Chapter 1

Logic

The ideal writing style ascribed to by mathematicians is that in writing mathematics, *less is more.* If we can convey the exact idea of a concept with 5 words instead of 10, then we will use 5. Thus, we will use the statement *Cardinal numbers form a well-ordered collection* over the wordier statement *The well-ordered property is enjoyed by the collection of cardinal numbers.* The second statement is mathematically correct, but it is more than we need to convey the idea.

I have tried to practice this ideal while writing the mathematics in this book. The only exceptions to this ideal are made on the basis of decisions on the educational value of sentence structure, the anecdotal comments, or discussions of this sort that occur between mathematical discourse. Sometimes it is good to sacrifice some mathematical austerity in the interest of getting an important point across to the reader. As the reader will clearly see, this economy of words in mathematical writings is not exercised in the text of a discussion. Discussions and intermediate anecdotes contain examples and illustrations that are the only tools we have to illustrate a concept. Since I have sacrificed a good bit of mathematical rigor in favor of clarity, examples and illustrations are necessary if I am to get some subtle ideas across to the reader. This form of personalized writing style is unavoidable when discussing advanced ideas from mathematics in the popular press.

We have a bit of a mountain to climb in this book, so please be patient. Perhaps you can sit down in an overstuffed chair or at a table and open the book. Maybe you have a pencil and paper handy. That's a good idea. Some of these topics need to be diagrammed.

And certainly you have a cup of beverage, coffee would be my choice. Now turn on that lamp overhead and blend in that final inspiring ingredient: cream in your coffee. Good luck.

1.1 Axiomatic Method

The Axiomatic Method is how mathematicians apply logic. It is how we advance from one topic to the next, and so this is how future generations will discover more sophisticated forms of mathematics. The section will be brief, but it is how the mathematics in each successive chapter is treated.

Axioms are mathematical statements that we assume are True. We do not prove axioms, they come to us as statements whose Truth we do not deduce. The use of axioms first comes to us from the Greek slave Euclid circa 300 BC in his book *The Elements*. *The Elements* begins by stating five axioms and five postulates to be taken as primitive Truths. By assuming these 10 statements, Euclid was building a foundation on which logic would be used to deduce the mathematics in the reminder of *The Elements*. Today, the method of applying logic to a small set of primitive Truths is called the *axiomatic method*. It is the way mathematics has been practiced for the last 2300 years. It has lasted essentially unchanged since Euclid wrote it, including the many editions printed in the various lands in which *The Elements* was read and studied. It is how mathematics will progress to find larger thoughts using today's theorems.

For example, Euclid defines a *right angle* as the bisection of a line, and then he assumes that two right angles are equal. Today, we would say that it is obvious that any two right angles are equal, and so it was with Euclid's contemporaries. You might even suggest that you can prove it, but when you do you are assuming that any two lines represent an angle of the same measure, π. You have assumed what you wanted to prove. Euclid did not have angular measure, so he could not talk about π radians, but he knew what he was assuming. Thus the fourth postulate of *The Elements* assumes that any two right angles have equal measure.

A more subtle axiom is the fifth postulate, today called the *parallel postulate.* This was an attempt to describe the interior angles of two lines cut by a transversal. After thousands of years of investigation the parallel postulate has evolved into the equivalent form that we know today. It states that *through a point* P *not on a line* L *there is a unique line* L' *that is parallel to* L. Today's plane geometry is based on this parallel postulate. It is an interesting topic for further reading that one can change the parallel postulate into two different parallel postulates, and that each has its own use.

1.2 Tabular Logic

Formal logic is the logic used in Computer Science to design and construct the guts of your computer and its central processing chip. You have used this logic every time you analyzed regions in a Venn diagram.

And then there is Aristotle's logic. This is *the* logic used by rational men to form rational arguments. This logic is used to form arguments to prove that something is, or to prove that something is not. While we will examine formal logic, we are most interested in Aristotle's logic. Before we use Aristotle's logic to construct arguments we will introduce elementary or primitive logical statements.

The statements P, Q, and R are variables. They represent all statements from the language we are speaking. They do not exclude values unless we state so. Thus, P represents something simple like *The sky is blue*, or $1 \neq 0$, or something more complicated like

> *The sum of the squares of the lengths of the legs in a*
> *right triangle is the square of the length of the hypothenuse.*

Even that last statement is a possible value for P.

Aristotle's Logic begins with these statements and combines them using the elementary logical operations *not, and, or.* There might be other logical operations but they can be expressed as combinations of these three. The logical state of a statement formed by using P and Q is determined by the entries in a few tables.

The operation *not* simply changes the logical state of P from one logical state into the other. In tabular form, *not* can be described by the following.

P	$not\,P$
T	F
F	T

In the first column of this table, we are considering all possible logical states for P. True T and False F are all of the logical states that P can achieve in this book. (We consider only binary logic here.) The second column is the logical state of the statement $not\,P$. We should be clear about this. We begin with a table and *define* what *not* means. That defined meaning reflects exactly what you have used *not* for in your life. We do not begin with a word *not* and then try to make up a table for it. We have tried to define a logical operation here, and that is what the table does.

Notice that the table for *not* does just what we first stated *not* will do. It takes a logical states for P and changes it from True to False, or from False to True. Follow the logic for P today.

1. $P =$ *The sky is blue* is a True statement.

2. $not\,P =$ *The sky is not blue* is a False statement.

Of course, if it is a slate gray sky, or if we are on Mars, then *The sky is blue* will be a False statement. This is what you mean when you say to someone *What is the color of the sky in your Universe?* You are asking for the logical state of the statement *The sky is blue.* You are asking that individual for a logical foundation from which the two of you can intelligently converse.

Our operation *not* has a familiar property that comes from that early English class you had. Given a statement P, then $not\,not\,P$ has the same logical states as P. That is, a double negative does not change the logical state of P. In terms of a table, we have

P	$not\,P$	$not\,not\,P$
T	F	T
F	T	F

The first and third columns of the table show that P and $not\,not\,P$ have the same logical state. If P is True, then $not\,not\,P$ is True, and if P is False, then $not\,not\,P$ is False. Notice that the first way we described the double negative, the table we gave for it, and the last couple of lines all describe the same operation. From input to ultimate output, *not not* does not change the logical state of the input statement. It does change P, though, doesn't it. If we let $P =$ *The sky is blue* then one reading of *not not* P is *It is False that the sky is not blue.* This last statement will have the same logical state as P, but it is awkward in its presentation. We will avoid the double negative whenever possible but we will find that at times we are forced to deal with it.

The operation *and* combines two statements P and Q and makes a compound statement P *and* Q. The statement P *and* Q is True exactly when both P and Q are True. So, of course, the other possible combinations of T and F for P and Q yield logical states False. Thus, if one or more of P, Q is False then the statement P *and* Q is False. If we let $P =$ *The sky is blue* and if we let $Q =$ *I am human* then today P *and* Q is a True statement. *The sky is blue and I am human* is a True statement on the day this is written. But if it is a slate gray sky, then *The sky is blue and I am human* is a False statement. If I come from Mars then *The sky is blue and I am human* is a False statement. If I am writing this on Mars in January of 1900 then *The sky is blue and I am human* is a False statement because both $P =$ *The sky is blue* and $Q =$ *I am human* are False statements.

The table of values T, F for *and* will make the above discussion short and mechanical. That table is

P	Q	P *and* Q
T	T	T
T	F	F
F	T	F
F	F	F

In other words, P *and* Q is True exactly when both P and Q are True. In any other situation, P *and* Q is False.

The first two columns of the above table gives us all of the possible pairs of logical states T, F for P and Q, and in the third column we read the corresponding logical states for the compound statement P *and* Q. Notice that P *and* Q is a True statement exactly when both P and Q are True. Otherwise, P *and* Q is a False statement. This is an effective shorthand since once we know that

P *and* Q is True exactly when both P and Q are True, then the logical states in the rest of the table fall into place.

Let us see how the logic of our discussion proceeds. It is elementary, but it also shows us what the undercurrent of our thought process is.

1. Let $P =$ *The sky is blue* and let $Q =$ *I am human.*

2. P is True, and Q is True.

3. P *and* Q is then *True*.

4. Thus *The sky is blue and I am human* is True.

You may have skipped all of those thoughts but this list of thoughts fills in all of those nagging details about the logic of the compound statement. Actually, this linear discussion shows how logic is part of the structure of your language. We just don't think in that much detail, now do we?

The third operation is the *or* operation. This operation takes two statements P and Q and assigns a True logical state to P *or* Q when at least one of them is True. Or to put it another way, P *or* Q is False exactly when both P and Q are False. The rest of the cases T, F for P and Q yield a True statement P *or* Q.

So if we let $P =$ *The sky is blue*, and if we let $Q =$ *I am human*, then P *or* Q is a True statement. That is, *The sky is blue or I am human* is a True statement. That is because Q is True. The logical state of P in this case does not matter. If it is a slate gray sky, then *The sky is blue or I am human* is still a True statement. The True statement *I am human* makes the compound statement *The sky is blue or I am human* a Truth. But if I am writing this on Mars in January of 1900, then *The sky is blue and I am human* is a False statement because both $P =$*The sky is blue* and $Q =$*I am human* are False statements.

The table for the operation *or* will again make the above discussion short and mechanical.

P	Q	P *or* Q
T	T	T
T	F	T
F	T	T
F	F	F

In other words, P *or* Q is False exactly when both P and Q are False. The other values T in the third column of the table are then forced.

The first two columns give us all possible pairs of logical states for the statements P and Q. The third column gives us the logical states of P *or* Q that correspond to the first two columns of the table. Notice that the logical states in column three show that P *or* Q is False exactly when both P and Q are False.

The next way to combine statements P and Q we will call *implication*. We write

$$P \Rightarrow Q$$

when we want to say that P *implies* Q. In its simplest form, $P \Rightarrow Q$ is False exactly when P is True and Q is False. In every other instance the statement $P \Rightarrow Q$ is True. Its tabular description follows from this verbal description.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

From the table defining *implication*, we see several important properties. The first row of the table for $P \Rightarrow Q$ gives us the most important argument in mathematics, that of deductive reasoning. The first row shows us that if we make no mistake, that is, if $P \Rightarrow Q$ is True, then the Truth of P implies the Truth of Q. Thus, if P is True and if we make no mistakes, that is, if $P \Rightarrow Q$ is True, then Q is True. We can find new Truths from old Truths in this way.

The last row shows us that if we make no mistakes in our argument, then a Falsehood Q comes from a Falsehood P. Hence, if $P \Rightarrow Q$ is True and if Q is False, then P is False. We will often work with this argument. It is called the *indirect proof*.

The first line of the table allows us to deduce the second line. The Truth of P implies the Truth of Q if we make no mistakes in our argument $P \Rightarrow Q$. Therefore, if P is True and if Q is False, then we made a mistake somewhere, and $P \Rightarrow Q$ is False. If you have deduced a Falsehood from a Truth then you have made a mistake. That is what the False logical state of $P \Rightarrow Q$ stands for: a mistake. Thus, the implication $P \Rightarrow 1{=}0$ is False if P is True. The conclusion $1 = 0$ is False, so the implication is a Falsehood.

There is another interesting possibility for $P \Rightarrow Q$. A Falsehood P will imply anything. If we begin our argument with a False premise P, then subsequent deductions Q do not possess a predictable logical state. These deductions Q can be either True or False. If P is a Falsehood then $P \Rightarrow Q$ is True no matter what the logical states of Q is. Thus, if P is False, you can deduce that $1 = 0$, that *All opinions are valid*, and that there is a Universal Set. But, as we will prove later, each of these is a Falsehood. The conclusion drawn will have no logical weight whatsoever because your premise P was False

After all, we can deduce that there are no prime numbers if we assume that $1 = 0$, but of course the conclusion is False. The argument goes like this. Assume that $1 = 0$. Then $1 + 1 = 0 + 0$ and so $2 = 0$. In this manner, we can prove that $n = 0$ for each $n \in \mathbb{N}$. That is correct. From the premise $1 = 0$ we can prove that there are no other natural numbers but 0. Since 0 is not a prime number, we have proved that there are no prime numbers. This is the kind of foolishness we can arrive at by proceeding from a False premise. However, the steps in our argument were all True, so that the implication $1{=}0 \Rightarrow$ *there are no prime numbers* is a True statement. Think about that for awhile.

Exercise 1.2.1 Let P, Q, and R be statements. Make Truth Tables for the compound statements in the following exercises.

1. $not(P \, or \, Q)$
2. $not(P \, and \, Q)$
3. $P \, and \, (Q \, and \, R)$
4. $P \, or \, (Q \, or \, R)$
5. $((not P) \, or \, Q) \, and \, (P \, or \, (not Q))$

1.3 Tautology

A *tautology* is a logical statement R that is always True. When its table is established, the output logical states, those values in the rightmost column, are all T. Let us examine a few tautologies.

Consider $P\,or\,(not\,P)$. We can see that this is tautological by observing that by the table for the *not* operation, either P or $not\,P$ is a Truth. That is, one of P and $not\,P$ is True. Examining the table for *or* shows us that $P\,or\,(not\,P)$ is then True. In its tabular form, we have

P	$not\,P$	$P\,or\,(not\,P)$
T	F	T
F	T	T

Thus, $P\,or\,not\,P$ is a tautology. $P\,or\,not\,P$ is a True statement given any logical state for P. In other words, as the table suggests, no matter which statement is used for P, the output $P\,or\,(not\,P)$ is a True statement. For example, *The sky is blue or the sky is not blue* is True, as is $1 = 0$ *or* $1 \neq 0$. Also, the statement *There is a Universal Set or there is no Universal Set* is True. We may not know what a Universal Set is, but we know that the statement *There is a Universal Set or there is no Universal Set* is True. That is, it either is or it is not.

In the same way, the statement $P\,and\,(not\,P)$ is False because by definition of *not*, the statements P and $not\,P$ have different logical states. For instance, if P is True, then $not\,P$ is False. Then by the the definition of *and*, $P\,and\,(not\,P)$ is False. The next table shows all of this in tabular form.

P	$not\,P$	$P\,and\,(not\,P)$
T	F	F
F	T	F

For example, let X be a any statement. Given the statement $P =$ X *is valid* then $not\,P = X$ *is not valid.* Hence, the statement

$$X \text{ is valid and } X \text{ is not valid}$$

is a False statement. We will encounter this kind of Falsehood often as this chapter moves along.

A more common logical comparison of statements is the following. Let X and Y be statements. We say that X and Y are *logically equivalent* iff X and Y have the same logical states. In terms of a Truth Table, the two right-hand columns in the table for X and Y have the same sequence of T's and F's. We saw above that P and *not not* P have the same logical values, so

> The statements P and *not not* P are logically equivalent

A more complex example of logically equivalent statements is formed from an implication and its *contrapositive*. The contrapositive of the implication $P \Rightarrow Q$ is the implication

> $$not Q \Rightarrow not P.$$

Notice the reversal in the roles of P and Q. The Truth Table of the contrapositive of $P \Rightarrow Q$ is given below.

P	Q	$not P$	$not Q$	$not Q \Rightarrow not P$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Let us show that the implication and its contrapositive are logically equivalent. We use a sizable Truth Table.

P	Q	$not Q$	$not P$	$P \Rightarrow Q$	$not Q \Rightarrow not P$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

Notice that the two right most columns have exactly the same entries in the same order. Thus, $P \Rightarrow Q$ and $not Q \Rightarrow not P$ are logically equivalent. They are not the *same* statement. They differ in their sentence structure. They differ in the way they are written in English. This is important. Logically equivalent statements do not have to look alike at all.

Consider the statement

> Let $a > 0$ be a natural number. If a^2 is odd, then a is odd.

Its contrapositive is True when the implication is True and its contrapositive is False when the implication is False. That contrapositive is

> Let $a > 0$ be a natural number. If a is even, then a^2 is even.

These two statements are logically equivalent even though their statements are different.

Something more symbolic is the logical expression

$$(not\,P)\ or\ Q.$$

It is a simple combination of the operations *not* and *or*, and yet we will see that it is logically equivalent to a familiar statement.

To begin, we argue verbally. The statement $(not\,P)\ or\ Q$ is a False statement exactly when the twostatements $not\,P$ and Q are False. This occurs when P is True and Q is False, and only when P and Q are in these logical states. For any other logical states, (not P) or Q is a True statement. There is a coincidence here. The implication $P \Rightarrow Q$ is False only when P is True and Q is False. Given any other logical states for P and Q, $P \Rightarrow Q$ is True. Hence (not P) or Q and $P \Rightarrow Q$ have the same logical states. They are then logically equivalent. The relevant Truth Table looks like this.

P	Q	$not\,P$	(not P) or Q	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We conclude that

> (not P) or Q and $P \Rightarrow Q$ are logically equivalent.

At this point, we will abandon the use of Truth Tables. One can use them to discuss the other logical tautologies that we will bring out presently, but we feel that in our present setting, they are Baroque. The reader should feel free to translate our lingual discussion into a tabular one. The exercise will do you good.

Some logical statements are combinations of two or more smaller statements. Let P, Q, and R be statements. Different ways to combine and manipulate these statements are from the following tautologies.

associative law	$P\ and\ (Q\ and\ R)$	$=$	$(P\ and\ Q)\ and\ R$
distributive law	$P\ and\ (Q\ or\ R)$	$=$	$(P\ and\ Q)\ or\ (P\ and\ R)$
distributive law	$P\ or\ (Q\ and\ R)$	$=$	$(P\ or\ Q)\ and\ (P\ or\ R)$
the biconditional	$P \Leftrightarrow Q$	$=$	$(P \Rightarrow Q)\ and\ (Q \Rightarrow P)$

The associative law, for example, states that there is no reason to use parentheses in conjuncted statements using the *and* operation. The statement

The sky is blue and the grass is green and I am human

is unambiguous in the calculation of its logical state. The distributive laws simply give us reasons to replace commas with parentheses. For example,

Either the sky is blue, or the grass is green and I am human

can be rewritten as

The sky is blue or the grass is green,
and the sky is blue or I am human

without changing the logical state of the compound statement.

The *biconditional* is a short way of writing that P implies Q, and also that Q implies P. The biconditional $P \Leftrightarrow Q$ is read P *if and only if* Q. This means *If* P *then* Q and *If* Q *then* P. It is common to write

$$P \text{ iff } Q$$

for $P \Leftrightarrow Q$.

For instance, we will prove that

$$1 + 1 = 2 \text{ iff } 1 + 1 + 1 = 3.$$

Proof: We must first prove that $1 + 1 = 2 \Rightarrow 1 + 1 + 1 = 3$. This is done by beginning with $1 + 1 = 2$, and then adding 1 to each side, which yields $1 + 1 + 1 = 2 + 1 = 3$.

Conversely, begin with $1 + 1 + 1 = 3$, and then subtract 1 from both sides, from which we have $1 + 1 = 1 + 1 + 1 - 1 = 3 - 1 = 2$. Thus, $1 + 1 = 2$, which is what we had to prove.

Therefore, having proved the implication $1+1 = 2 \Rightarrow 1+1+1 = 3$ and the implication $1 + 1 + 1 = 3 \Rightarrow 1 + 1 = 2$, we conclude that $1 + 1 = 2$ iff $1 + 1 + 1 = 3$. This completes the argument.

Let us prove another biconditional statement.

Let $a \in \mathbb{N}$. Then $a < 2a$ iff $a \neq 0$.

Proof: We must prove that *if* $a < 2a$ *then* $a \neq 0$ and that *if* $a \neq 0$ *then* $a < 2a$. Begin by proving *if* $a < 2a$ *then* $a \neq 0$. Suppose that $a < 2a$. Then, allowing that we all know the arithmetic, $0 = a - a < 2a - a = a$. Thus $a \neq 0$, which proves this half of the larger proof.

Conversely, we must prove *if* $a \neq 0$ *then* $a < 2a$. Begin with $a \neq 0$. Then $a > 0$, so that $2a = a + a > 0 + a = a$. Thus $a < 2a$ is proved. Therefore, by having proved both *if* $a < 2a$ *then* $a \neq 0$ and *if* $a \neq 0$ *then* $a < 2a$, we conclude that $a \neq 0$ iff $a < 2a$. This completes the larger proof.

There will be much more to say about the biconditional statement in the later pages of this book.

Another tautology that we will find useful in the rest of this book is called *DeMorgan's Laws for Logic*. It shows how the *not* operation combines with the *and* and the *or* operations. Symbolically, DeMorgan's Laws for Logic look like this.

DeMorgan's Law 1.3.1 *Let P and Q be statements. Then*

1. $\text{not}(P \,\text{and}\, Q) = (\text{not}\, P) \,\text{or}\, (\text{not}\, Q)$

2. $\text{not}(P \,\text{or}\, Q) = (\text{not}\, P) \,\text{and}\, (\text{not}\, Q)$

Note the change from *and* to *or* and from *or* to *and* in each of DeMorgan's Laws. Thus

Either the sky is not blue, or the grass is not green.

has the same logical state as

It is False that the sky is blue and the grass is green.

Furthermore,

The sky is not blue and the grass is not green.

has the same logical state as

It is False that the sky is blue or that the grass is green.

Exercise 1.3.2 Let P, Q, and R be statements.

1. Show that $P \, and \, (not Q)$ is logically equivalent to $not((not P) \, or \, Q)$.

2. Show that $P \, or \, (not Q)$ is logically equivalent to $not((not P) \, and \, Q)$.

3. Find a shorter statement that is logically equivalent to $not(not((not P) \, or \, Q)) \, or \, R)$.

4. Give a highly detailed proof that *If* $a = ab$ *then* $b = 1$.

5. Give a highly detailed proof that *If* $x^2 - 1 = 0$ *then* $x \in \{-1, 1\}$.

1.4 Logical Strategies

The first logical observation is that there is a statement that is always False, no matter what the logical value of P is. The statement $P \, and \, (not P)$ is a Falsehood. It is False F all of the time. We discussed this in the previous section.

Example 1.4.1 1. Let P be the statement *The sky is blue.* Then *the sky is blue and the sky is not blue* is a Falsehood.

2. Let P be the statement *There is a mountain.* Then the statement *There is a mountain and there is no mountain* is a Falsehood.

3. Let P be the statement *This is True.* Then *This is True and this is not True* is a Falsehood.

We continue our discussion with *implications.* Let P and Q be statements. It is good to know that $P \Rightarrow Q$ is True or False, but it is better to know how one uses the implication $P \Rightarrow Q$ to argue correctly. For example, let us reexamine the first line of the Truth Table for $P \Rightarrow Q$. That first row is T, T, T. It can be read as follows. If we start with a True premise P, and if we make no mistakes in our argument, then our conclusion Q is True. You might expect this, since every deductive argument is based on it.

Example 1.4.2 1. Here is a Greek classic. Begin with the statement P, *Socrates is a man.* It is True that *If he is a man then he is mortal.* We conclude Q that *Socrates is mortal* is True. I know that P is True, and that $P \Rightarrow Q$ is True. From this I deduce that Q is True.

2. Let P be the statement *I see the sky on Earth* (on a sunny day). It is True that *If I see the sky on Earth then the sky is blue.* Therefore I conclude Q that *The sky I see is blue.* I have assumed P and observed that $P \Rightarrow Q$ is True. From this I deduce that Q is True.

Example 1.4.3 1. Let P be the True statement *The sky is not blue.* Argue correctly as follows. If the sky is not blue, then we are not on Earth. Conclude the statement Q that *We are not on Earth* is a True statement.

2. Something more advanced goes like this. Let $a, b \in \mathbb{N}$ be nonzero. We prove that $a \geq ab$.

Proof: We will write one True implication after another to form our argument. We will then conclude that $a < ab$. Begin with a True statement. Since $a, b \in \mathbb{N}$ are nonzero, $a, b \geq 1$. Since $a \geq 1$, multiplying $1 \leq b$ by a implies that $a = a \cdot 1 \leq ab$. Thus, $a \leq ab$ is a True statement, which is what we had to prove.

A more subtle argument shows us the strength of logic when arguments are concatenated. Consider the statements

$$\begin{aligned} &P \\ &P \Rightarrow Q \\ &Q \Rightarrow R \\ &R \end{aligned}$$

and assume that P is a True statement, that $P \Rightarrow Q$ is a True statement, and that $Q \Rightarrow R$ is a True statement. Make no assumptions about R. Argue as follows.

From the Truth of P and the Truth Table of $P \Rightarrow Q$, we deduce that Q is a True statement. Then Q and $Q \Rightarrow R$ are True, so we may deduce that R is a True statement.

We began with the Truth of P. The Truth of the statements $P \Rightarrow Q$ and $Q \Rightarrow R$ allows us to make a correct argument $P \Rightarrow Q$ *and then* $Q \Rightarrow R$. We then deduce the Truth of R. This justifies our use of this compound argument in our deliberations.

More generally, we have shown that if your argument begins with a True statement P, and if, as above, the compound argument consists of one True implication after another, then you have formed a longer True argument. You then deduce the Truth of the last statement R in your argument.

Example 1.4.4 This example shows how the above discussion can be applied to longer arguments.

a) Assume that $10 < 2^{10}$. (This is the statement P.)

b) Because $10 < 2^{10}$, we see that $2 \cdot 10 < 2 \cdot 2^{10}$. (This is $P \Rightarrow Q$.)

c) Because $20 = 2 \cdot 10$, we have $11 < 2 \cdot 10$. (This is $Q \Rightarrow R$.)

d) Because $2 \cdot 10 < 2 \cdot 2^{10}$, it follows that $2 \cdot 10 < 2^{1+10} = 2^{11}$. (This is $R \Rightarrow S$.)

e) The conclusions in lines c) and d) combine to show that $11 < 2 \cdot 10 < 2^{11}$.

We conclude that $11 < 2^{11}$.

We will not use this level of detail in future arguments.

Exercise 1.4.5 Find the error in the following arguments. These errors make the arguments False.

1. (a) This amoeba lives on Earth.
   (b) All life on Earth has hair.
   (c) Thus, this amoeba has hair.

2. (a) Assume that $x \in \mathbb{R}$ satisfies $x^2 - x = 0$.
   (b) If $x^2 - x = 0$ then $x^2 = x$.
   (c) $x^2/x = x$ and $x/x = 1$.
   (d) Thus, $x = 1$.

3. (a) I swim in the sea.
   (b) If I am a fish, then I swim in the sea.
   (c) I swim in the sea, so I am a fish.
   (d) Thus, I am a fish.

1.5 Implications From Implications

The implication $P \Rightarrow Q$ comes with what is called the *converse*. *The converse of* $P \Rightarrow Q$ is $Q \Rightarrow P$. Let us write down the Truth Table for $Q \Rightarrow P$ and compare it with $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$Q \Rightarrow P$
T	T	T
T	F	T
F	T	F
F	F	T

As you can see, the implication and the converse do not have the same Truth Table. The value of the implication $P \Rightarrow Q$ in the third row is T, while the logical value of the implication $Q \Rightarrow P$ in the third row is F. Thus, the converse implication $Q \Rightarrow P$ can be False even when $P \Rightarrow Q$ is True. For this reason, the converse cannot be assumed to be a True statement even when the original implication is True. Therefore, we must be careful to avoid the classic error of using the converse of an implication. Some examples will help.

Example 1.5.1 1. Let P be *The sky is blue*, and let Q be *The world is flat*. Then $P \Rightarrow Q$ is *If the sky is blue then the world is* *flat.* This is a False statement. Its converse is $Q \Rightarrow P$. *If the world is flat then the sky is blue.* The Truth of this converse is established by looking at line 3 in the Truth Table for $\Rightarrow$.

2. The converse of the implication *If today is Monday then my schedule is clear* is *If my schedule is clear then today is Monday.* The implication may be True, but the converse is False if my schedule is clear on Sunday.

Suppose that we have an implication $P \Rightarrow Q$. If the implication is True and if Q is False, then line 4 of the Truth Table for $P \Rightarrow Q$ shows us that P is False. This leads us to an important implication known as the *contrapositive.*

$$not Q \Rightarrow not P$$

We discussed the contrapositive beginning on page 18. The implication $P \Rightarrow Q$ and its contrapositive $not Q \Rightarrow not P$ are logically equivalent statements.

Example 1.5.2 1. Let P be *My GPS is working* and let Q be *I am not lost.* The implication is *If my GPS is working then I am not lost.* Its contrapositive is *If I am lost then my GPS is not working.* Notice that both the implication and its contrapositive are True, provided that I always use my GPS.

2. Let P be *My spell-check program is running,* and let Q be *I misspell all the time.* The implication *If my spell-check program is running then I do not misspell all the time* has contrapositive *If I misspell all the time then my spell-check program is not running.* Notice that both the implication and its contrapositive are True.

3. The implication *If the sky is not blue then this is not Earth* has as contrapositive *If this is Earth then the sky is blue.* Once again the implication and its contrapositive are True.

The next form of argument is meant to demonstrate how to use a counterexample as a means of proof. The idea is as follows. If I tell you that *All colors are blue,* then you can show that this is a Falsehood by producing some color that is not blue. One nonblue color would be red. The existence of red shows that *All colors are blue* is False. You just proved *Some colors are not blue.*

This approach to proof is called *proof by contradiction*. They all proceed in the same way. You are given a statement that claims either *All of something has a property* or *Nothing has a property.* To show that these two statements are Falsehoods, and to prove a statement about something having a property, you produce a counterexample to the claim. Some examples will help.

Example 1.5.3 1. *All numbers are even* is claimed. You produce the number 3, which is not even. This 3 is a counterexample to the claim that *all* numbers are even. Thus *All numbers are even* is False. You have thus proved that *Some numbers are not even.*

2. *All people are Truth sayers* is claimed. That is, it is claimed that every person tells the Truth all of the time. You react by producing a known False statement and then speaking it. You speak *1 = 0*, thus uttering a False statement. The claim is proved False. You have proved that *Someone will sometimes speak Falsehoods.*

3. *All statements are False* is claimed. You state that *The number line has no end.* This Truth is a counterexample to the claim. You have shown that *Some statements are True.*

4. *Nothing is interesting* is claimed. The claim is that nothing in this world is interesting at all. You say, *I find that the lack of interesting facts in this world is interesting,* thus showing that something is interesting. With this counterexample to the claim, you have shown that *Somethings are interesting.*

Exercise 1.5.4 Prove these statements using the proof by counter example.

1. All math professors are male.
2. All numbers are positive.
3. If $a \leq a^2$ then $1 < a$.
4. $\tan(\theta) \leq 1$ for all $\theta \in (-\pi/2, \pi/2)$.
5. All men are liars.
6. All men are Truth sayers.
7. Left alone things do not change.

1.6 Universal Quantifiers

Statements about *All* or *Nothing* of something are called *universal quantifiers.* We will discuss proofs of universal quantifiers in this section. In so doing, we will show that a certain kind of statement is always False. These statements are common when speaking about logical ideas. We will show that these Falsehoods not only look alike but they also have similar proofs.

A logical strategy that comes from the bottom row

P	Q	$P \Rightarrow Q$
F	F	T

of the Truth Table we used to define the implication $P \Rightarrow Q$ is given as follows. *Suppose that you make no mistakes in your argument, but that you conclude that Q is False. Then your premise P is also a Falsehood.* To employ this proof, you begin by assuming that P is True. Then you make a careful argument with no mistakes. Once you have come upon a Falsehood, Q, you can conclude that P is False. We will continually use this strategy in this section.

The *universal* and *existential quantifiers* are logical statements used to define the logical value of infinitely many statements without writing down infinitely many statements. Let $P(t)$ be a statement that is defined in terms of a parameter t. For example, $P(n)$ might be $n < n+1$, or $P(x)$ might be *If* $x \neq 0$ *then* $\frac{1}{x}$ *is defined.* The *universal quantifier* allows us to discuss the logical values of $P(n)$ and $P(x)$ all at once. It is the infinite counterpart of the *and* operation. Suppose that $t \in I$ for some index set I. Then

$\forall t \in I, P(t)$ is read *for all t in I, $P(t)$ is True.*

You may delete the *is True* as the language permits. The universal quantifier $\forall t \in I, P(t)$ is True only when $P(t)$ is True for every $t \in I$. Thus, $\forall n \in \mathbb{N}, n < n+1$ is a True universal quantifier since $n < n+1$ is True for all $n \in \mathbb{N}$.

The *existential quantifier* is the infinite equivalent of the *or* operation. Given a statement $P(t)$ with parameter $t \in I$ then

$\exists t \in I, P(t)$ is read *there exists a t in I such that P(t) is True.*

You may drop *is True* as the language permits. The existential quantifier $\exists t \in I, P(t)$ is True when there is some $t \in I$ such that $P(t)$ is True. (The words $P(t)$ *is True for some* t mean that $P(t)$ is True for at least one value t and at most all t.)

The *negation* of $\forall t \in I,\ P(t)$ is argued as follows. It is False that $P(t)$ is True for all $t \in I$, so $P(t)$ is False for some $t \in I$. Put succinctly,

$\exists t \in I,\ \textit{not}\, P(t)$ is the logical negation of $\forall t \in I,\ P(t)$.

Thus, the negation of a universal quantifier is an existential quantifier. For instance, $\forall x \in \mathbb{R}, \sqrt{x}$ *is real* is a False statement because $\sqrt{-1}$ is not real. The logical negation of $\forall x \in \mathbb{R}, \sqrt{x}$ *is real* is $\exists x \in \mathbb{R},\ \sqrt{x}$ *is not real.*

The logical negation of $\exists t \in I,\ P(t)$ is rationalized as follows. Suppose it is False that $\exists t \in I,\ P(t)$. Then no statement $P(t)$ is True, or equivalently $P(t)$ is not True for *any* $t \in I$. Given $t \in I$ then $P(t)$ is False, so that $\textit{not}\, P(t)$ is True for each $t \in I$. Symbolically we would write as follows.

$\forall t \in I,\ \textit{not}\, P(t)$ is the logical negation of $\exists t \in I,\ P(t)$.

Example 1.6.1 1. Let $P(n)$ be the statement $2n = 2^n$. Since $2 \cdot 2 = 2^2$, the existential statement $\exists n \in \mathbb{N},\ 2n = 2^n$ is a True statement, while $2 \cdot 3 \neq 2^3$ shows that $\forall n \in \mathbb{N},\ 2n = 2^n$ is False.

2. Fix a continuous function f of a real variable on the closed bounded interval $[a, b]$, and let F denote an arbitrary function on $[a, b]$. If $P(F)$ is the statement $\dfrac{dF}{dx} = f$, then $\exists F, P(F)$ is True by The Fundamental Theorem of Calculus. The function $F(x) = \int_a^x f(t)\ dt$ has derivative $f(x)$.

Exercise 1.6.2 In each of the following exercises, negate the False quantifier.

1. $\forall x \in \mathbb{R},\ x^2 > 0$

2. $\exists x, y, z \in \mathbb{N},\ x^3 + y^3 = z^3$

3. $\exists$ continuous function f on $[0,1]$, $\dfrac{d}{dx}\displaystyle\int_0^1 f(t)dt \neq f(x)$.

4. $\exists$ a set $\mathbf{C}$, $\mathbf{C}$ contains all sets.

5. $\forall$ opinion P, P is valid.

1.7 Fun With Language and Logic

Here are some examples that will illustrate how deduction is used in mathematics. To understand these examples, we must first examine how to diagram a sentence.

Let Y be some quality of statements. So Y could be, but is not restricted to, the values *True, valid, known, recursive, hard to read, impossible to understand.* Let Q be the statement *This statement has quality* Y. We would say that Q and *This statement has quality* Y are the same. Furthermore, since *This statement* refers to Q, Q is the same as *This statement* Q *has quality* Y, which is also the same as Q *has quality* Y. The above discussion is easily translated into the following table, each of whose statements are the same.

1. Q

2. *This statement has quality* Y

3. *This statement* Q *has quality* Y

4. Q *has quality* Y

Some clever arguments can be formed by interchanging these equivalent ways to write Q.

Example 1.7.1 This problem goes back to ancient Greece, sometime before 300 BC. Epimenides is a philosopher from the island of Crete. He walks into a room of Greek scholars and cries out, *All*

Cretans are liars. Then he says, *I am lying.* The Greek scholars proceed to determine the logical value of *I am lying*, deciding that *I am lying* is both a lie and the Truth, an intolerable situation in those days.

We will give a modern approach to the Epimenides Paradox that explains the paradox. I have hired a Cretan public speaker to speak the quotations in this discussion. Assume that *All Cretans are liars.* Consider the statement Q: *This statement when spoken by a Cretan is not a lie.* Since our Cretan spoke Q, Epimenides' declaration states that Q is a lie. But Q states that *This statement* Q *is not a lie*, so that Q is not a lie. But the statement R: Q *is a lie* has logical negation *not*R: Q *is not a lie.* Recall that R *and* (*not*R) is a Falsehood from page 9. We have argued to a Falsehood, so our premise is False. That is, *All Cretan's are liars* is a Falsehood.

Let us review how we argued. We assumed that the statement P : *All Cretans are liars* is True. We then made no errors in our implication, and we deduced the two statements R: Q *is a lie* and its logical negation *not*R: *Q is not lie.* Since this conclusion of ours R *and* (*not*R) is then False, we began with a False premise. That is, *All Cretans are liars* is a Falsehood.

Returning to the ancient paradox, we proceed from the False premise *All Cretans are liars.* Thus, we can deduce many things, but we have no means of showing that the conclusions are True. Specifically, we claim that we have deduced that *I am lying* is paradox being neither True nor False. But the Truth is that we cannot identify the logical state of *"I am lyin" is a paradox.* Beginning with a Falsehood the way we did makes any analysis of the logical state of *"I am lying" is a paradox* impossible. This illustrates just how badly facts can be confused when we proceed from a False premise.

Example 1.7.2 Suppose we assume that *All is known.* That is, if there is a fact to be learned, then it is already known. Consider the statement Q : *This fact is not known.* By our assumption, Q is known. By reading Q we see that *The fact* Q *is not known*, so that Q is not known. We have deduced a statement R: Q *is known* and its logical negation *not*R: Q *is not known*, which we combine to form the Falsehood R *and* (*not*R) as on page 9. We conclude that we began with a False premise, so that *All is known* is False.

Suppose that in some future time we have deduced a body of knowledge B. Then B represents all of the known facts, which by the above paragraph cannot be all facts. Specifically, knowledge is neither finite nor bounded. Consequently, there will always be some fact to be discovered. Hence, unless we willingly stop work, researchers and scholars cannot work themselves out of a job. There will always be some unknown fact to be researched.

Example 1.7.3 Suppose we assume that *All opinions are valid.* Consider the statement Q: *This opinion is not valid.* Since Q is an opinion, our supposition implies that Q is valid. But Q itself states that this opinion Q is not valid. Thus, we have deduced R: Q *is valid* and its logical negation *not*R: Q *is not valid.* When combined we have deduced the Falsehood R *and* (*not*R). Thus, our premise *All opinions are valid* is False. This can be interpreted as stating that not every opinion advanced is worth listening to.

Example 1.7.4 The above examples illustrate a general form of statement and argument. Let Y be one of the properties in the set {True, known, valid, complicated, assumed, hard to understand, worth listening to}. Suppose we assume that *All self-referential statements are of type* Y. Consider the statement Q: *This statement is not of type* Y. Because Q is a self-referential statement, it is True that Q *is of type* Y. But Q itself states that the statement Q *is not of type* Y. Thus, we have deduced R: Q *is of type* Y and its logical negation *not*R: Q *is not of type* Y. The Falsehood R *and* (*not*R) shows us that *All statements are* Y is False.

Example 1.7.5 A *perfect logician* is a person who knows all of logic. Let us decide the logical state of *There is a perfect logician* PL. Assume that there is a perfect logician PL. We begin our argument with the statement Q: *This statement of logic is not known to* PL. Then this logical statement, Q, is not known to PL. Furthermore, by assumption, PL is a perfect logician, so Q is known to PL. The contradiction is that we have deduced Q *is not known to* PL and its logical negation Q *is known to* PL. Thus *There is a perfect logician* PL is a Falsehood, and hence there can be no perfect logicians.

Let us apply the previous example.

Example 1.7.6 At the time of this writing, *The World's Hardest Logic Puzzle* has been a popular stop for those who surf the web. The puzzle begins with 200 perfect logicians on an island, 100 of them are blue eyed, and 100 of them are brown eyed. Once a perfect logician has determined his eye color through the use of logic alone, he will leave the island. The problem is to determine when all of these perfect logicians have left the island. That's it. That is all that we assume in this version of the puzzle. There are some internet versions of this puzzle that include much more detail than this version, but they and their solutions follow from our solution given below. In other words, once we solve this problem then we can solve any other version of it.

The solution is that the problem begins by assuming that there are 200 perfect logicians, while we have proved that there are *no* perfect logicians. Thus the premise of the problem is False. You can therefore deduce anything you want, but you have no way of knowing which deduction is True. Thus, you might deduce that 200 people leave the island Friday, you might deduce that 200 people leave the island instantly, or you might deduce that seven of them never leave the island. But we cannot know the Truth of these conclusions as we proceed from a False premise.

This kind of indirect argument will appear often in the succeeding chapters. The readers should familiarize themselves with it.

The Q statements used above are examples of *self-referential statements*. These are statements that refer to themselves. One fun example of a self-referential statement is the following story that in the end refers to itself.

Mythology 1.7.7 There once was a girl who liked to travel from town to town, telling this story about herself. One day, while traveling in the dense forest, she entered a small village in a small clearing. She told them that she was hungry and tired, and then asked if she could exchange a telling of her story for some food and a place to sleep. But the villagers knew that only evil came from the dense forest, so they threw garbage at her, and chased her in large numbers. She was so overcome by these people that she stumbled and fell into a great blaze just outside the village. There she went up in a dark cloud of smoke. This is always how her story ended, though, with her death in a fiery place. It seems that the myth and the miss had this end in common.

We say that a statement P is a *paradox* if it is neither True nor False. While it is True that *I am lying* is a paradox in the context of the conversation *I am a Cretan, All Cretans are Liars, I am lying,* what follows is a statement that is a paradox without being contained in such a conversation. That statement is called the *Liar's Paradox* or *Satan's Statement.*

> *Satan's Statement*: This statement is False.

In the context of the conversation

All statements are False
This statement is False,

Satan's Statement is True. In the context of the conversation

The next statement is True
This statement is False

Satan's Statement is a Falsehood. These examples contrast the commonly held erroneous belief that Satan's Statement is False in every conversation that contains it. In Section 9.2 we will give a correct proof that shows that Satan's statement is neither True nor False, thus making it a True paradox.

Exercise 1.7.8 Using Q statements, prove that the statements 1 through 5 below are Falsehoods. Ignore simpler arguments if they should exist.

1. All statements are valid.

2. All men are liars.

3. Each statement is False.

4. All that is written is known.

5. All of Algebra in known.

6. Define: A perfect physicist is one who knows all of physics. Show that *There is a perfect physicist* is a Falsehood.

7. Define: A perfect mathematician is one who knows all of mathematics. Show that *There is a perfect mathematician* is a Falsehood.