Chapter One

Introduction

The aim of the present book is to provide a rigorous proof of a conjecture of Deser and Schwimmer, originally formulated in [21]. This work is a continuation of the previous two papers of the author [2, 3], which established the conjecture in a special case and introduced tools that laid the groundwork for the resolution of the full conjecture. The present volume is complemented by the two papers [4, 6], in which certain techical lemmas that are asserted and used here are proven. Thus this book, together with the papers [2, 3] and [4, 6], provides a confirmation of the Deser-Schwimmer conjecture.

In this chapter, we first provide a formulation of the conjecture and present some applications; we also discuss its close relation with certain questions in index theory and in Cauchy-Riemann (CR) and Kähler geometry. Then, we broadly outline the strategy of the proof and very briefly present the tasks that are undertaken in each of the subsequent chapters. Each chapter contains its own separate introduction, which provides a more detailed description of the ideas and proofs there.

1.0.1 Formulation of the problem

To formulate the conjecture of Deser and Schwimmer we first recall a classical notion from Riemannian geometry, that of a scalar Riemannian invariant: In brief, given a Riemannian manifold (M, g), scalar Riemannian invariants are *intrinsic, scalar-valued* functions of the metric g. More precisely:

DEFINITION 1.1. *Let $L(g)$ be a formal polynomial expression in the (formal) variables $\partial^{(k)}_{r_1\dots r_k} g_{ij}, k \geq 0$ and $(\det g)^{-1}$ (here the indices $r_1, \dots, r_k, i, j$ take values $1, \dots, n$.) Given any coordinate neighborhood $U \subset \mathbb{R}^n$ and any Riemannian metric g expressed in the form $g_{ij}dx^i dx^j$ in terms of the coordinates $\{x^1, \dots, x^n\} \in U$, let L^U_g stand for the function that arises by substituting the values $\partial^{(k)}_{r_1\dots r_k} g_{ij}$, $(\det g)^{-1}$ into the formal expression $L(g)$.*

We say that $L(g)$ is a Riemannian invariant of weight K if

1. *Given any two Riemannian metrics g, g' defined over neighborhoods $U, U' \subset \mathbb{R}^n$ that are isometric via a map $\Phi : U \to U'$, then $L^U_g(x) = L^{U'}_{g'}(\Phi(x))$ for every $x \in U$. (This property is called the intrinsicness property of $L(g)$.)*

2. *Given a Riemannian metric g defined over $U \subset \mathbb{R}^n$ and $t \in \mathbb{R}_+$, then $L^U_{g'}(x) = t^K L^U_g(x)$ for $g' = t^2 g$, for some given $K \in \mathbb{Z}$. (We then say that $L(g)$ has weight K.)*

In view of the first property, a Riemannian invariant $L(g)$ assigns a well-defined,[1] scalar-valued function to any Riemannian manifold (M, g).

We next recall a classical theorem that essentially goes back to Weyl [38]; it asserts that any scalar Riemannian invariant can be expressed in terms of complete contractions of covariant derivatives of the curvature tensor. To state this result precisely, we recall some basic facts from Riemannian geometry:

Given a Riemannian metric g defined over a manifold M, consider the curvature tensor R_{ijkl} and its covariant derivatives $\nabla^{(m)}_{r_1 \dots r_m} R_{ijkl}$ (these are thought of as $(0, m+4)$ tensors.) This gives us a list of *intrinsic tensors* defined over M.

A natural way to form intrinsic *scalars* out of this list of intrinsic *tensors* is by taking tensor products and then contracting indices using the metric[2] g^{ab}: First we take a (finite) number of tensor products, say,

$$\nabla^{(m_1)}_{r_1 \dots r_{m_1}} R_{i_1 j_1 k_1 l_1} \otimes \cdots \otimes \nabla^{(m_s)}_{r_1 \dots r_{m_s}} R_{i_s j_s k_s l_s}, \tag{1.1}$$

thus obtaining a tensor of rank $(m_1 + 4) + \cdots + (m_s + 4)$. Then, we repeatedly pick out pairs of indices in the above expression and contract them against each other using the metric g^{ab}. In the end, we obtain a scalar. We denote such *complete contractions* by $C(g)$.[3]

It follows from the intrinsicness of the metric tensor, the curvature tensor and its covariant derivatives [22], that any such complete contraction is a scalar Riemannian invariant of weight $-[(m_1 + 2) \cdots + (m_s + 2)]$. Thus, taking linear combinations of complete contractions of a given weight w, we can construct scalar Riemannian invariants of weight w. We will denote such linear combinations by $\sum_{l \in L} a_l C^l(g)$ (here L is the index set of the complete contractions $C^l(g)$, and the a_l, $l \in L$ are their coefficients.) *Remark:* In the above each l is a *label* that serves to distinguish the different complete contractions $C^l(g), l \in L$. L is the set of all these labels.

Now, a classical result in Riemannian geometry (essentially due to Weyl, [38]) is that the converse is also true: Given any Riemannian invariant $F(g)$ there exists a (nonunique) linear combination of complete contractions in the form (1.1), $\sum_{l \in L} a_l C^l(g)$ so that for every manifold (M, g) the value of $F(g)$ is equal to the value of the linear combination $\sum_{l \in L} a_l C^l(g)$. Thus from now on we *identify* Riemannian invariants with linear combinations of the form

$$F(g) = \sum_{l \in L} a_l C^l(g), \tag{1.2}$$

[1] Meaning coordinate-independent.

[2] This is a $(2, 0)$-tensor.

[3] A rigorous, if somewhat abstract, definition of a complete contraction appears in the introduction of [2].

where each $C^l(g)$ is a complete contraction (with respect to the metric g) in the form

$$\mathrm{contr}(\nabla^{(m_1)}R \otimes \cdots \otimes \nabla^{(m_a)}R). \tag{1.3}$$

(For brevity we do not write out the indices of the tensors involved.)

We remark that a complete contraction is determined by the *pattern* according to which different indices contract against each other. Thus, for example, the complete contraction $R_{abcd} \otimes R^{abcd}$ is different from the complete contraction $R^a{}_{bad} \otimes R_s{}^{bsd}$. The notation (1.3) of course does not encode this pattern of which index is contracting against which.

The Deser-Schwimmer conjecture: The conjecture deals with conformally invariant *integrals* of Riemannian scalars.

DEFINITION 1.2. *Consider a Riemannian invariant $P(g)$ of weight $-n$ (n even.) We say that the integral $\int_{M^n} P(g)dV_g$ is a global conformal invariant if the value of $\int_{M^n} P(g)dV_g$ remains invariant under conformal rescalings of the metric g.*

In other words, $\int_{M^n} P(g)dV_g$ is a global conformal invariant if for any $\phi \in C^\infty(M^n)$ we have $\int_{M^n} P(e^{2\phi}g)dV_{e^{2\phi}g} = \int_{M^n} P(g)dV_g$.

To state the Deser-Schwimmer conjecture, we recall the following:

DEFINITION 1.3. *A local conformal invariant of weight $-n$ is a Riemannian invariant $W(g)$ for which $W(e^{2\phi}g) = e^{-n\phi}W(g)$ for every Riemannian metric g and every function $\phi \in C^\infty(M^n)$.*

Furthermore, a Riemannian vector field $T^i(g)$ is a linear combination $T^i(g) = \sum_{q\in Q} a_q C^{q,i}(g)$, where each $C^{q,i}(g)$ is a partial contraction (with one free index i) in the form

$$C^{q,i}(g) = \mathrm{pcontr}(\nabla^{(m_1)}R \otimes \cdots \otimes \nabla^{(m_a)}R), \tag{1.4}$$

with $\sum_{t=1}^{a}(m_t + 2) = n - 1$.[4] (Notice that for each such vector field, the divergence $\mathrm{div}_i T^i(g)$ is a Riemannian invariant of weight $-n$.) Finally, we recall that $\mathrm{Pfaff}(R_{ijkl})$ stands for the Pfaffian of the curvature tensor.[5]

The Deser-Schwimmer conjecture [21] asserts the following:

CONJECTURE 1.4. *Let $P(g)$ be a Riemannian invariant of weight $-n$ such that the integral $\int_{M^n} P(g)dV_g$ is a global conformal invariant. Then there exists a local conformal invariant $W(g)$, a Riemannian vector field $T^i(g)$, and a constant* (const) *so that $P(g)$ can be expressed in the form*

$$P(g) = W(g) + \mathrm{div}_i T^i(g) + (\mathrm{const}) \cdot \mathrm{Pfaff}(R_{ijkl}). \tag{1.5}$$

[4] In this notation, q is a label that serves to distinguish the different partial contractions. The index i denotes the free index in each of the partial contractions; in other words, i denotes the one index in each partial contraction (1.4) that does not contract against any other index.

[5] Recall the Chern-Gauss-Bonnet theorem, which says that for any compact orientable Riemannian n-manifold (M^n, g) we must have $\int_{M^n} \mathrm{Pfaff}(R_{ijkl})dV_{g^n} = \frac{2^n \pi^{\frac{n}{2}} (\frac{n}{2}-1)!}{2(n-1)!}\chi(M^n)$.

We recall the theorem we proved in [2] and [3]:

THEOREM 1.5. *[A] Let $\int_{M^n} P(g)dV_g$ be a global conformal invariant, where $P(g)$ is in the special form*

$$P(g) = \sum_{l \in L} a_l \operatorname{contr}^l(R_{i_1 j_1 k_1 l_1} \otimes \cdots \otimes R_{i_{\frac{n}{2}} j_{\frac{n}{2}} k_{\frac{n}{2}} l_{\frac{n}{2}}}) \tag{1.6}$$

(i.e., each of the complete contractions above has $\frac{n}{2}$ undifferentiated factors R_{ijkl}.) Then $P(g)$ can be expressed in the form

$$P(g) = W(g) + (\text{const}) \cdot \operatorname{Pfaff}(R_{ijkl}).$$

In the present book (combined with [4, 6]), we prove the Deser-Schwimmer conjecture in its entirety.

THEOREM 1.6. *Conjecture 1.4 is true.*

1.0.2 Applications of the result

One main challenge that motivated the study of this problem was to understand the local algebraic structure of Branson's Q-curvature. This mysterious quantity was introduced by Thomas Branson in [13];[6] it is a Riemannian scalar defined for each even dimension n. Its integral is a global conformal invariant; thus, theorem 1.6 directly applies to yield a decomposition for the Q-curvature in each even dimension. Its particularly simple transformation law under conformal rescalings makes it a useful tool in geometric analysis, particularly in dimension four, but also in higher ones. Moreover, the decomposition of Q-curvature guaranteed by Theorem 1.6 was used by Chang, Qing, and Yang [20] in relating it to a notion originally introduced in string theory, that of the renormalized volume of conformally compact Einstein manifolds.

We also wish to mention the recent work of A. Juhl [34, 35], where he obtains new remarkable insight on the significance of Q-curvature from an entirely fresh point of view. While our Theorem 1.6 applies to *all* global conformal invariants, and thus in particular implies a decomposition of the form (1.5) for the Q-curvature, it is very difficult to apply to find the *specific* local conformal invariant $W(g)$ and the divergence $\operatorname{div}_i T^i(g)$ for Q-curvature. It is worth wondering whether the recursive understanding of Q-curvature obtained by Juhl[7] could give further insight into the nature of the decomposition (1.5) for the Q-curvature.

Q-curvature. The Q-curvature is a scalar Riemannian invariant, defined for each even dimension n. It was introduced by T. Branson in [13], through

[6] We refer the reader to [15] for a detailed account of the development and motivation behind Q-curvature.

[7] Through recursive local formulae in all the even dimensions.

considerations on functional determinants. It appears naturaly in many contexts of Riemannian geometry: E.g., in the heat kernel asymptotics of the conformal Laplacian [36] and in the scattering theory on Poincaré-Einstein manifolds developed by Graham and Zworski [30]. It also provides a clearer understanding of the renormalized volume of such manifolds in [20]. In conformal geometry in particular, by virtue of the Deser-Schwimmer conjecture, Q-curvature can be thought of as a generalization of the Gauss curvature to higher dimensions. (An analog of Q-curvature exists in CR geometry, introduced in [26]; the understanding of this object is more elusive at present.)

In dimension n, Q-curvature $Q^n(g)$ is a scalar Riemannian invariant of weight $-n$. Under conformal rescalings, it transforms via the GJMS operators P^n_g [28], which are nth order analogues of the conformal Laplacian,[8]

$$Q^n(e^{2\phi}g) = e^{-n\phi}[Q^n(g) + 2P^n_g(\phi)]. \tag{1.7}$$

One of the main properties of Q-curvature is that the integral $\int_{M^n} Q^n(g)dV_g$ over closed manifolds (M^n, g) is a global conformal invariant.

Thus, our Theorem 1.6 implies immediately that

$$Q^n(g) = W(g) + \mathrm{div}_i\, T^i(g) + C \cdot \mathrm{Pfaff}(R_{ijkl}), \tag{1.8}$$

where $W(g)$ is a local conformal invariant and $\mathrm{div}_i\, T^i(g)$ is a divergence of a vector field. It can be seen that in this case $C \neq 0$.

Therefore, (1.7) and (1.8) allow one to view Q-curvature as a higher order generalization of the Gauss curvature in dimension 2.

The strongest results to date on the Q-curvature have been obtained in dimension four, where the value of the global conformal invariant $\int_{M^4} Q^4(g)dV_g$ has been proven to relate to the topology of the underlying manifold: In [19], Chang-Gursky and Yang showed that if a closed 4-manifold (M^4, g) satisfies $\int_{M^4} \mathrm{Scal}(g) > 0$ and[9] $\int_{M^4} Q(g)dV_g > 0$, then there exists a $\phi \in \mathcal{C}^\infty(M^4)$ such that the Ricci curvature of the conformaly rescaled metric $e^{2\phi}g$ is positive, $\mathrm{Ric}(e^{2\phi}g) > 0$; in particular, the fundamental group of M^4 must be finite.

It is unclear yet whether these strong results can be generalized to higher dimensions. One possible avenue for a better understanding of the Q-curvature in high dimensions via recursive formulas has been recently explored by Juhl [34, 35]. His entirely fresh perspective on this question uses the principle of holography, where one views a Riemannian manifold (M^n, g) as a boundary at infinity of a Poincaré-Einstein metric.

Renormalized volumes and areas. An important application of Q-curvature is in understanding the *renormalizations* of volume and area in the

[8] The operators P^n_g have leading order symbol $\Delta_g^{\frac{n}{2}}$ and are themselves conformally covariant; i.e., they satisfy $P^n_{e^{2\phi}g}(f) = e^{-n\phi}P^n_g(f)$ for all smooth functions ϕ, f.

[9] $\mathrm{Scal}(g)$ is the scalar curvature of the metric g.

context of Poincaré-Einstein manifolds [31]. These are manifolds with boundary (X^{n+1}, g^{n+1}), that are *conformally compact* and *Einstein*. Conformally compact means that given any defining function[10] x for ∂X^{n+1}, the metric $x^2 g^{n+1}$ extends smoothly to ∂X^{n+1}; Thus the boundary $M^n := \partial X^{n+1}$ inherits a conformal class of metrics $[g]$ which arise by restricting $x^2 g^{n+1}$ to M^n, for all choices of defining function x. The Einstein constant is assumed to be $Ric(g^{n+1}) = -n g^{n+1}$.

The renormalized volume of such manifolds arises as the *finite part* in the expansion of volume with respect to certain special defining functions[11] x. Calculating the volume of $\{x \geq \epsilon\}$ and letting $\epsilon \to 0$, Graham-Witten calculated that for n odd,

$$\mathrm{Vol}(\{x \geq \epsilon\}) \sim C_0 \cdot \epsilon^{-n-1} + C_2 \epsilon^{-n+1} + \cdots + C_{n-2}\epsilon^{-1} + V + O(\epsilon). \qquad (1.9)$$

Remarkably, the number V turns out to be *independent* of the choice of the special defining function x;[12] it is the renormalized volume of (X^{n+1}, g^{n+1}). An analogous picture emerges in the renormalization of area for *complete minimal submanifolds* in (X^{n+1}, g^{n+1}). Consider smooth submanifolds $Y^{k+1} \subset X^{n+1}$ with a smooth boundary in $\partial Y^{k+1} \subset \partial X^{n+1}$. Then one can consider the expansion of the volumes of these submanifolds. When k is odd, Graham-Witten showed

$$\mathrm{Vol}(Y^{k+1} \bigcap \{x \geq \epsilon\}) \sim C_0 \cdot \epsilon^{-k-1} + C_2 \epsilon^{-k+1} + \cdots + C_{k-2}\epsilon^{-1} + A + O(\epsilon). \quad (1.10)$$

Again, the number A turns out to be independent of the choice of the defining function; A is the *renormalized area* of Y^{k+1}.

Now, Chang, Qing, and Yang [20] obtained a concrete understanding of the renormalized volume via the Q-curvature, based on its decomposition (1.8). They showed that

$$V = \int_{X^{n+1}} W(g^{n+1}) dV_{g^{n+1}} + C \cdot \chi(X^{n+1}).$$

In the above the integral $\int_{X^{n+1}} W(g^{n+1}) dV_{g^{n+1}}$ is *convergent*; $\chi(X^{n+1})$ is the Euler number[13] of X^{n+1}. An analogous formula is known only for submanifolds in dimension $k+1 = 2$ (for minimal surfaces); this was obtained in [7] and the proof uses the classical Gauss-Bonnet theorem on surfaces with boundary.

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[10] I.e., $x \in \mathcal{C}^\infty(\overline{X^{n+1}})$, $x = 0$ on ∂X^{n+1} and $dx \neq 0$ on ∂X^{n+1}.

[11] See [31] for details. These special defining functions are (up to high order in their Taylor expansion) in one to one correspondence with the metrics in the conformal class $[g]$.

[12] Thus by virtue of the previous footnote, it is independent of the choice of metric in the conformal class $[g]$.

[13] A related result was obtained by Albin in [1], where he obtains the dependence of V on $\chi(X^{n+1})$, but he also has renormalized integrals on the RHS, rather than the convergent $\int_{X^{n+1}} W(g^{n+1}) dV_{g^{n+1}}$.

There is no well-defined notion of renormalized volume (or area) when n (or k) is even; in that case the volume expansion contains a logarithmic term:

$$\mathrm{Vol}(\{x \geq \epsilon\}) \sim C_0 \cdot \epsilon^{-n-1} + C_2\epsilon^{-n+1} + \cdots + C_{n-2}\epsilon^{-2} + L \cdot \log \epsilon + V + O(\epsilon), \tag{1.11}$$

$$\mathrm{Vol}(Y^{k+1} \bigcap \{x \geq \epsilon\}) \sim C_0 \cdot \epsilon^{-k-1} + C_2\epsilon^{-k+1} + \cdots + C_{k-2}\epsilon^{-1} + K \cdot \log \epsilon + A + O(\epsilon.) \tag{1.12}$$

Now, V, A are *not* independent of the choice of the defining function x. It is the terms L, K that enjoy this property. In fact, the terms L, K encode the dependence of V, A on the choice of metric in the conformal class $[g]$: If we let $\phi \in C^\infty(M^n)$ and consider the one-parameter family of metrics $g_t := e^{2t\phi}g$ in $[g]$ and by V_t the terms in the volume expansion (1.11) that correspond to the special defining function x_t of g_t then:

$$\frac{dV_t}{dt}|_{t=0} = \int_{M^n} \phi \cdot U(g) dV_g, \tag{1.13}$$

where $U(g)$ is a Riemannian invariant of weight $-n$. Then $L = \int_{M^n} U(g)dV_g$. $U(g)$ is a *conformal anomaly* in that it represents the failure of V_t to be defined independently of the choice of metric in the conformal class $[g]$. This is in fact a general feature of anomalies in quantum field theories: The anomalies themselves are invariant under the symmetry they break. Thus in this context the anomalies give rise to the global conformal invariant $L = \int_{M^n} U(g)dV_g$.

In the case of renormalized volume, Graham-Zworski showed that $L = C \cdot \int_{M^n} Q^n(g)dV_g$; in the case of area renormalization for submanifolds, no such formula is known except when $k = 2$.

The analog of volume renormalization in complex geometry was discovered by Hirachi [33]; given any strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, one can consider a volume element $B(z, \overline{z})dV$, where $B(z, \overline{w})$ is the Bergman kernel[14] for Ω, with respect to the standard volume element dV for Ω. The volume expansion of this element is analogous to the Poincaré-Einstein setting; given a special defining function x for $\partial\Omega$ we find

$$\mathrm{Vol}(\{x \geq \epsilon\}) = \sum_{j=0}^{n-1} C_j \epsilon^{j-n} + L(\partial\Omega) \log \epsilon + O(1.) \tag{1.14}$$

Here $L(\partial\Omega)$ is a term that depends only on the CR structure of $\partial\Omega$. (The above volume expansion corresponds to the classical asymptotics of the Szegö kernel of Ω; see the next section.) In fact, $L(\partial\Omega)$ can be expressed as the integral of a local invariant of the CR structure on $\partial\Omega$, $L(\partial\Omega) = \int_{\partial\Omega} \psi_\theta \theta \wedge (d\theta)^{n-1}$; the integrand ψ_θ appears naturally in the asymptotic expansion of the Szegö kernel, as we discuss below. Boutet de Monvel showed in [12] that $L = 0$; however, the local algebraic structure of ψ_θ is still not understood.

[14]Thus $B(z, \overline{z})$ is Bergman kernel evaluated on the diagonal $\overline{w} = \overline{z}$.

1.1 RELATED QUESTIONS

We first discuss the relation between this work and the study of local invariants of geometric structures (mostly Riemannian and conformal); next, we note certain instances in other areas of mathematics where one wishes to understand the algebraic structure of local invariants whose *integrals* are known to exhibit invariance properties under natural transformations.

Broad discussion. The theory of *local* invariants of Riemannian structures (and indeed, of more general geometries, e.g., conformal, projective, or CR) has a long history. The original foundations of this field were laid in the work of Hermann Weyl and Élie Cartan; see [38, 18]. The task of writing out local invariants of a given geometry is intimately connected with understanding which polynomials in a space of tensors with given symmetries remain invariant under the action of a Lie group. In particular, the problem of writing down all local Riemannian invariants[15] reduces to understanding the invariants of the orthogonal group.

In more recent times, a major program was laid out by C. Fefferman in [23] aimed at finding all scalar local invariants in CR geometry. This was motivated by the problem of understanding the invariants that appear in the asymptotic expansions of the Bergman and Szegö kernels of strictly pseudoconvex CR manifolds, in a way similar to which Riemannian invariants appear in the asymptotic expansion of the heat kernel. The study of the local invariants in the singularities of these kernels led to important breakthroughs in [10] and more recently by Hirachi in [32]. This program was extended to conformal geometry by Bailey, Eastwood, and Graham in [24]. Both these geometries belong to a broader class of structures, the *parabolic geometries*; these are structures that admit a principal bundle with structure group a parabolic subgroup P of a semi-simple Lie group G, and a Cartan connection on that principle bundle (see the introduction in [16].) An important question in the study of these structures is the problem of constructing all their local invariants, which can be thought of as their *natural, intrinsic* scalars.

In the context of conformal geometry, the first (modern) landmark in understanding *local conformal invariants* was the work of Fefferman and Graham in 1985 [24], where they introduced the *ambient metric*. This allows one to construct conformal invariants of any order in odd dimensions and up to order $\frac{n}{2}$ in even dimensions. A natural question is then whether *all* invariants arise via this construction.

The subsequent work of Bailey-Eastwood-Graham [10] proved that this is indeed true in odd dimensions; in even dimensions, they proved that the result holds when the weight (in absolute value) is bounded by the dimension.

[15] The *scalar-valued* invariants considered in Definition 1.1 are particular cases of such local invariants.

The ambient metric construction in even dimensions was recently extended by Graham-Hirachi [29]; this enables them to identify in a satisfactory manner *all* local conformal invariants, even when the weight (in absolute value) exceeds the dimension.

An alternative construction of local conformal invariants can be obtained via the *tractor calculus* introduced by Bailey-Eastwood-Gover in [9]. This construction bears a strong resemblance to the Cartan conformal connection and to the work of T.Y. Thomas in 1934 [37]. The tractor calculus has proven to be very universal; tractor bundles have been constructed [16] for an entire class of parabolic geometries. The relation betweeen the conformal tractor calculus and the Fefferman-Graham ambient metric has been elucidated in [17].

The present work, while pertaining to the question above (given that it ultimately deals with the algebraic form of local *Riemannian* and *conformal* invariants), nonetheless addresses a different type of problem. We here consider Riemannian scalars $P(g)$ for which the *integral* $\int_{M^n} P(g)dV_g$ remains invariant under conformal changes of the underlying metric; we then seek to understand the possible algebraic form of the *integrand* $P(g)$, ultimately proving that it can be decomposed in the way that Deser and Schwimmer asserted. It is thus not surprising that the prior work on the construction and understanding of local *conformal* invariants plays a central role in this endeavor, in Chapter 3 of this book and in [4]. We will explain in the introduction of Chapter 3 how some of the local conformal invariants that we *identify* in $P(g)$ would be *expected* (given the properties of the ambient metric but also the insight obtained in [10]), while others are much less obvious.

On the other hand, our resolution of the Deser-Scwimmer conjecture also relies heavily on a deeper understanding of the algebraic properties of the *classical* local Riemannian invariants. The fundamental theorem of invariant theory (see Theorem B.4 in [10] and also Theorem 2 in [2]) is used extensively throughout this book. However, the most important algebraic tools on which our method relies are certain main algebraic propositions presented in chapters 2 and 3.[16] These are purely algebraic propositions that deal *exclusively* with *local Riemannian invariants*. While the author was led to led to these propositions out of the strategy that he felt was necessary to solve the Deser-Schwimmer conjecture, they can be thought of as results of independent interest. The *proof* of these propositions, presented in the second half of this book is in fact not particularily intuitive. It is the author's sincere hope that deeper insight will be obtained in the future as to *why* these algebraic propositions hold.

Hirachi's global CR invariant. A question analogous to the Deser-Schwimmer conjecture arises in the context of understanding the asymptotic expansion of the Szegö kernel of strictly pseudoconvex domains Ω in $\mathbb{C}^n$ (or, alternatively, of abstract CR-manifolds.) Hirachi [33] showed that the leading

[16] A summary of these is provided in subsection 2.1 below.

term $\psi_\theta|_{\partial\Omega}$ of the logarithmic singularity of the Szegö kernel[17] $S_\theta(z,\overline{w})$,

$$S_\theta(z,\overline{z}) = \phi_\theta(z)\rho^{-n}(z) + \psi_\theta(z)\log\rho(z),$$

exhibits a global invariance, which is very similar to the one we discuss here; in particular, $L = \int_{\partial\Omega}\psi_\theta(z)\theta\wedge(d\theta)^{n-1}$ is invariant under conformal rescalings[18] of the contact form θ. In [12] Boutet de Monvel showed that Hirachi's global invariant L vanishes on all (integrable) compact CR manifolds. A natural question is whether $\psi_\theta(z)$ must be a divergence.

The Catlin-Tian-Yau-Zelditch expansion. A further problem related to the Deser-Schwimmer conjecture arises in Kähler geometry. The problem is to understand the algebraic structure of the coefficients in the Catlin-Tian-Yau-Zelditch expansion; this is a local version of the classical Riemann-Roch theorem regarding the dimension of the space of holomorphic sections of high powers of ample line bundles (L,h) over complex manifolds X. One constructs a density of states function $\sum_{j=0}^{d_N}||S_j^N(z)||^2_{h_N}$ over X out of an orthonormal basis $\{S_0^N(z),\ldots,S_{d_N}^N(z)\}$ of the space of holomorphic sections $H^0(X,L^{\otimes N})$; this function then admits an asymptotic expansion

$$\sum_{j=0}^{d_N}||S_j^N(z)||^2_{h_N} = a_0(z)N^n + a_1(z)N^{N-1} + \ldots,$$

where the terms $a_i(z)$ are local invariants of the Kähler metric $g_{i\overline{j}}$ that the Ricci curvature $\mathrm{Ric}(h) = \omega_g$ induces on the base manifold X; see [39] for a detailed discussion. By *integrating* the above expansion over X, one recovers the Riemann-Roch theorem. The analogy with the Deser-Schwimmer conjecture thus lies in the fact that the coefficients $a_l(z)$ are local invariants of the Kähler metric $g_{i\overline{j}}$ whose *integral* $\int_X a_l(z)\omega_g^n$ over the base manifold remains invariant under Kähler deformations of $g_{i\overline{j}}$. A question here would be to understand the precise algebraic form of the local invariants $a_l(z)$ and their relation to the underlying Chern characters.

Index theory. Questions similar to the Deser-Schwimmer conjecture arise naturaly in index theory; a good reference for such questions is [11]. For example, in the heat kernel proof of the index theorem (for Dirac operators) by Atiyah-Bott-Patodi [8], the authors were led to consider integrals arising in the (integrated) expansion of the heat kernel for general Dirac operators over Riemannian manifolds and sought to understand the local structure of the integrand.[19] In that setting, however, the fact that one deals with a *specific*

[17] The singularity occurs on the diagonal $\overline{w} = \overline{z}$ as $z \to \partial\Omega$; it is expressed with respect to a defining function $\rho(z)$ for the boundary, which also induces a contact form θ on $\partial\Omega$.

[18] It is in fact *also* invariant under strictly pseudoconvex deformations of the domain Ω.

[19] We note that the geometric setting in [8] is more general than the one in the Deser-Schwimmer conjecture. In particular one considers vector bundles, equipped with an auxiliary connection, over compact Riemannian manifolds; the local invariants thus depend *both* on the curvature of the Riemannian metric *and* the curvature of the connection.

integrand that arises in the heat kernel expansion plays a key role in the understanding of its local structure. This is true both of the original proof of Patodi, Atiyah-Bott-Patodi [8] and of their subsequent simplifications and generalizations by Getzler, Berline-Getzler-Vergne; see [11].

The closest analogous problem to the one considered here is the work of Gilkey and Branson-Gilkey-Pohjanpelto, [27, 14]. In [27], Gilkey considered Riemannian invariants $P(g)$ for which the *integral* $\int_{M^n} P(g)dV_g$ on any given (smooth, closed) manifold M^n has a given value, *independent of the metric* g. He proved that $P(g)$ must then be equal to a divergence, plus possibly a multiple of the Chern-Gauss-Bonnet integrand if the weight of $P(g)$ agrees with the dimension in absolute value. In [14] the authors considered the problem of Deser-Schwimmer for locally conformally flat metics and derived the same decomposition (for *locally conformaly flat metrics*) as in [27]. Although these two results can be considered precursors of ours, the methods there are entirely different from the ones here; it is highly unclear whether the methods of [27, 14] could be applied to the problem at hand.

1.2 OUTLINE OF THIS WORK

1.2.1 A one-page summary of the argument

The Deser-Schwimmer conjecture is proven by a multiple induction. At the roughest level, the induction works as follows: We express $P(g)$ as a linear combination of complete contractions,

$$P(g) = \sum_{l \in L} a_l C^l(g), \tag{1.15}$$

each $C^l(g)$ in the form (1.3).

The different complete contractions $C^l(g)$ appearing above can be grouped up into categories according to certain algebraic features of the tensors involved.[20] Accordingly, we divide the index set L into subsets $L^1, \dots, L^T$ so that the terms indexed in the same index set L^t belong to the same category (and vice versa); then, we write

$$P(g) = \sum_{t=1}^{T} \sum_{l \in L^t} a_l C^l(g). \tag{1.16}$$

We also introduce a *grading* among the set of categories: A given category of complete contractions is better or worse than any other given category. For future reference, the best category of complete contractions contains the ones with $\frac{n}{2}$ factors.

Assume that in (1.16), for each pair $1 \leq \alpha < \beta \leq T$ the category of complete contractions indexed in L^β is worse than the category of complete contractions

[20] This is made more precise in below.

indexed in L^α. (Therefore, in particular the worst category of complete contractions in (1.16) is the one that corresponds to the terms $C^l(g), l \in L^T$.)

The main step of our induction is to prove that *unless the complete contractions $C^l(g), l \in L^T$ are in the best category*,[21] there exists a local conformal invariant $W(g)$ and a divergence of a vector field $\mathrm{div}_i\, T^i(g)$ so that

$$\sum_{l\in L^T} a_l C^l(g) - W(g) - \mathrm{div}_i\, T^i(g) = \sum_{l\in L^{\mathrm{new}}} a_l C^l(g), \tag{1.17}$$

where the complete contractions $C^l(g), l \in L^{\mathrm{new}}$ in the RHS belong to categories that are all better than the category of $\sum_{l\in L^T} a_l C^l(g)$.

Observe that once this main step is proven, we can iteratively apply it to derive that there exists a local conformal invariant $\tilde{W}(g)$ and a divergence $\mathrm{div}_i\, \tilde{T}^i(g)$ so that

$$P(g) - \tilde{W}(g) - \mathrm{div}_i\, \tilde{T}^i(g) = \tilde{P}(g), \tag{1.18}$$

where $\tilde{P}(g)$ is a linear combination of terms with $\frac{n}{2}$ factors. *Furthermore, $\int_{M^n} \tilde{P}(g) dV_g$ is also a global conformal invariant.* Therefore, invoking the main theorem of [3], we derive that $\tilde{P}(g)$ can be written in the form

$$\tilde{P}(g) = W'(g) + (\mathrm{const}) \cdot \mathrm{Pfaff}(R_{ijkl}), \tag{1.19}$$

where $W'(g)$ is a local conformal invariant.[22] Thus, combining (1.18) and (1.19), we derive the Deser-Schwimmer conjecture.

1.2.2 An outline of this book

Important remark. It is important to clarify how Riemannian invariants are viewed in this book: As explained below Definition 1.1, a Riemannian invariant is in principle an operator that assigns to every Riemannian manifold (M, g) a function $P(g) \in \mathcal{C}^\infty(M)$. However, the work of Weyl allows us to *identify* $P(g)$ with a (non-unique) *formal algebraic expression*.

As we have seen, given any Riemannian invariant $P(g)$, there exists a formal linear combination of complete contractions, $\sum_{l\in L} a_l C^l(g)$, (where each $C^l(g), l \in L$ is in the form (1.3)) such that for every Riemannian manifold (M, g) and any $x \in M$, $P(g)$ evaluated at x equals $\sum_{l\in L} a_L C^l(g)$ evaluated at x.

Each of these complete contractions $C^l(g)$ contains a finite number (say a) of factors $\nabla^{(m_i)}R$, $1 \le i \le a$, each with $m_i + 4$ indices. Every such complete contraction is fully determined once we specify the *pattern* according to which the various indices contract against each other (in short, which index contracts against which.)

Throughout this work, we always think of Riemannian invariants as linear combinations of complete contractions of tensors. This will enable us to use

[21] I.e., unless $P(g)$ is in the form (1.6).

[22] $\mathrm{Pfaff}(R_{ijkl})$ is the Pfaffian of the curvature tensor (i.e., the Gauss-Bonnet integrand.)

the rich algebraic structure in each such linear combination, by examining the algebraic properties of the various complete contractions $C^l(g)$. Examples of such algebraic properties would be the total number of factors $\nabla^{(m)}R$ in a given complete contraction in the form (1.3) *or* how many pairs of indices that contract against each other belong to the same factor. In fact, later we consider more general complete and partial contractions, which will involve a finite number of factors of different forms. (For example there will be iterated covariant derivatives of the Weyl tensor $\nabla^{(a)}W$, the Schouten tensor $\nabla^{(b)}P$, and of scalar-valued functions $\nabla^{(c)}\phi$ in other complete and partial contractions that we will be considering.)

This work can be naturally divided into two parts. The first part consists of chapters 2 and 3 and [4] and establishes the Deser-Schwimmer conjecture, subject to proving certain propositions of a purely algebraic nature, namely, Propositions 2.28, 3.27 and 3.28 which concern the classical theory of Riemannian invariants. The second part consists of chapters 4, 5, 6 and 7 and [6] and proves these main algebraic propositions.

The second part is thus logically independent of the first one. Now, although these main algebraic propositions are purely statements about Riemannian invariants, the motivation behind them is to prove the Deser-Schwimmer conjecture. Nonetheless, the author believes that they can be thought of as results of independent interest. In particular, it is conceivable that similar results on the structure of local invariants of other geometries (e.g., Kähler or CR) could contribute to understanding the decompositions of global invariants in these geometries, such as those described in the Related Questions discussion.

Now, let us outline the tasks undertaken in each of the chapters here: In chapter 2 we flesh out the strategy of *iteratively* decomposing any $P(g) = \sum_{l\in L} a_l C^l(g)$ for which $\int P(g)dV_g$ is a global conformal invariant. We make precise the notions of better and worse complete contractions in $P(g)$ and then spell out (1.17), via Propositions 2.7, 2.8. In particular, using the well-known decomposition of the curvature tensor into its trace-free part (the Weyl tensor) and it trace part (the Schouten tensor[23]), we reexpress $P(g)$ as a linear combination of complete contractions involving differentiated Weyl tensors and differentiated Schouten tensors, as in (2.47). To define the worst terms in $P(g)$, we then pick out the terms with the smallest number of factors in total; among those we pick out the ones with the smallest number of differentiated Weyl tensors. *These are the worst complete contractions in* $P(g)$. We then claim, in Propositions 2.7 and 2.8, that we can subtract a divergence and a local conformal invariant from $P(g)$ and *cancel out* those worst terms, modulo introducing new ones, with either more factors or with the same number of factors in total, but more differentiated Weyl tensors. This is the content of (1.17).

In Chapter 2 we also prove (1.17) when the worst terms involve at least one differentiated Schouten tensor. In that case, we need only a divergence

[23]This can be thought of as a trace modification of the Ricci tensor.

for (1.17); no local conformal invariant is needed or used. The proof relies on considering the conformal variation $I_g(\phi)$ of $P(g)$[24] and observing a one-to-one correspondence between the worst piece in $P(g)$ and a certain sum of terms in $I_g(\phi)$. We then employ the super divergence formula from [2] to $I_g(\phi)$ and derive a useful, local formula that expresses this sum of terms in $I_g(\phi)$ as a divergence of vector-valued operators. The challenge is then to use this local formula *for* $I_g(\phi)$ to prove (1.17) (which refers to $P(g)$.) To accomplish this, a new iteration is performed. The necessity for this new iteration lies in the fact that the worst piece in $P(g)$ involves differentiated Weyl tensors, while the super divergence formula can be applied only when we split the Weyl tensors into a curvature part and a Ricci part. To close the iterative argument, we must resort to the main algebraic proposition 2.28.

In Chapter 3 we prove (1.17) when the worst terms in $P(g)$ involve only factors of the differentiated Weyl tensor. This case is much harder than the previous one; in particular, in this case we need both a local conformal invariant $W(g)$ and a divergence $\mathrm{div}_i T^i(g)$ to prove (1.17). One obvious difficulty is how, upon inspection of $P(g)_{\mathrm{worst-piece}}$, to *separate* the piece that must be cancelled out by a local conformal invariant from the piece that is cancelled out by a divergence. (We could phrase this question as "What is the nature of the decomposition (1.5) in $P(g)$?") In a first step, we prove that we can first explicitly construct a local conformal invariant and a divergence and subtract them from $P(g)_{\mathrm{worst-piece}}$, to be left with a *new* worst piece (again with only differentiated Weyl curvature terms), which has some *additional* algebraic properties. In a second step, we show that this new worst piece can be cancelled out by subtracting a divergence.

The challenges in proving this second step are severe; they are described in the introduction of Chapter 3. Again, the proof relies on applying the super divergence formula in [2] to the conformal variation $I_g(\phi)$ of $P(g)$; the proof is again iterative, essentially for the same reason as in Chapter 2; it also relies on the main algebraic propositions 3.27 and 3.28. One additional difficulty here is that it is harder to identify the worst piece of $P(g)$ by examining $I_g(\phi)$. The main difficulty, however, is that *after* we apply the super divergence formula to $I_g(\phi)$, we end up with terms that arise *not* from the worst piece in $P(g)$, but from other pieces in $P(g)$, over which we have no control. We are able to close the argument, due to the very specific algebraic properties of the *new* worst piece in $P(g)$, which are guaranteed by the first step.

Chapter 4 is the beginning of the second part of this book, where the main algebraic propositions 2.28, 3.27 and 3.28 are proven. In Chapter 4 in particular,[25] we set up a highly complicated proposition (the fundamental proposition

[24] This is the operator $I_g(\phi) := e^{n\phi} P(e^{2t\phi} g) - P(g)$.

[25] This chapter is a somewhat modified reproduction of [5]; it is included here for the reader's convenience, since it is a central part of this work.

4.13), which is to be proven by an elaborate induction on four parameters. The main algebraic propositions are special cases of this fundamental proposition; in fact, they are the ultimate and penultimate steps in the induction with respect to certain of the parameters. The proof of the inductive step is reduced to three main Lemmas, 4.16, 4.19 and 4.24, which are proven in the following chapters.

The fundamental proposition is purely a statement on Riemannian invariants. Very loosely, the objects that enter into the fundamental proposition are specific types of intrinsic tensor fields, locally constructed over any Riemannian manifold. The building blocks of these tensor fields are tensors with specific symmetries and antisymmetries, primarily iterated covariant derivatives of the curvature tensor, and iterated covariant derivatives of scalar-valued functions. We build intrinsic tensor fields out of these objects by taking tensor products and then contracting pairs of indices and leaving some indices free. We also introduce a formal notion of pseudo-divergence of such objects, which we call the X-divergence:

The "real" divergence of a tensor field (partial contraction) as above with respect to one of its free indices equals a *sum* of partial contractions each with one less free index; the X-divergence arises from the "real" divergece by just *discarding* some terms. The fundamental proposition then says that if the iterated X-divergence of a tensor field of rank α vanishes, then the α-tensor field must *itself* be an X-divergence of a tensor field of higher rank. Thus, very loosely speaking, the fundamental propositon can be interpreted as a zero cohomology theorem for our formal expressions with respect to this notion of X-divergence.

At the end of [5] we also include a note written by Travis Schedler and Paul Christiano, which reexpresses the main algebraic proposition 2.28 in the language of graph theory, in the hope that the statement will be accessible to a broader audience.

In chapters 5, 6 and 7 we prove the main Lemmas 4.16, 4.19 and 4.24. Specificaly, in Chapter 5 we take up the proof of Lemmas 4.16 and 4.19, which are easier than Lemma 4.24. The proof of these two lemmas relies on the study of the first conformal variation of the assumption of Proposition 4.13. This study involves many complicated calculations and also the appropriate use of the inductive assumption of Proposition 4.13.

In Chapter 6 we take up the proof of Lemma 4.24, which is harder than the previous two; Lemma 4.24 has two cases, A and B. The strategy goes as follows: We first repeat the ideas from chapter 5 and derive a new local equation from the assumption of Proposition 4.13. However, we find that this new equation is *very far* from proving the claim of Lemma 4.24. We then have to return to the hypothesis of Proposition 4.13 and extract an entirely new equation from its conformal variation. We proceed with a detailed study of this new equation (again using the inductive assumption of Proposition 4.13); the result is a second new local equation which *again* is very far from proving the claim of our lemma. We then formally manipulate this second new local equation and add it to the first one and observe certain *miraculous cancellations*, which give us new local

equations, that we collectively call the grand conclusion. Lemma 4.24 in case A then immediately follows from the grand conclusion.

Case B of Lemma 4.24 is derived in Chapter 7. The proof follows by distinguishing numerous subcases; we derive our claim via a different argument in each of the subcases. A new element here is that we combine the two local equations derived in Chapter 6, with the grand conclusion to obtain *systems* of equations, from which we then derive Lemma 4.24 in case B.

1.2.3 Why is this proof so long?

This is a valid question. In fact, one can pose this question separately for the two parts of this book. Recall from the previous subsection that this work can be naturally divided into two parts: Part A consists of Chapters 2 and 3 and [4] and establishes the Deser-Schwimmer conjecture subject to proving the main algebraic Propositions 2.28, 3.27 and 3.28. Part B consists of Chapters 4, 5, 6 and 7 and [6] (it is thus the bulk of this work) and *proves* these propositions. The author feels that (broadly speaking) the iterative strategy pursued in part A is forced on us, given the tools available at present and the nature of the problem.

It seems that an iterative scheme to solve this problem is inevitable, and (given our main tools, the super divergence formula from [2] and the ambient metric construction of Fefferman-Graham) it is inevitable that the proof must make use of the main algebraic propositions. However, the author feels that the present proof of the main algebraic propositions in part B is unintuitive, and thus unsatisfactory. He hopes that a better proof of these main algebraic propositions will be found in the future.

Part A: An iterative scheme and the challenges it addresses. As explained above, given any global conformal invariant $\int P(g)dV_g$, we set up an algorithm that successively subtracts divergences and local conformal invariants from the integrand $P(g)$ and at each step simplifies $P(g)$. Given the algebraic/combinatiorial complexity of Riemannian invariants, as well as the possibility of expressing a given Riemannian invariant $P(g)$ as a linear combination in the form (1.2) *in numerous ways*, it seems inevitable to the author that the decomposition (1.5) can be proven only by such an iterative scheme.

The precise nature of the scheme we pursue here is necessitated by the tools we have at our disposal. In that sense, the author feels that the algorithm that he has chosen, his fairly involved yet simple use of the Fefferman-Graham ambient metric, and the fact that he has to appeal to the main algebraic propositions to complete the proof are to some extent dictated by the problem. The very existence of local conformal invariants forces us to study the conformal variation of $\int_{M^n} P(g)dV_g$ to *identify* a divergence in the integrand $P(g)$. The necessity of a fairly elaborate iteration to *simplify* $P(g)$ is forced by the nature of the tools we have to study this conformal variation $I_g(\phi)$. In particular, to use the conformal variation $I_g(\phi)$ to recover $P(g)$ itself, we choose to express $P(g)$ via complete contractions in the (differentiated) Weyl and Schouten tensors; but

then the main tool we have to express $I_g(\phi)$ as a divergence forces us to again decompose the (differentiated) Weyl tensors into Riemann-curvature and Ricci terms. As we explain in the introduction of Chapter 2, the only apparent way then to close the argument and construct a divergence for which (1.17) holds is to resort to a new iteration, which relies on the main algebraic proposition 2.28.

A further key challenge one must face is how to *separate* the local conformal invariant piece in $P(g)$ from the divergence piece. We present here one possible decomposition, which makes elaborate use of the Fefferman-Graham ambient metric construction. It *could be* that one can make further use of the ambient metric, constructing more conformal invariants and simplifying $P(g)$ without resorting to the super divergence formula. The author was not able to do this, in part due to the difficulty of calculations in the ambient metric. The approach taken here necessitated the introduction of the main algebraic propositions 3.27 and 3.28, which are simple modifications of Proposition 2.28.

As discussed, the main algebraic propositions can be loosely thought of as zero cohomology results for formal algebraic expressions, which are linear combinations of partial contractions built out of tensors of two types: either (differentiated) curvature tensors (which enjoy well-known symmetries and antisymmetries) or differentiated scalars (these are essentially symmetric tensors.) The main algebraic propositions say that if the iterated X-divergence[26] of such a μ-tensor vanishes, then the tensor itself must be an X-divergence of another such a tensor with rank $\mu + 1$. It is the proof of these main algebraic propositions that could perhaps be derived in a simpler, more intuitive way.

Part B: A long, complicated proof of the main algebraic propositions. Part B of this book is entirely devoted to proving the main algebraic Propositions 2.28, 3.27 and 3.28. This proof is long, and unsatisfactory in the sense that it does not provide insight into why these results must hold. The proof relies on a multi-parameter induction; the most important parameter in this induction is the *rank*. The key idea in the induction is to generalize the main algebraic propositions to a more involved statement, the fundamental Proposition 4.13, and then (roughly) to prove that this fundamental proposition 4.13 for tensors of rank μ implies the fundamental proposition 4.13 for tensors[27] of rank $\mu - 1$. The first reason why part B is so long is that it relies *again* (and perhaps unnecessarily this time) on the study of the conformal variation of a local equation (2.103), which is the assumption of Proposition 4.13. This analysis relies heavily on computation and by necessity has to be much more detailed than the analysis of $I_g(\phi)$ in chapters 2 and 3 and [4].

[26]The X-divergence is properly defined in the introduction of the next chapter; it is *not* the (iterated) divergence of the tensor field, but rather a specific algebraic expression that can be picked out from the iterated divergence of the tensor field.

[27]Since for tensors of rank zero the assumption coincides with the conclusion, this iterative argument proves our claim.

The key point that we observe is that the first conformal variation of an iterated X-divergence preserves a lot of the underlying algebraic structure, once we pick out a certain carefuly chosen linear combination, which vanishes separately. However, as explained in Chapters 6 and 7 the analysis can become very complicated, and in fact our conclusion does not follow from a straightforward study of the newly derived local equation. In Chapter 6 we resort to picking out *two* equations from the conformal variation of (2.103), we analyze them separately, and then we combine the resulting new equations to derive our conclusion.

The specifics of this proof are outlined in the introductions of Chapters 5, 6, 7. This vague sketch above indicates that our proof of the main algebraic propositions provides very little intuition regarding *why* they hold. It is the author's hope that further insight will be gained in the future into this question. This it is hoped both simplify this proof of the Deser-Schwimmer conjecture and also make it possible to study integral invariants of other geometries.