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Humanity Awakening: Sensing Form and Creating Structures

The earliest ancestors of humans began to emerge about 7 million years ago, and the human species has existed for about 100,000 years. For most of this period humans were preoccupied with acquiring food and securing shelter. They lived in caves, made stone tools and weapons, and hunted and foraged for food. Some 10,000 years ago, an important transition occurred. By that time, the Ice Age that had started about 60,000 years ago was over. The ice sheets that covered Europe and Asia had receded to make room for forests, plains, and deserts. Seeing how plants sprang forth and grew in nature, these humans began to cultivate their own. Over time, they emerged from their caves, built their own primitive dwellings, and scratched an existence from the soil. Remains of some early huts show that they were constructed with skeletons of pine poles and bones and covered with animal skins. In time, villages formed, bread was baked, beer was brewed, and food was stored. The crafts of weaving, pottery, and carpentry developed, and basic goods were exchanged. Words expressed very concrete things and the constructions of language were simple. Copper was discovered, then bronze was made, and both were shaped into tools and weapons. Pottery and woven fabrics began to be decorated with geometrical patterns that reflected numerical relationships. Trading activity increased in radius and languages increased in range. With the continuing development of crafts, food production, and commerce, the need to express "how many?" and "how much?" in a spoken and also symbolic way became increasingly relevant so that a concept of number emerged.

Larger communities were a later development. One of them unearthed in the plains of Anatolia (in today's Turkey) consisted of a dense clustering of dwellings. Access to them was gained across their roofs. Mud-brick walls and timber frameworks enclosed rectangular spaces that touched against neighboring ones to form the town's walled perimeter. Interspersed between the houses were shrines that contained decorative images of animals and statuettes of deities. The settlements that began to develop along the world's great rivers at around 5000 B.C. profited from the arteries of communication and commerce that connected them. They became economically thriving, literate, urban communities. Those in Mesopotamia (in today's Iraq) and those on the upper and lower Nile in Egypt would become the cradles of Western civilization. The fertile plains near these rivers gave rise to a large-scale agriculture that required organization and storage facilities. Irrigation projects and efforts to control flooding drove technological advances. In this environment the practice of mathematics began. It was a mathematics of basic arithmetic. It had almost no symbolism and did not formulate general methods. It computed elementary areas and volumes and was strictly a tool for solving particular practical problems. Architecture developed in response to the requirements of commerce and agriculture, the need to honor the gods on whose good will success depended, and the rulers' insistence on a secure afterlife. Cities had storehouses, sprawling configurations of temples, and elaborate tomb complexes. Depending on their purpose, these structures were built with sun-baked bricks, stone columns and wooden beams, and massive stone slabs. A sense of aesthetics found expression in ornamented glazed tiles, decorative terra cotta elements, and monumental statues of rulers and deities.

Figure 1.1. An owl traced in the Chauvet Cave. Photo by HTO

Sensing Form and Conceiving Number

As humans began to be more acutely aware of their surroundings, watchful waiting turned into awareness, and fear and instinct became caution and reflection. Humans became curious about the physical world, began to observe similarities in things, and noticed regularities and sequences. They observed the points of light of the night sky, the line drawn by sea and sky at the horizon, the circular shapes of the moon, sun, and irises of eyes. They became conscious of the perpendicularity of the trunk of a tree against a flat stretch of land, the angles between the trunk and its branches, and the triangular silhouettes of pine trees against an illuminated sky. They wondered about the arcs of rainbows, shapes of raindrops, designs of leaves and blossoms, curves of horns, beaks and tusks, spirals of seashells, the oval form of an egg, and the shape of fish and starfish. They looked up at the vast reaches of sky and the moving clouds, sun, and moon within it, and became aware of the spatial expanse of their environment. The cave paintings executed about 30,000 years ago (Plate 2) demonstrate a wonderful ability of early humans to record what they observed. In fact, they show us that they had a heightened sense of their surroundings, a capacity to reflect about what they saw and experienced, and a sense for composition. Humans became aware of nature's organizational structures: the leaf configuration of a fern, the branching patterns of a bare tree in the winter, the arrangement of seeds in their housings, the intertwined form of a bird's nest, the hexagonal repetition of honeycombs, and the netted arrangement of a spider web. They began to gain a sense for basic shapes such as those depicted in Figures 1.1 and 1.2. When humans noticed a common aspect about a group of three trees, three

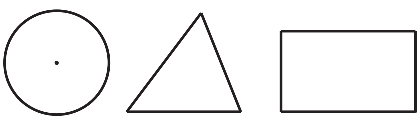

Figure 1.2

Figure 1.3. Bone discovered in the village Ishango, Democratic Republic of Congo, Africa. Photo from the Science Museum of Brussels

grazing zebras, three chirping birds, three mushrooms, and three roars of a lion, they began to gain a sense of number. The earliest records of the practice of counting are from 15,000 to 30,000 years old. The bone shown in Figure 1.3 is an example.

The important transition from the gathering and hunting for food to the cultivation of crops and the domestication of animals began about 10,000 years ago. Humans left their caves, began to erect primitive dwellings, and clustered in villages for protection. With ropes and sticks, they could trace out lines and circles. The living quarters that they laid out, whether circular yurts or rectangular huts, borrowed forms and structures from nature. They baked bread, built granaries, conceived of the wheel and axle, and made carts. Trade began and spoken language grew more sophisticated. When they encountered the need to count objects, estimate distances, and measure lengths, they used fingers, feet, and paces to do so. Elementary pottery, weaving, and carpentry developed. These early efforts of designing, building, and shaping cultivated a sense of planar and spatial relationships. The connection in our language between "stretch" and "straight" and between "linen" and "line" provides some evidence for the links between these early crafts and early geometry. Textiles were decorated with geometric designs of the sort depicted in Figure 1.4. They provide evidence of an increased awareness of order, pattern, symmetry, and proportion.

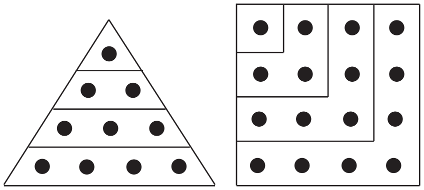

Figure 1.4

Powerful natural images and events such as threatening weather formations, angry thunderstorms, devastating floods, and volcanic eruptions were attributed to supernatural forces. Myths and primitive religions responded to a basic need to explain these phenomena. The realization that seasonal weather patterns and life cycles of plants follow a rhythm that is related to the variation in the height of the midday Sun in the sky gave importance to the tracking of celestial phenomena within the passage of time. Architectural structures organized these beginnings of astronomy. Stonehenge in southern England is an example. Started around 3000 B.C. and added to for a millennium and more, it featured huge stone slabs arranged in vertical

Figure 1.5. Stonhenge, southern England. Photo by Josep Renalias

pairs with horizontal slabs across the top. The slabs were arranged in a careful pattern that included a circular arrangement 100 feet across. The fact that this pattern was aligned with solar phenomena tells us that Stonehenge served as a prehistoric observatory that made the predictions of the summer and winter solstices (the longest and shortest days of the year) possible. Figure 1.5 shows some of the several dozen slabs that remain. The largest weigh as much as 20 tons (1 ton = 2000 pounds). Stonehenge also testifies to the incredible ability of its builders to move and shape huge megaliths (in Greek, *mega* = great, *liths* = stones) and to arrange them for intelligent purposes.

Rising Civilizations

Urban settlements became possible when agricultural surpluses allowed some people to assume specialized roles (priests, merchants, builders, and craftsmen) that were not directly tied to the production of food. Starting in the fifth millennium B.C., more advanced societies evolved along the banks of the great rivers Tigris and Euphrates, Nile, Indus, Huang, and Yangtze. These mighty rivers served as channels of communication and trade and brought raw materials from neighboring uplands. Extensive irrigation systems spread floodwaters to the low-lying, fertile plains, and made it possible to grow an abundance of crops. Large structures such as levees, dams, canals, reservoirs, and storage facilities rose to restrain and regulate the flow of water and to order agricultural production. Elaborate temples were built to appease the gods on whose good will the success of these efforts was thought to depend. Monumental burial complexes were designed and constructed to provide Egypt's ruling pharaohs and other important citizens with a smooth

Figure 1.6. The great pyramids of Giza, Egypt. Photo by Ricardo Liberato

and comfortable transition to and existence in the afterlife. Structures made of wood and earthen bricks did not survive the forces of time and erosion, but there are impressive remains of great stone structures. One of these, a tomb complex built near ancient Egypt's capital Memphis around 2500 B.C., consisted of a step pyramid with a burial chamber, a reconstruction of the pharaoh's palace, courtyards, altars, and a temple, all surrounded by a 33-foot wall laid out in a rectangle with a perimeter of one mile. The pyramid and sections of the wall survive. The greatest pyramids were built around the same time and not far away near today's city of Giza. They are shown in Figure 1.6. Their construction was an amazing feat. Without any machinery beyond ramps, levers, and strong ropes, and no metal harder than copper, thousands of Egyptian laborers cut massive blocks of stone, moved them to the site, and stacked them into precise position. The largest of these pyramids, the pyramid of the pharaoh Khufu (on the far right in Figure 1.6) rises to a height of 481 feet from a square base with 755-foot sides. The tip of the pyramid is almost exactly over the center of its square base. The pyramid was built with 2.3 million blocks weighing from 2.5 to 20 tons. The lowest layer of blocks rests on the limestone bedrock of the area and supports about 6.5 million tons. The building materials used most frequently by the Egyptians in their monuments are limestone and sandstone. Both are formed by sedimentation. Limestone consists of calcium carbonate. Sandstone is usually harder. It consists of sand, commonly quartz fragments, cemented together by various substances. The structural qualities of limestone and sandstone depend on the particular deposit, but both can be carved and cut without great difficulty. The sizable burial chambers deep in the interior of the pyramids were constructed with granite so that they could resist the enormous loads of the blocks above them. Granite, a rock formed by the crystallization and solidification of molten lava from the hot core of the Earth, is much harder and stronger than limestone and sandstone.

Large constructions such as the pyramids necessitated organizational capacity, enhanced technological expertise, and record keeping that needed to be promoted by growing central administrations. All this required a richer symbolic representation of language and an enhanced development of mathematics. The mathematics of these river civilizations originated as a practical science that facilitated the computation of the calendar, the surveying of lands, the coordination of public projects, the organization of the cycle of crops and harvests, and the collection of taxes. In the hands of a class of administrator priests, the practice of arithmetic and measuring and the study of shapes and patterns evolved into the beginnings of algebra and geometry.

The most advanced of the great river civilizations were the people of the "fertile crescent" of Mesopotamia (in Greek, *meso* = between, *potamia* = rivers) between the Tigris and Euphrates (of today's Iraq). They introduced a positional numerical notation for integers and fractions based on 60 and used wedge-shaped symbols to express $1, 60, 60^2 = 3600$, and $60^{-1} = \frac{1}{60}$, and $60^{-2} = \frac{1}{60^2}$. Traces of this system survive to this day in the division of the hour into 60 minutes, a minute into 60 seconds, and a circle into $6 \times 60 = 360$ degrees. The mathematicians of the Babylonian dynasty that followed around 2000 B.C. solved linear, quadratic, and even some cubic equations. In particular, they knew that the solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

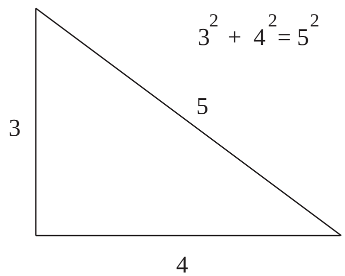

Figure 1.7

The Babylonians knew the theorem now called the Pythagorean Theorem. For any right triangle with side lengths a, b, and c, where c is the length of the hypotenuse, the equality $a^2 + b^2 = c^2$ holds. A clay tablet cast between 1900 and 1600 B.C. gives testimony to the achievements of the Babylonians. Now referred to as Tablet 322 of the Plimpton collection, it lists triples, in other words threesomes, of whole numbers a, b, and c, with the property that $a^2 + b^2 = c^2$. As such triples represent the sides of right triangles, they provide specific instances of the Pythagorean Theorem. The triple of numbers $a = 3$, $b = 4$, and $c = 5$ is an example (see Figure 1.7). Another is $a = 5$, $b = 12$, and $c = 13$. Some of the very large triples listed on the tablet strongly suggest that the Babylonians had a recipe for generating such threesomes of numbers. The Babylonians had formulas for the areas of standard planar figures and volumes of some simple solids. They also analyzed the positions of the heavenly bodies and developed a computational astronomy with which they predicted solar and lunar eclipses.

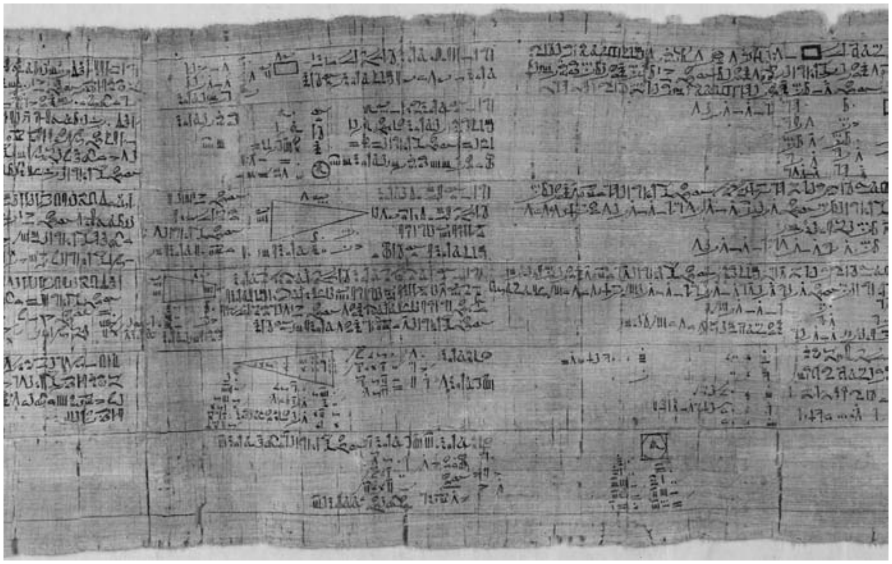

Figure 1.8. The Rhind papyrus, British Museum, London. Photo © Trustees of the British Museum

The Egyptian Rhind papyrus is a long scroll that dates from around 1600 B.C. A portion of it is shown in Figure 1.8. The Rhind papyrus (papyrus is a plant product that served as the paper of that time) gets its name from the Scotsman A. Henry Rhind, who bought it in Egypt in the nineteenth century. Its introduction promotes it to be "a thorough study of all things, insight into all that exists, knowledge of all obscure secrets." But it is simply a handbook of practical mathematical exercises of the sort that arose in commercial and administrative transactions. The 85 mathematical problems it presents and the elaborate theory of fractions on which many of the solutions rely give a good idea of the state of Egyptian mathematics at the time. It also provides the approximation $(\frac{16}{9})^2 = \frac{256}{81} \approx 3.1605$ for the ratio $\frac{c}{d}$ of the circumference c of a circle to its diameter d. (The symbol $\approx$ means "is approximately equal to.") Today, this ratio is designated by π (and better approximated by $\pi \approx 3.1416$). The papyrus also contains some practical advice: "catch the vermin and the mice, extinguish noxious weeds; pray to the God Ra for heat, wind, and high water."

As physical witness to the growing sophistication of thought, architecture advanced as well. The walled city of Babylon with its imposing temples and soaring towers was well known for its architectural splendors. The Greek historian Herodotos traveled widely in the Mediterranean region in the fifth century B.C. and recorded what he saw. About Babylon he wrote that "in magnificence, there is no other city that approaches it." Unfortunately, very little remains. One of the main gates to the inner city was built early in the sixth century B.C. and dedicated to the goddess Ishtar. Its central passage featured a high semicircular arch, walls covered with blue glazed tiles, and

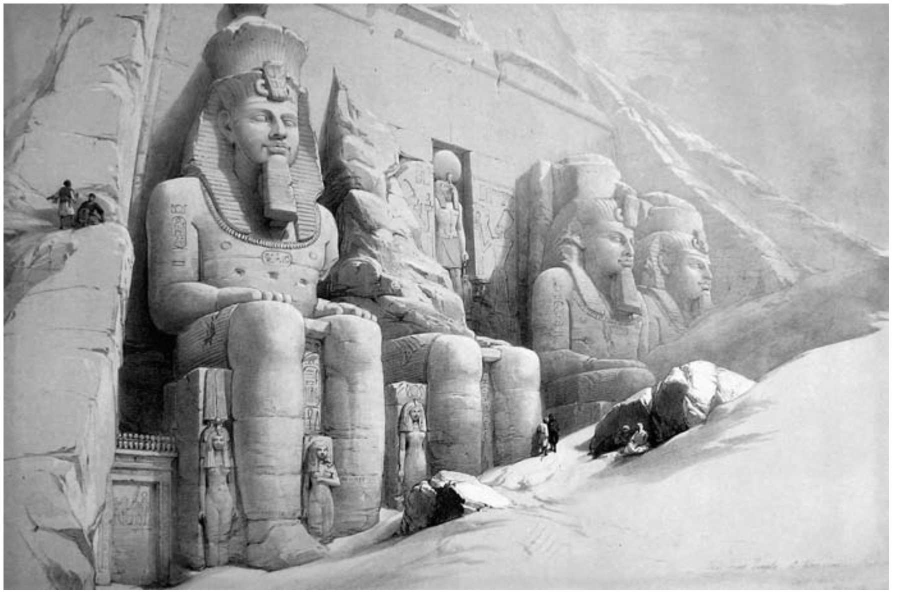

Figure 1.9. The facade of the Great Temple at Abu Simbel, Egypt. Lithograph by Louis Haghe, 1842–1849, from a painting by David Roberts, 1838–1839

doors and roofs of cedar. The middle section of the Ishtar Gate (only a small part of the ancient gate complex) has been reconstructed from materials excavated from the original site. A full 47 feet tall, it stands in the Pergamon Museum in Berlin. A similar reconstruction exists near the original site in Baghdad, Iraq.

Egypt's buildings withstood the challenge of time better than those of Babylon. The rise of the sun god Amun in the middle of the second millennium to the position of primary state god inspired the building of stone temple complexes that were grander and more elaborate than before. An impressive example is the great temple of Amun, built near today's city of Karnak from the middle of the sixteenth century to the middle of the fourteenth century B.C. A succession of pharaohs ordered the construction of monumental entrance gates, obelisks, colossal statues, and grand ceremonial halls. The largest of these halls was built in the reign of the powerful Rameses II. It measured 165 feet by 330 feet and was tightly packed with tall and massive columns that supported the heavy stone slabs of its roof. Plate 3 tells us that enough of the structure is preserved to give today's visitor a sense of its former size and grandeur.

The great temple of Abu Simbel that the same Rameses II had built in his honor in the fourteenth century B.C. in southern Egypt is another example. Figure 1.9 shows the temple carved into a sandstone cliff on the banks of the Nile, its facade dominated by statues of the great man himself in ceremonial pose. These massive statues are 67 feet high and weigh 1200 tons. Smaller figures immortalize the queen and lesser dignitaries. The depiction confirms that sandstone can be brittle. The ancient temple has a modern history. It was rediscovered around 1815 as it emerged from the shifting sands that had buried most of it. When the temple was threatened in the 1960s by the rising waters of the artificial lake created by the dam being constructed near Aswan in Upper Egypt, the United Nations organized a monumental effort to save it. The temple's facade and its elaborate interior (reaching 200 feet into the rock) were cut into sections weighing many tons each, moved carefully block by block, and reassembled, exactly as they had been, on higher ground a few hundred feet away.

The very brief survey of early mathematics and architecture presented above is a story of the developing ability of humans to recognize shape, pattern, and structure in their surroundings and their later efforts to impose shape, pattern, and structure on the activities that impacted their existence. However, the mathematics and architecture of ancient civilizations were driven by different forces. Mathematics arose primarily in response to the practical need to organize and order production, commerce, and their underlying infrastructures. The primary purpose of architecture on the other hand was to give powerful visual expression to the importance and grandeur of the rulers and their gods.

Problems and Discussions

The problems below are related to matters discussed in the text. They provide an opportunity for thinking about and maneuvering through some basic mathematics.

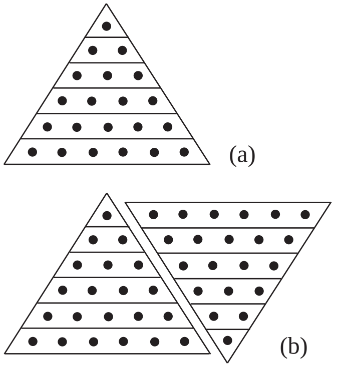

Figure 1.10

Problem 1. Consider the diagram in Figure 1.10a. Count the dots from the top down to get $1 + 2 + 3 + 4 + 5 + 6$. Turn to the diagram in Figure 1.10b and notice that

$$2(1 + 2 + 3 + 4 + 5 + 6) =$$
$$(1 + 6) + (2 + 5) + (3 + 4) + (4 + 3) + (5 + 2) + (6 + 1) = 6 \cdot 7.$$

So $1 + 2 + 3 + 4 + 5 + 6 = \frac{1}{2}(6 \cdot 7)$. Use the same strategy to show, for any positive integer n, that $1 + 2 + \cdots + (n - 1) + n = \frac{1}{2}n(n + 1)$.

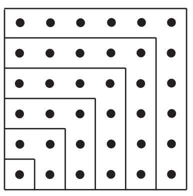

Figure 1.11

Problem 2. It seems to be the case that any sum of consecutive odd numbers starting with 1 is a square. For example, $1 + 3 = 2^2$, $1 + 3 + 5 = 3^2$, $1 + 3 + 5 + 7 = 4^2$, and $1 + 3 + 5 + 7 + 9 = 5^2$. The diagram of Figure 1.11 shows that $1 + 3 + 5 + 7 + 9 + 11 = 6^2$. Let n be any positive integer. Consider the term $2k - 1$. Plugging in $k = 1, k = 2, \ldots, k = n$, provides a list $1, 3, 5, \ldots, 2n - 1$ of the first n odd integers. Show that their sum $1 + 3 + 5 + \cdots + (2n - 1)$ is equal to n^2. [Hint: Try the strategy used in the solution of Problem 1.]

Problem 3. Take any two positive integers m and n with $n > m$. Now form the positive integers a, b, and c, by setting $a = n^2 - m^2$, $b = 2nm$, and $c = n^2 + m^2$. This is a recipe for generating numbers a, b, and c that satisfy $a^2 + b^2 = c^2$. Taking $m = 1$ and $n = 2$ gives us $a = 4 - 1 = 3$, $b = 2 \cdot 2 = 4$, and $c = 4 + 1 = 5$. Because $3^2 + 4^2 = 5^2$, the recipe works in this case. Taking $m = 2$ and $n = 3$ gives us $a = 9 - 4 = 5$, $b = 2 \cdot 6 = 12$, and $c = 9 + 4 = 13$. The fact that $5^2 + 12^2 = 169 = 13^2$ tells us that the recipe works in this case as well. Verify the recipe in general, and then use it to list five additional triples (a, b, c) of positive integers that satisfy $a^2 + b^2 = c^2$.

Problem 4. Study the three diagrams of Figure 1.12. A right triangle with sides a, b, and c is given. The diagram at the center is a configuration of two squares arranged in such a way that the four triangular regions that they determine are each equal to the given triangle. Make use of the diagrams to write a paragraph that verifies the Pythagorean Theorem.

Problem 5. The Pythagorean Theorem was also known to the Chinese. The essential information of the old Chinese diagram depicted in Figure 1.13a is captured by Figure 1.13b. It depicts four identical right triangles (each with sides of lengths a, b, and c) arranged inside a square. Determine the size of the inner square and use the diagram to verify the Pythagorean Theorem.

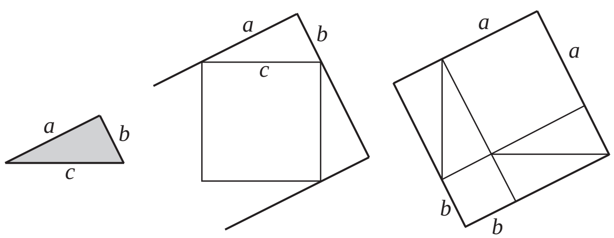

Figure 1.12

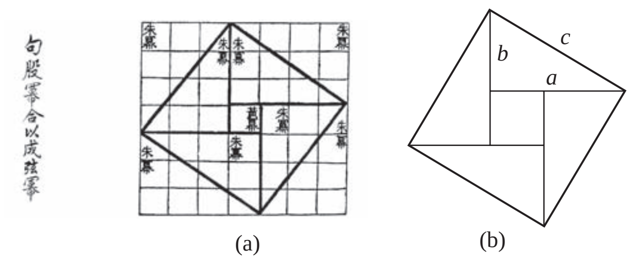

Figure 1.13. (a) is the Chinese Pythagorean Theorem from Joseph Needham, *Science and Civilization in China: Vol. 3, Mathematics and the Sciences of the Heavens and Earth*, Cave Books Ltd., Taipei, 1986, p. 22

Discussion 1.1. Solving the Quadratic Equation. The solutions of the equation $ax^2 + bx + c = 0$ (with $a \neq 0$) are given by the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Today's verification of this formula is a consequence of a procedure known as *completing the square*. The procedure consists of several algebraic steps that are illustrated below in the case of the quadratic polynomial $6x^2+28x-80$.

First factor out the coefficient of the x^2 term. So $6x^2+28x-80=6(x^2+\frac{28}{6}x-\frac{80}{6})$. Focus on $\frac{28}{6}x=\frac{14}{3}x$, divide $\frac{14}{3}$ by 2 to get $\frac{14}{6}=\frac{7}{3}$, and square this to get $\frac{49}{9}$. Now rewrite $6(x^2+\frac{14}{3}x-\frac{80}{6})$ as $6(x^2+\frac{14}{3}x+\frac{49}{9}-\frac{49}{9}-\frac{80}{6})$. Regroup to get $6[(x^2+\frac{14}{3}x+\frac{49}{9})-\frac{49}{9}-\frac{80}{6}]$. Because $(x^2+\frac{14}{3}x+\frac{49}{9})=(x+\frac{7}{3})^2$, you now have

$$6x^2+28x-80=6\left(x^2+\tfrac{28}{6}x-\tfrac{80}{6}\right)=6\left[\left(x+\tfrac{7}{3}\right)^2-\tfrac{49}{9}-\tfrac{80}{6}\right]=6\left[\left(x+\tfrac{7}{3}\right)^2-\tfrac{169}{9}\right].$$

Having rewritten $6x^2+28x-80$ as $6[(x+\frac{7}{3})^2-\frac{169}{9}]$, you have completed the square for the quadratic polynomial $6x^2+28x-80$. Notice that it is now easy to solve $6x^2+28x-80=0$ for x. Divide $6[(x+\frac{7}{3})^2-\frac{169}{9}]=0$ by 6 to get $(x+\frac{7}{3})^2-\frac{169}{9}=0$. So $(x+\frac{7}{3})^2=\frac{169}{9}$, and hence $x+\frac{7}{3}=\pm\sqrt{\frac{169}{9}}=\pm\frac{13}{3}$. Therefore, $x=-\frac{7}{3}\pm\frac{13}{3}$. So $x=2$ or $x=-\frac{20}{3}$.

Problem 6. Repeat the steps above to complete the square for $4x^2-8x-12$. Use the result to solve $4x^2-8x-12=0$ for x.

Problem 7. Complete the square for the polynomial $-5x^2+3x+4$. Then use the result to solve $-5x^2+3x+4=0$ for x. Try the same thing for $-5x^2+3x-4$. [Note: The solution of the equation $-5x^2+3x-4$ requires square roots of negative numbers. Such *complex numbers* will not be considered in this text and we will regard such equations to have no solutions.]

Problem 8. Verify by completing the square that the solutions of $ax^2+bx+c=0$ (with $a\neq 0$) are given by the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. What happens when $a=0$?

Problem 9. Let x and d be any two positive numbers. Study the diagrams in Figure 1.14 and write a paragraph that discusses their connection with the completing the square procedure.

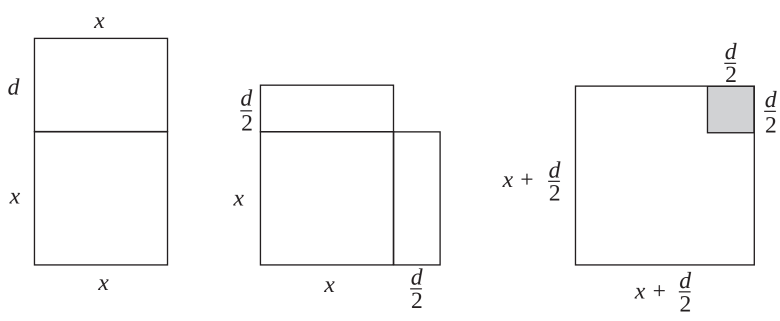

Figure 1.14