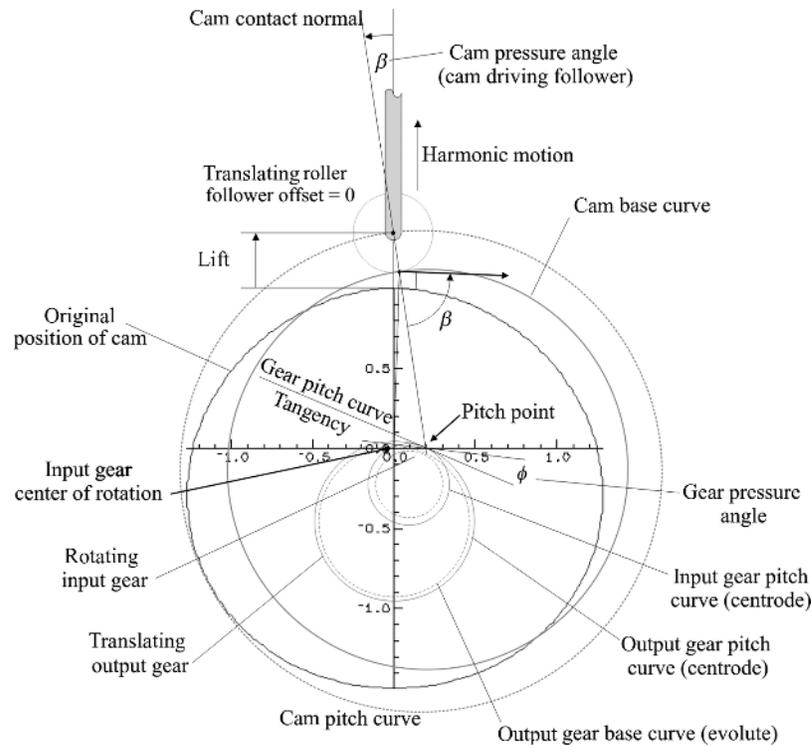


Figure 1.33 Cam and gear nomenclature

involute tooth profile. The pitch curve is the loci of the center of the roller follower, whereas in the gearing community the pitch curve is the centrode.

1.15.2 Rotary/Translatory Motion Transmission

One of the more common applications of transforming a rotary motion to a translating motion involves a *rack and pinion*, where the I/O relationship is linear. A sketch of the two gear elements used to accomplish such a task are shown in Figure 1.34a. The rack is the “linear” element and the pinion is the “circular” element. Special cases of motion transformation from a rotary input to a translating output are shown in Figure 1.34b. Here, the so-called pinion is noncircular and the mating rack is nonlinear. One restriction on such forms of motion transformation is that the range or amount of rotation of the pinion is limited. In order for the output to sustain an unlimited number of rotations of the input then the I/O relationship must be cyclic. If the I/O relationship is sinusoidal, one possible method of satisfying such specifications involves an external circular gear and an internal circular gear. The axis of rotation for the external gear is parallel to its central axis and displaced an amount equal to the external gear’s radius, while the internal gear is restricted to oscillate as shown in Figure 1.34c. Each position of the input contacts the output in two distinct positions for one

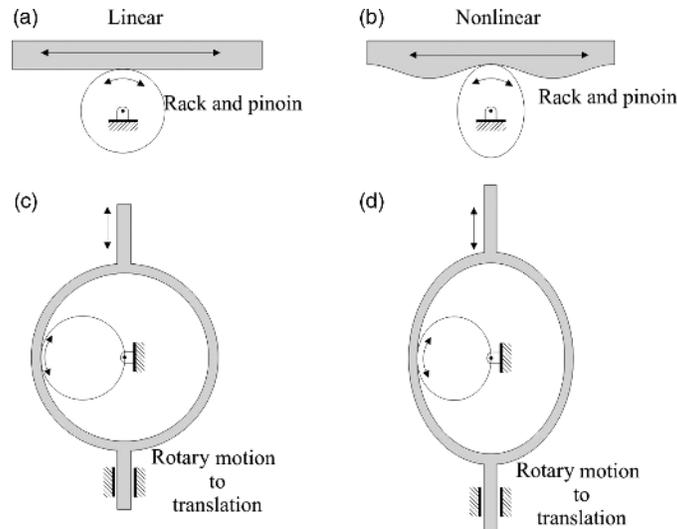


Figure 1.34 Rotary to translation motion for (a) linear translation, (b) nonlinear translation, (c) harmonic oscillation, and (d) nonlinear oscillation

complete cycle of the input. When the I/O relationship is not symmetric, each position of the input can contact the output in only one position as demonstrated in Figure 1.34d.

1.16 Schematic Illustration of Gear Types

Different gear types and their corresponding names have been used to illustrate motion transmission between two axes. Oftentimes gear specialists will use different names when referring to a particular gear type. The various names used to identify these different gear types and a schematic illustration of their general form are given in Table 1.2.

1.17 Mechanism Trains

Up to this point the analysis of mechanisms has been restricted to three link single dof mechanisms or gear pairs. Typically, gear pair reductions are limited to 10:1. Many situations in practice occur where three link gear pairs are cascaded to create a new and more useful mechanism (an introduction to many other mechanisms capable of a particular type of motion generation are cataloged by Chironis (1991)) where it is possible to have an overall gear reduction of 1000:1. Next to three link mechanisms, mechanism trains are one of the oldest mechanisms. A clever combination of gear pairs (e.g., transmissions and differentials) can be combined to effectively accomplish certain tasks. A more recent treatment regarding mechanism trains is presented by Müller (1982). The same principles can be equally applied to other mechanism trains where the basic elements are not restricted to planar three link 1-dof mechanisms comprising circular or noncircular gear elements.

Table 1.2 Schematic representation of different gear types

Illustration	Name/gear type
	Cylindrical gearing Helical gearing Parallel axis gearing Planar gearing Spur gearing
	Crossed axis gearing Crossed helical gearing Nonenveloping gearing
	Bevel Conical gearing Miter gearing Beveloid™ gearing Zero1™
	Double-enveloping gearing Globoidal gearing Hourglass gearing Single-enveloping gearing Worm gearing
	Hypoid gearing Rear axis gearing Skew axis gearing Spatial gearing

1.17.1 Compound Drive Trains

An example of a compound⁷ six-element gear train is shown in Figure 1.35 where four of its elements consists of two noncircular gear pairs. The mobility m of the gear train is $\mathfrak{M} = 1$ (i.e., $\mathfrak{M} = 3(6 - 1) - 2(5) - 1(4) = 1$). This six elements are labeled where gears 3 and 4 are one body and gears 6 and 7 are another body. For the gear train shown, the input is the noncircular gear element 2 and the output is the noncircular gear element 8. The net I/O relationship G for the gear train depends upon the I/O relationship g between each gear pair within the gear train. The I/O relationship between each pair of meshing gears is denoted by a pair of subscripts corresponding to the two gears in mesh. For example, the I/O relationship between the noncircular gear elements 2 and 3 is g_{23} , where the speed ω_{30} of the output gear

⁷ A compound gear train exists when more than one meshing gear element is attached to a single shaft. A simple gear train has a single gear attached to each shaft.

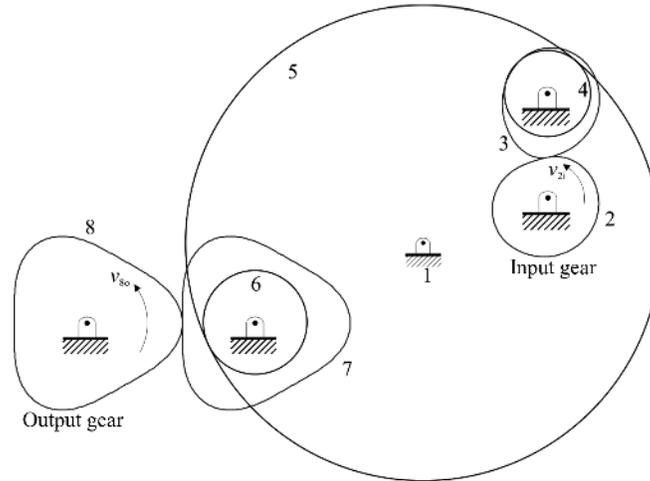


Figure 1.35 A compound eight-element gear train incorporating noncircular gear pairs

3 relative to ground is $\omega_{3o} = g_{23} \omega_{2i}$. The functional relation $f_{23}, (v_{2i})$ between the angular position v_{3o} of the output gear 3 relative to the angular position v_{2i} of the input gear 2 is $v_{3o} = f_{23}(v_{2i})$ where $g_{23} = f'_{23}(v_{2i})$. Recognizing that the output from gear element 3 is the input to gear element 4 and that the output from gear element 6 is the input to gear element 7, the functional relationship between the output angular position v_{8o} and the input angular position v_{2i} of the gear train is

$$v_{8o} = f_{78}(f_{56}(f_{45}(f_{23}(v_{2i})))) \quad (1.14)$$

Differentiating the aforementioned expression with respect to the angular position v_{2i} of the input, the angular displacement dv_{8o} of the output relative to the angular displacement dv_{2i} is

$$\frac{dv_{8o}}{dv_{2i}} = f'_{78}(f_{56}(f_{45}(f_{23}(v_{2i})))) f'_{56}(f_{45}(f_{23}(v_{2i}))) f'_{45}(f_{23}(v_{2i})) f'_{23}(v_{2i}), \quad (1.15)$$

where the *net I/O relationship* G for the entire gear train is (here g_{45} and g_{56} are constant and independent of the input v_{2i})

$$G \equiv \frac{dv_{8o}}{dv_{2i}} = g_{78}(f_{56}(f_{45}(f_{23}(v_{2i})))) g_{56} g_{45} g_{23}. \quad (1.16)$$

For constant I/O relationships g_{78} and g_{23} (i.e., circular gears), g_{78} and g_{23} are independent of the angular position v_{2i} and g reduces to

$$G = g_{78} g_{56} g_{45} g_{23}. \quad (1.17)$$

Acknowledging that two gears in mesh must have the same diametral pitch, the net I/O relationship for compound gear trains incorporating only circular gear elements can be expressed in terms of the number of teeth on each gear; hence,

$$G = \frac{N_7 N_5 N_4 N_2}{N_8 N_6 N_5 N_3} = \frac{\text{product of number of teeth on each input gear}}{\text{product of number of teeth on each output gear}}. \quad (1.18)$$

Notice that the gear element 5 acts as both an input and output and its size does not affect G . Such gear elements that do not affect the net I/O relationship G and are referred to as *idler* gears. The effect of each idler gear is a change in direction of rotation of the output or final gear element. Also, idler gears are used as intermediate elements to accommodate a desired center distance between two shafts. The sign of G must be carefully considered for an assumption of a positive g for two external gears of a compound gear train fails to identify the sign of G .

1.17.2 Epicyclic Gear Trains

Example of an *epicyclic gear trains* or EGTs are depicted in Figures 1.36 and 1.37. One of EGTs most attractive feature is its high power-to-weight ratio. The special (EGT) shown in the Figure 1.37 has a gear element 5 as an internal gear whose axis of rotation is collinear with the axis of rotation for the gear element 2. When the axes of rotation for the input and output elements of an EGT are coaxial it is referred to as a *reverted* EGT. EGTs are also frequently referred to as *planetary gear trains* (PGTs). The special gear train shown consist of a central gear, an intermediate gear, an arm (sometimes referred to as a spider or carrier), and an internal ring gear or annulus. The reason this arrangement is referred to as a planetary gear train is because the intermediate gear resembles a planet or satellite orbiting the sun or central gear. Alternately, the arrangement is also referred to as an EGT is because of the epicyclic motion of the planet gear relative to the sun gear. Usually, EGTs incorporate more than one planet as shown in Figure 1.37. Multiple planets do not affect the kinematic relationships between the various elements. Typically, PGTs incorporate three planet gears. Use of multiple planets reduces the load on each planet and also distributes the loads transmitted to both the sun and the ring gear. The distribution of load transmitted between the sun and the ring gear is known as *power branching*. Use of multiple planets for load sharing can reduce the load per face width at each mesh and eliminate the radial thrust on each element.

The difference between the compound gear train shown in Figure 1.35 and the EGT shown in Figure 1.36 is that all axes of rotation for the gear elements of the compound gear train are held stationary. When all three elements (i.e., the sun gear, the arm, and the ring gear) of a PGT rotate, it is referred to as a *differential*. The kinematic structural composition of PGTs was studied by Lévai (1968), and he demonstrated that there are 34 different PGTs where each type can be derived from a single PGT consisting of two sun gears, a planet gear, and an arm.

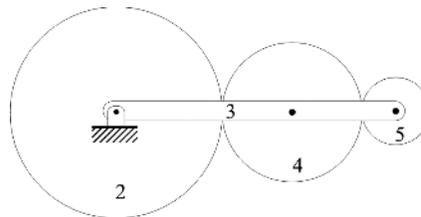


Figure 1.36 The speed relationship between the various elements of a planetary gear train is obtained by determining the velocity of the pitch point between each element

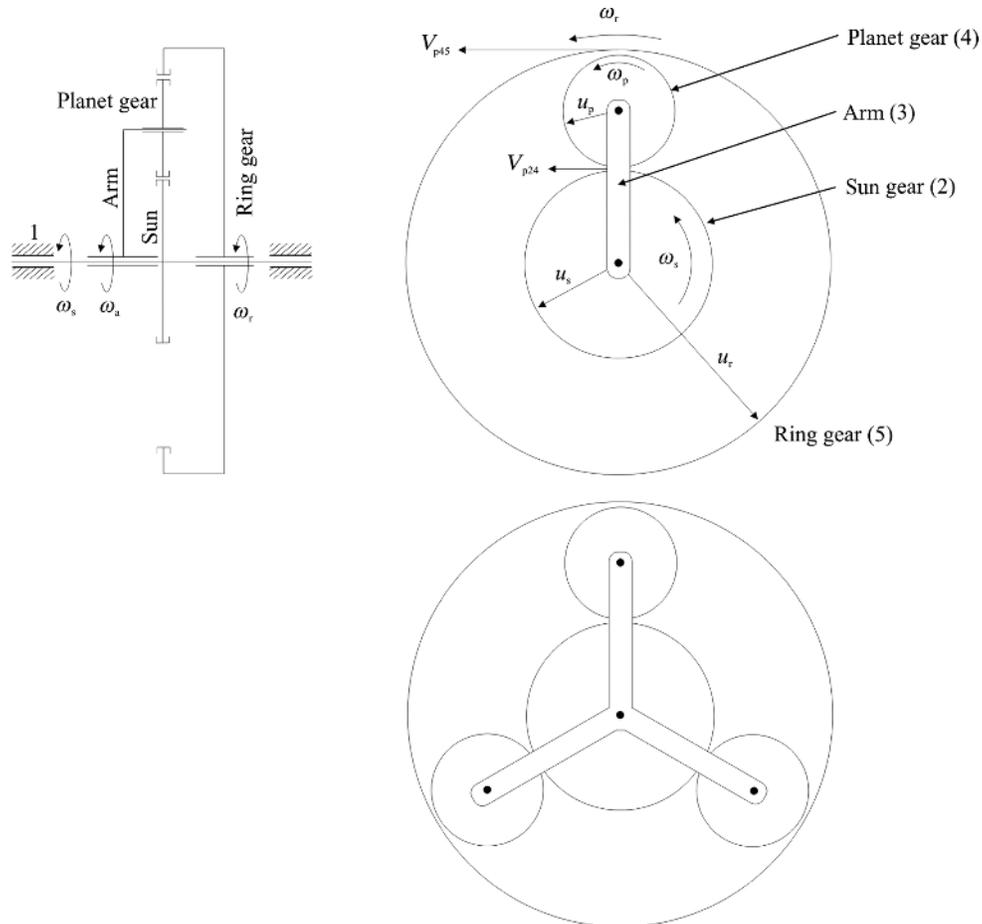


Figure 1.37 Schematic diagram for the kinematic analysis of the PGT shown in Figure 1.36

Before proceeding to develop the I/O relationship of an EGT, the mobility of an EGT is presented. The EGT shown in Figure 1.38 consists of five elements: ground 1, a sun gear 2, an arm 3, a planet gear 4, and a ring gear 5. The number of joints within the EGT is six:

1. One between the sun and ground
2. One between the arm and ground
3. One between the ring and ground
4. One between the arm and the planet
5. One between the sun and the planet
6. One between the planet and the ring

The freedom between the planet and both the sun and the ring is two; hence, the mobility of the EGT becomes (Equation (1.2))

$$\mathfrak{M} = 3(5 - 6 - 1) + (1 + 1 + 1 + 1 + 2 + 2) = 2. \quad (1.19)$$

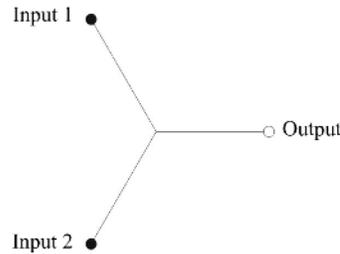


Figure 1.38 Three nodes to represent the inputs and outputs of a PGT

A mobility of two indicates that two independent inputs are needed to uniquely define the angular position of the output. For the EGT shown in Figure 1.37, there are usually three different possible combinations of inputs each resulting in a corresponding output. Representation of the two inputs and the single output is represented diagrammatically using a circle and three appendages as shown in Figure 1.38. The two inputs to the EGT are indicated by the darkened ends of the appendages, whereas the output of the EGT is indicated by the undarkened end of the third appendage. The considered EGT can also be interpreted as a mechanism consisting of a single input and two outputs. The relation between the two outputs of the EGT must be an equal split of the input power. This equal split of power is known as *power division* and is not to be confused with power branching.

A schematic representation of a EGT is introduced as illustrated in Figure 1.38 in order to facilitate the displacement analysis. The *Instant Center Method*, *Formula Method*, and *Tabular Method* (Graphical approach and Graph theory) are three methods often used to obtain the speeds relations between the various elements of a EGT. The instant center method will be demonstrated. The speed relationship between the elements of the EGT is obtained by first recognizing that no slippage occurs at the pitch points p_{24} and p_{45} . The speed $\mathbf{V}_{p_{24}}$ of the pitch point p_{24} relative to ground can be then expressed

$$\mathbf{V}_{p_{24}} = \omega_2 u_2. \quad (1.20a)$$

Also, the speed $\mathbf{V}_{p_{24}}$ of the pitch point p_{24} relative to ground is obtained using the vector loop equation; thus,

$$\mathbf{V}_{p_{24}} = \omega_3(u_2 + u_4) - (\omega_3 + \omega_4)u_4. \quad (1.20b)$$

Equating the above two expression for the speed $\mathbf{V}_{p_{24}}$ and solving for the angular speed ω_4 of the planet gear gives

$$\omega_4 = \frac{u_2(\omega_3 - \omega_2)}{u_4}. \quad (1.21)$$

Similarly, the speed $\mathbf{V}_{p_{45}}$ of the pitch point p_{45} relative to ground yields

$$\mathbf{V}_{p_{45}} = \omega_5(u_2 + 2u_4) = \omega_3(u_2 + u_4) + (\omega_3 + \omega_4)u_4. \quad (1.22)$$

Substituting the expression obtained in Equation (1.21) for the angular speed ω_4 of the planet gear into the aforementioned relation yields the following expression for the speed relationship for the considered EGT:

$$(u_2 + 2u_4)\omega_5 - 2(u_2 + u_4)\omega_3 + u_2\omega_2 = 0. \quad (1.23a)$$

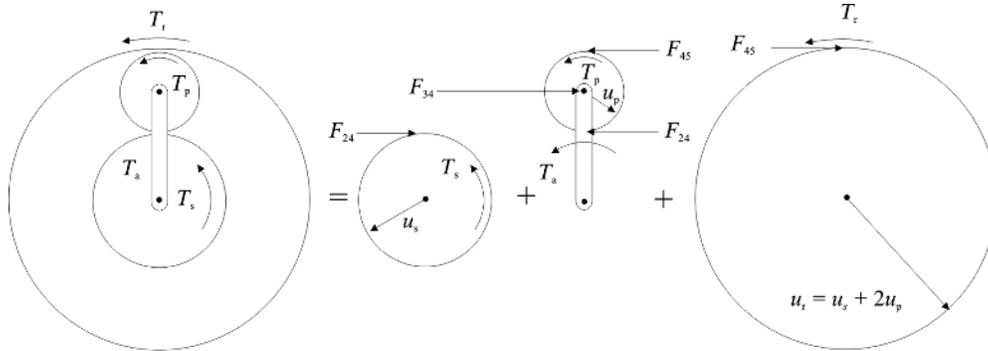


Figure 1.39 Load relationship for a PGT is obtained by a summation of forces

At this point, it is descriptive to identify each element in the PGT as the sun gear, arm, planet gear, or ring gear rather than elements 2, 3, 4, or 5, respectively; hence, the aforementioned relationship becomes

$$(u_s + 2u_p)\omega_r - 2(u_s + u_p)\omega_a + u_s\omega_s = 0. \quad (1.23b)$$

Recognize that if the angular speed ω_a of the arm is zero, then $(u_s + 2u_p) = u_r$ and the aforementioned speed relationship for the EGT is the same as that obtained for the net I/O relationship G for a compound gear train (i.e., $N_r/N_s = G = \omega_r/\omega_s$). As demonstrated earlier for gear pairs with constant I/O relationships g , an additional expression for the relationship of a mechanism train can be obtained by replacing the pitch radius of each gear element by the number of teeth on that gear; thus, Equation (1.23b) becomes

$$(N_s + 2N_p)\omega_r - 2(N_s + N_p)\omega_a + N_s\omega_s = 0. \quad (1.23c)$$

The static torque relationship (i.e., for nonaccelerating elements) between the elements of the EGT shown in Figure 1.37 is determined by disassembling it as shown in Figure 1.39 and evaluating the torques at each element. The contact force F_{sp} between the sun gear and the planet gear, and the contact force F_{pr} between the planet gear, and the ring gear, correspond to the pitch points p_{sp} and p_{pr} , respectively. A summation of forces about the sun gear, the ring gear, and the arm are, respectively,

$$T_s = u_s F_{sp}, \quad (1.24a)$$

$$T_r = (u_s + 2u_p) F_{pr}, \quad (1.24b)$$

$$T_a = -(u_s + u_p)(F_{sp} + F_{pr}). \quad (1.24c)$$

Solving Equations (1.24a) and (1.24b) for F_{sp} and F_{pr} , and substituting the results into Equation (1.24c) yields the following torque relationship:

$$u_s(u_s + u_p)T_r + u_s(u_s + 2u_p)T_a + (u_s + u_p)(u_s + 2u_p)T_s = 0 \quad (1.25a)$$

$F_{sp} = F_{pr}$ for $\omega_p = \text{constant}$ and the following torque relations also exists:

$$T_a = -2 \frac{(u_s + u_p)}{u_s} T_s = -2 \frac{(u_s + u_p)}{(u_s + 2u_p)} T_r. \quad (1.26)$$

Replacing the pitch radius of each gear element by the number of teeth on that gear the aforementioned torque relationship for the EGT is

$$N_s(N_s + N_p)T_r + N_s(N_s + 2N_p)T_a + (N_s + N_p)(N_s + 2N_p)T_s = 0. \quad (1.25b)$$

A subtle difference between the speed relationship of Equation (1.23c) and the torque relationship of Equation (1.25b) is that the coefficients associated with the angular speeds ω_s , ω_a , and ω_r are different from the coefficients associated with their respective torques T_s , T_a , and T_r . Expanding the torque relationships in Equations (1.25a) and (1.25b) reveals that the summation of torques T_s , T_s , and T_r is zero (i.e., $\Sigma T = T_s + T_a + T_r = 0$). The speed and torque relations for the EGT are presented in Table 1.3. Each relationship is expressed in terms of the pitch radius of each gear element of the EGT and also in terms of the number of teeth on each gear element of the EGT.

Often it is desirable to distribute the load exerted on each element of an EGT by incorporating multiple planets (power branching). In order to evenly space multiple planet gears around the periphery of the sun gear, the number of teeth on the sun gear, each of the planet gears, and the ring gear are not arbitrary. Acknowledging that each planet gear meshes with both the sun gear and the ring gear, the circular pitch c_p of each gear of the EGT must be identical; hence,

$$\frac{2\pi u_s}{N_s} = \frac{2\pi u_p}{N_p} = \frac{2\pi u_r}{N_r} = c_p. \quad (1.27)$$

If n is the number of planet gears and θ_s is the angle subtended between successive planet gears, integer multiple k_s of the circular pitch must be equal to the arc length along the pitch curve of the sun gear; therefore,

$$u_s \left(\frac{2\pi}{n} \right) = c_p k_s = \left(\frac{2\pi u_s}{N_s} \right) k_s,$$

or upon rearranging

$$k_s = \frac{N_s}{n}.$$

Likewise, a similar relation must exist for the ring gear; thus,

$$k_r = \frac{N_r}{n}.$$

Since the sum of two integers is an integer, the following relation must also be an integer if the planet gears are to be evenly spaced along the outer periphery of the sun gear:

$$k_s + k_r = \frac{N_s + N_r}{n} = \frac{2(N_s + N_p)}{\text{Number of planet gears}} = \text{Integer}. \quad (1.28)$$

Table 1.3 Speed and torque relationships for the EGT shown in Figure 1.37

Input	Output	Speed relationship	Torque relationship
Sun gear Arm	Ring gear	$\omega_5 = \left(\frac{2(u_2 + u_4)}{u_2 + 2u_4}\right) \omega_3$	$T_5 = \left(\frac{u_2 + 2u_4}{u_2 + u_4}\right) T_3 - \left(\frac{u_2 + 2u_4}{u_2}\right) T_2$
		$-\left(\frac{u_2}{u_2 + 2u_4}\right) \omega_2$	$T_5 = -2\left(\frac{u_2 + 2u_4}{u_2 + u_4}\right) T_2$
			$T_5 = -\left(\frac{u_2 + 2u_4}{u_2}\right) T_3$
	Arm	$\omega_5 = \left(\frac{2(N_2 + N_4)}{N_2 + 2N_4}\right) \omega_3$	$T_5 = \left(\frac{N_2 + 2N_4}{N_2 + N_4}\right) T_3 - \left(\frac{N_2 + 2N_4}{N_2}\right) T_2$
		$-\left(\frac{N_2}{N_2 + 2N_4}\right) \omega_2$	$T_5 = -2\left(\frac{N_2 + 2N_4}{N_2 + N_4}\right) T_2$
			$T_5 = -\left(\frac{N_2 + 2N_4}{N_2}\right) T_3$
Sun gear Ring gear	Arm	$\omega_3 = \left(\frac{u_2}{2(u_2 + u_4)}\right) \omega_2$	$T_3 = \left(\frac{u_2 + u_4}{u_2}\right) T_2 + \left(\frac{u_2 + u_4}{u_2 + 2u_4}\right) T_5$
		$+\left(\frac{u_2 + 2u_4}{2(u_2 + u_4)}\right) \omega_5$	$T_3 = -\left(\frac{u_2}{u_2 + 2u_4}\right) T_5$
			$T_3 = -2\left(\frac{u_s + u_p}{u_s}\right) T_2$
	Arm	$\omega_3 = \left(\frac{N_2}{2(N_2 + N_4)}\right) \omega_2$	$T_3 = \left(\frac{N_2 + N_4}{N_2}\right) T_2 + \left(\frac{N_2 + N_4}{N_2 + 2N_4}\right) T_5$
		$+\left(\frac{N_2 + 2N_4}{2(N_2 + N_4)}\right) \omega_5$	$T_3 = -\left(\frac{N_2}{N_2 + 2N_4}\right) T_5$
			$T_3 = -2\left(\frac{N_s + N_p}{N_s}\right) T_2$
Arm Ring gear	Sun gear	$\omega_2 = \left(\frac{2(u_2 + u_4)}{u_2}\right) \omega_3$	$T_2 = \left(\frac{u_2}{u_2 + u_4}\right) T_3 - \left(\frac{u_2}{u_2 + 2u_4}\right) T_5$
		$-\left(\frac{u_2 + 2u_4}{u_2}\right) \omega_5$	$T_2 = -\frac{1}{2}\left(\frac{u_s}{u_s + u_p}\right) T_3$
			$T_2 = -\frac{1}{2}\left(\frac{u_2 + u_4}{u_2 + 2u_4}\right) T_5$
	Arm	$\omega_2 = \left(\frac{2(N_2 + N_4)}{N_2}\right) \omega_3$	$T_2 = \left(\frac{N_2}{N_2 + N_4}\right) T_3 - \left(\frac{N_2}{N_2 + 2N_4}\right) T_5$
		$-\left(\frac{N_2 + 2N_4}{N_2}\right) \omega_5$	$T_2 = -\frac{1}{2}\left(\frac{N_s}{N_s + N_p}\right) T_3$
			$T_2 = -\frac{1}{2}\left(\frac{N_2 + N_4}{N_2 + 2N_4}\right) T_5$

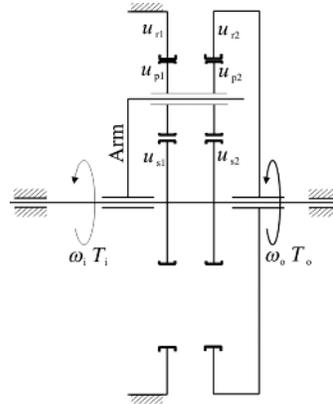


Figure 1.40 Compound PGTs using Fergusson's paradox to illustrate circulating power

1.17.3 Circulating Power

A myriad of mechanisms can be devised by incorporating multiple EGTs into a single mechanism, also known as *compound* EGTs. As demonstrated, the PGT is a 2-dof mechanism where in certain circumstances the two input can be obtained from a single input by creating branches or loops where the combined effect of torque and speed are high. This combination of torque and speed is referred to as *circulating power* or *internally transmitted power*. These meshes where the combination of high speed and torque do not affect the net I/O relationship G of the mechanism train. Circulating power is important and can determine the load capacity of the compound EGT. High circulating power can reduce the efficiency of a compound EGT due to increased mesh losses. The term “circulating power” can be misleading in that either high torques do not exist for the static case or that input power does not equal output power. The concept of circulating power is the basis of locked torque tests used to evaluate the performance of gear pairs.

An interesting application of such a compound EGT is based on the schematic illustration shown in Figure 1.40. The compound EGT shown in Figure 1.40 can be used to obtain very high speed reductions. The two EGTs shown utilize the same sun gear and arm; however, the radii u_{r1} and u_{r2} of the two ring gears are different. Initially, it might appear that the radii u_{r1} and u_{r2} must be the same since the sun gear for the two EGTs are identical. The ability for the compound EGT to operate in spite of $u_{p1} \neq u_{p2}$ is known as *Fergusson's mechanical paradox*⁸ (Nakada, 1952). The fact that the compound EGT is functional is attributed to the property that involute profiled gears continue to provide uniform motion transmission between two parallel axes independent of small changes in center distance. When the center distance for two cylindrical gears with involute tooth profiles is increased (a profile shift), then the pitch radii and the diametral pitch for the gear pair must also increase. This feature enables a single gear (with involute profiles) to mesh with two other gears, each with a slightly different number of teeth and mounted along the same axis.

⁸ James Fergusson, 1773. Note date of publication and timing of Euler's proposed use of involute for tooth profile! One of Fergusson's application is a geared Orrery (after Charles Boyle, Earl of Orrery) or astronomical devices for showing earth and moon's motion around the sun.

Earlier in this section, a single EGT was described as a two input, one output device. As indicated by the feathered marks shown in Figure 1.38, the ring gear 1 is held stationary. Thus, the speed of the arm ω_{a1} is uniquely determined by the speed of the sun gear ω_{s1} . The output ω_{r2} of the compound EGT is the ring gear 2. The two inputs to the second EGT are the angular speed ω_{a2} of the arm and the angular speed ω_{s2} of the sun gear. Recognize the angular speeds ω_{s1} and ω_{s2} of both the sun gears are the same. Also, the angular speeds ω_{a1} and ω_{a2} of the arms are identical. Since the pitch radii u_{s1} and u_{r2} are not the same although N_{r1} and N_{r2} are the same, the second ring gear can be thought of as subjected to a profile shift. Here, the sun gear for the compound EGT is in essence two identical gears with different pitch radii and diametral pitches. The smaller the difference $N_{r2} - N_{r1}$ in the number of teeth between the two ring gears, the higher the speed reduction between the input and the output. The difference $N_{r2} - N_{r1}$ must be an integer equal to the number of locations where the center lines of teeth between the two ring gears coincide, and hence, is the maximum allowable number of “identical” planet gears.

The circulating power \mathbb{P}_c depends on the overall gear ratio G of the compound mechanism. Based on Equation (1.23c) an expression of the net gear ratio G can be expressed as

$$G = \frac{1}{1 + 2(N_{p2}/N_s)} \left(\frac{1 + (N_{p2}/N_s)}{1 + (N_{p1}/N_s)} - 1 \right), \quad (1.29)$$

and as a result the output torque T_o becomes

$$T_o = \frac{T_i}{G}. \quad (1.30)$$

Based on Equations (1.24a), (1.24b) and (1.24c) the corresponding torque T_{s2} on the “second” sun gear is

$$T_{s2} = \frac{1}{1 + 2(N_{p2}/N_s)} T_o. \quad (1.31)$$

For a given input speed ω_i and torque T_i , the input power becomes $T_i\omega_i$. The circulating power \mathbb{P}_c for the compound EGT is the ratio between the maximum power at the sun gear to that of the net power at the sun gear. Thus,

$$\mathbb{P}_c = \frac{T_{s2}\omega_{s2}}{T_i\omega_i} \left[\frac{1 + (N_{p2}/N_s)}{1 + (N_{p1}/N_s)} - 1 \right]^{-1}. \quad (1.32)$$

1.17.4 Harmonic Gear Drives

Harmonic gear drives are a compact method of achieving a high speed reduction. Such drives were developed by C.W. Musser and are frequently used in robotic manipulators. The harmonic drive typically consists of three elements: a wave generator, a flexible external gear element, and a rigid internal gear element. The anatomy of such a harmonic gear drive is depicted in Figure 1.41. Speed reductions typically range from 30:1 to 300:1. Harmonic gear drives are compact, simple, and have low backlash, whereas their disadvantage is its low torsional rigidity due to the flexible external gear element. A general trend is that the efficiency of harmonic gear drives decreases as the reduction ratio increases.

The flexible external gear element is smaller in diameter than the rigid internal gear element, resulting in having two fewer teeth on its outer circumference. It is held in an oval (usually elliptical) shape by the wave generator, and its teeth engage with the teeth on the rigid internal

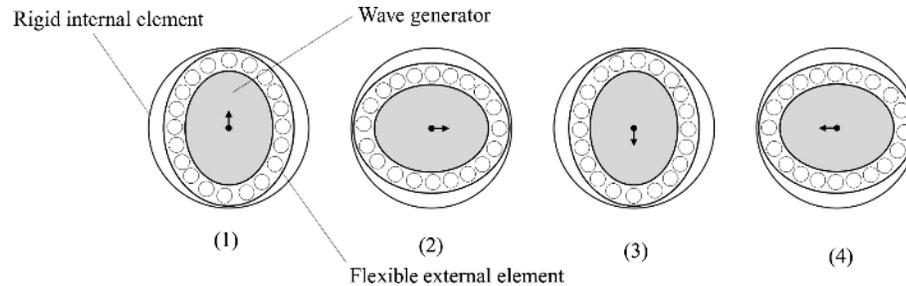


Figure 1.41 Anatomy of a harmonic gear drive

gear element across the major axis of the oval wave generator. The tooth mesh rotates with the major axis of the oval wave generator. When the wave generator rotates 180° Clockwise (CW), the flexible external gear element rotates CCW by one tooth relative to the rigid internal gear element. Each rotation of the wave generator causes the flexible gear element two-teeth counter clockwise (CCW) relative to the rigid external gear element. The gear ratio g for such harmonic drives is

$$g = \frac{N_f}{N_r - N_f}, \quad (1.33)$$

where

- N_f number of teeth on the flexible external gear element, and
- N_r number of teeth on the rigid internal gear element.

Errors in transmission function are typically inherent in harmonic drives. More appropriately, nonlinear motion exists as the gear teeth are usually based on circular external and circular internal gears elements. Distortions in the external gear element occur as a result of its flexibility. Conjugate action no longer occurs as a result of this distortion. This “distortion” can be reduced by designing the tooth based on the instantaneous radius of curvature of the ellipse in the distorted configuration.

1.17.5 Noncircular Planetary Gear Trains

PGTs are 2-dof mechanisms and consequently multiple noncircular gear elements can be combined within a single PGT (see Katori, 1998). PGTs with noncircular gear elements can yield “extreme” functional relations using “reasonable” noncircular gear elements. These “extreme” functional relations cannot be “reasonably” obtained using a single noncircular gear reduction. Such a special PGT was proposed by Mundo (2006) where he divided the PGT motion generation into two phases: one phase involving a nonlinear velocity relation g_{rp} between the ring gear and the planet gear and a second phase involving a nonlinear velocity relation g_{ps} between the planet gear and the sun gear. Mundo used a Fourier series expression to specify the instantaneous gear ratio accordingly:

$$g_{rp}(v_i) = m + \sum_{k=1}^N [a_k \cos(kmv_i) + b_k \sin(kmv_i)]. \quad (1.34)$$

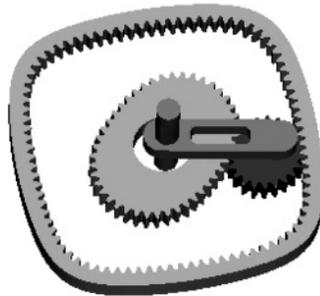


Figure 1.42 PGT with noncircular gear elements (reproduced by permission of MMT, Mundo, 2006)

The ring gear is the input gear element in this case and the planet is the output element. The restriction on Equation (1.34) is that its integral from 0 to 2π must be rational. The resulting planet radius and the sun gear's center of rotation define the sun gear radius. In general, the center of the sun gear and the ring gear are not coincident. Depicted in Figure 1.42 is an example of such a PGT. Mundo suggested the PGT as part of a bicycle driveline to match the optimal nonconstant pedal torque provided by the human body with constant torque to the wheels.

1.18 Summary

This chapter presents a review of the kinematic geometry of gearing and introduces some of the concepts of kinematic synthesis and analysis within the context of conventional cylindrical gearing. These concepts include mobility, the Arhbold-Kennedy instant center theorem, and the Euler-Savary equation. The concept of mobility is presented to check the constraint of multiple gears in mesh and determine if they will move. It is also useful in establishing the need for profile modification in order to enable the bodies to move. Mobility is a mathematical concept used to determine if a gear system is a constrained structure or if it is unconstrained to move as desired. The Arhbold-Kennedy instant center theorem is briefly discussed that can be used to determine in general the location of the pitch point relative to the two moving bodies and to ground at any instant for a given motion specification about two parallel axes. The Euler-Savary equation is presented to determine the radii of curvature of tooth profiles at the contact that satisfy the Arhbold-Kennedy instant center theorem. Different forms of conjugate tooth profiles are also presented and subsequently used to illustrate addendum and dedendum contacts, backlash, and contact ratio. A tabulation is given of the advantages and disadvantages of some of the tooth profiles that are commonly used in practice. Another table is given for the speed and torque relationships for PGTs. Included is the synthesis of noncircular gears for torque balancing purposes. Also, a brief analysis of the kinematics and statics of mechanism trains is given along with the theoretical basis for the generation of circulating power in compound gear trains. Part II of this book develops the necessary relationships for the geometric design and manufacture of these seemingly different aspects of gearing.