2

Ranks and Cranks, Part I

2.1 Introduction

This somewhat lengthy chapter concerns some of the most important formulas from the lost notebook [283], which are contained in only a few lines. We first introduce some standard notation that will be used throughout this chapter (and most of this book). Secondly, we record the two formulas listed at the top of page 20 (one of which is repeated in the middle of page 18). After stating these formulas, we provide history demonstrating that these entries are the genesis of some of the most important developments in the theory of partitions during the twentieth and twenty-first centuries. Next, we offer two further claims found in the lost notebook. Lastly, we provide proofs for all four claims.

For each nonnegative integer n, set

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad (a)_\infty := (a;q)_\infty := \lim_{n\to\infty}(a;q)_n, \quad |q|<1.$$

Also, set

$$(a_1,\dots,a_m;q)_n := (a_1;q)_n \cdots (a_m;q)_n$$

and

$$(a_1,\dots,a_m;q)_\infty := (a_1;q)_\infty \cdots (a_m;q)_\infty. \tag{2.1.1}$$

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab|<1. \tag{2.1.2}$$

It satisfies the well-known Jacobi triple product identity [60, p. 10, Theorem 1.3.3], [12, p. 21, Theorem 2.8]

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.1.3}$$

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part III*,
DOI 10.1007/978-1-4614-3810-6_2,

Also recall that [55, p. 34, Entry 18(iv)] for any integer n,

$$f(a,b)=a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n,b(ab)^{-n}). \tag{2.1.4}$$

We now state the first of the two aforementioned remarkable entries from the lost notebook.

Entry 2.1.1 (pp. 18, 20). *Let* ζ_5 *be a primitive fifth root of unity, and let*

$$F_5(q):=\frac{(q;q)_\infty}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty}. \tag{2.1.5}$$

Then

$$\begin{aligned}F_5(q)=A(q^5)&-(\zeta_5+\zeta_5^{-1})^2qB(q^5)\\&+(\zeta_5^2+\zeta_5^{-2})q^2C(q^5)-(\zeta_5+\zeta_5^{-1})q^3D(q^5),\end{aligned} \tag{2.1.6}$$

where

$$A(q):=\frac{(q^5;q^5)_\infty G^2(q)}{H(q)}, \tag{2.1.7}$$

$$B(q):=(q^5;q^5)_\infty G(q), \tag{2.1.8}$$

$$C(q):=(q^5;q^5)_\infty H(q), \tag{2.1.9}$$

$$D(q):=\frac{(q^5;q^5)_\infty H^2(q)}{G(q)}, \tag{2.1.10}$$

with

$$G(q):=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \tag{2.1.11}$$

and

$$H(q):=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{2.1.12}$$

We remark that by the famous Rogers–Ramanujan identities [15, Chapter 10],

$$G(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}\qquad\text{and}\qquad H(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}.$$

The identity (2.1.6) is an example of a *dissection*. Since this and the following chapter are devoted to dissections, we offer below their definition.

Definition 2.1.1. *Let* $P(q)$ *denote any power series in* q. *Then the* t*-dissection of* P *is given by*

$$P(q)=:\sum_{k=0}^{t-1}q^kP_k(q^t). \tag{2.1.13}$$

Note that (2.1.6) provides a 5-dissection for $F_5(q)$, i.e., (2.1.6) separates $F_5(q)$ into power series according to the residue classes modulo 5 of their powers. In analogy with (2.1.6), we see that (2.1.17) in the next entry provides a 5-dissection for $f_5(q)$.

Of the dissections offered by Ramanujan in his lost notebook, some, such as (2.1.6), are given as equalities in terms of roots of unity; others are given as congruences in terms of a variable a. In Chapter 3, we establish Ramanujan's dissections in terms of congruences, while in this chapter we prove 5- and 7-dissections in the form of equalities for each of the rank and crank generating functions, whose representations are given, respectively, in (2.1.24) and (2.1.27) below. The precise definitions of the *rank* and *crank* of a partition will be given after we record the second of the two aforementioned fundamental identities.

In order to explicate our remark about congruences in the preceding paragraph, following Ramanujan in his lost notebook, we define the more general function

$$F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{2.1.14}$$

(Note that the notation (2.1.14) conflicts with that of (2.1.5); the right-hand side of (2.1.5) would be $F_{\zeta_5}(q)$ in the notation (2.1.14).) Set

$$A_n := a^n + a^{-n} \qquad \text{and} \qquad S_n := \sum_{k=-n}^{n} a^k.$$

Then [62, p. 105, Theorem 5.1]

$$F_a(q) \equiv A(q^5) + (A_1 - 1)qB(q^5) + A_2q^2C(q^5) - A_1q^3D(q^5) \pmod{S_2}. \tag{2.1.15}$$

Thus, we have replaced the primitive root ζ_5 by the general variable a. The congruence (2.1.15) is then a generalization of (2.1.6), because if we set $a = \zeta_5$ in (2.1.15), the congruence is transformed into an identity. An advantage of (2.1.15) over (2.1.6) is that we can put $a = 1$ in (2.1.15) and so immediately deduce the Ramanujan congruence

$$p(5n+4) \equiv 0 \pmod{5},$$

where $p(n)$ is the number of partitions of n. Although (2.1.15) appears to be more general than (2.1.6), in fact, *it is not*. It is shown in [62, pp. 118–119] that (2.1.15) can be derived from (2.1.6). In Section 3.8 of the following chapter we reproduce that argument, which is due to F.G. Garvan.

Entry 2.1.2 (p. 20). *Let ζ_5 be a primitive fifth root of unity, and let*

$$f_5(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n \left(\zeta_5^{-1}q;q\right)_n}. \tag{2.1.16}$$

Then

$$f_5(q) = A(q^5) + \left(\zeta_5 + \zeta_5^{-1} - 2\right)\phi(q^5) + qB(q^5) + \left(\zeta_5 + \zeta_5^{-1}\right)q^2C(q^5) \\ - \left(\zeta_5 + \zeta_5^{-1}\right)q^3\left\{D(q^5) - \left(\zeta_5^2 + \zeta_5^{-2} - 2\right)\frac{\psi(q^5)}{q^5}\right\}, \tag{2.1.17}$$

where $A(q)$, $B(q)$, $C(q)$, and $D(q)$ are given in (2.1.7)–(2.1.10), *and where*

$$\phi(q) := \sum_{n=0}^{\infty}\phi_n q^n := -1 + \sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n} \tag{2.1.18}$$

and

$$\frac{\psi(q)}{q} := -\frac{1}{q} + \sum_{n=0}^{\infty}\psi_n q^n := \sum_{n=0}^{\infty}\frac{q^{5n^2-1}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}. \tag{2.1.19}$$

Corollaries of the preceding entry appear in the middle of page 184 in the lost notebook. Since their proofs are immediate consequences of Entry 2.1.2, we offer them here.

Entry 2.1.3 (p. 184). *Write*

$$\sum_{n=0}^{\infty}\lambda_n q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(1+\frac{\sqrt{5}+1}{2}q+q^2)\cdots(1+\frac{\sqrt{5}+1}{2}q^n+q^{2n})}. \tag{2.1.20}$$

Then,

$$\sum_{n=0}^{\infty}\lambda_{5n+1}q^n = \frac{(q^5;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = (q^5;q^5)_\infty G(q), \tag{2.1.21}$$

$$\sum_{n=0}^{\infty}\lambda_{5n+2}q^n = -\frac{\sqrt{5}+1}{2}\frac{(q^5;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = -\frac{\sqrt{5}+1}{2}(q^5;q^5)_\infty H(q), \tag{2.1.22}$$

λ_{5n-1} *is identically zero.* (2.1.23)

Proof. In the definition (2.1.16), set $\zeta_5 = e^{4\pi i/5}$; therefore, $\zeta_5+\zeta_5^{-1} = -\frac{\sqrt{5}+1}{2}$. Using then the notation (2.1.20), equate coefficients of q^{5n+1} on both sides of (2.1.17). Divide both sides by q and lastly replace q^5 by q in the resulting identity to establish (2.1.21). Similarly, to prove (2.1.22), equate coefficients of q^{5n+2} on both sides of (2.1.17). Divide both sides by q^2 and replace q^5 by q. Finally, we note that the dissection (2.1.17) does not have any powers of the form q^{5n-1}, and so (2.1.23) is immediate. □

Before presenting the third and fourth entries for this chapter, as remarked above, it is appropriate to say something about these results, which lay hidden during one of the most interesting developments in the theory of partitions during the twentieth century.

In 1944, F. Dyson [127] published a paper filled with fascinating conjectures from the theory of partitions. Namely, Dyson began by defining the *rank* of a partition to be the largest part minus the number of parts. Dyson's objective was to provide a purely combinatorial description of Ramanujan's theorem that 5 divides $p(5n+4)$. In particular, Dyson conjectured that the partitions of $5n+4$ classified by their rank modulo 5 did, indeed, produce five sets of equal cardinality, namely $p(5n+4)/5$. He was also led to conjecture that the partitions of $7n+5$, classified by rank, split into seven sets each of cardinality $p(7n+5)/7$. This would prove the second Ramanujan congruence, namely, that 7 divides $p(7n+5)$. He also conjectured a generating function for ranks. If $N(m,n)$ denotes the number of partitions of n with rank m, then Dyson's observations make clear he knew that

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n}. \tag{2.1.24}$$

Observe that if we take $z=1$ in (2.1.24), then (2.1.24) reduces to the well-known generating function for $p(n)$,

$$\sum_{n=0}^{\infty} p(n)q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n^2},$$

which is due to Euler. If we set $z=-1$ in (2.1.24), we obtain Ramanujan's mock theta function $f(q)$.

Unfortunately, it turned out that the Ramanujan congruence

$$p(11n+6) \equiv 0 \pmod{11} \tag{2.1.25}$$

was *not* explicable in the same way that worked for $p(5n+4)$ and $p(7n+5)$. So Dyson conjectured the existence of an unknown parameter of partitions, which he whimsically called "the crank," to explain (2.1.25).

In 1954, A.O.L. Atkin and H.P.F. Swinnerton-Dyer [28] proved all of Dyson's conjectures; however, the crank remained undiscovered.

The real breakthrough in this study was made by Garvan in his Ph.D. thesis [146] at Pennsylvania State University in 1986. Garvan's thesis is primarily devoted to the Entries 2.1.1 and 2.1.2 given above. Observe that Entry 2.1.2 is devoted to a special case of the generating function (2.1.24) for ranks. Not only was Garvan able to prove these two entries, but he also deduced all of the Atkin and Swinnerton-Dyer results for the modulus 5 from Entry 2.1.2. As for Entry 2.1.1, Garvan defined a "vector crank," which did provide a combinatorial explanation for 11 dividing $p(11n+6)$, but did this via certain triples of partitions, i.e., vector partitions. Subsequently, Garvan and Andrews [17] found the actual crank. Namely, for any given partition π, let $\ell(\pi)$ denote the largest part of π, $\omega(\pi)$ the number of ones appearing in π, and $\mu(\pi)$ the number of parts of π larger than $\omega(\pi)$. Then the crank, $c(\pi)$, is given by

$$c(\pi) = \begin{cases} l(\pi), & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi), & \text{if } \omega(\pi) > 0. \end{cases} \tag{2.1.26}$$

For $n > 1$, let $M(m,n)$ denote the number of partitions of n with crank m, while for $n \leq 1$ we set

$$M(m,n) = \begin{cases} -1, & \text{if } (m,n) = (0,1), \\ 1, & \text{if } (m,n) = (0,0), (1,1), (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

The generating function for $M(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m,n) a^m q^n = \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty}. \tag{2.1.27}$$

As shown by Andrews and Garvan [17], the combinatorial equivalent of (2.1.27) is given by (2.1.26). Note that if we set $a = 1$ in (2.1.27), we obtain Euler's original generating function for $p(n)$,

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}.$$

Observe that Entry 2.1.1 provides an identity for a special instance of the generating function for cranks.

Thus, although Ramanujan did not record combinatorial definitions of the rank and crank in his lost notebook (in fact, there are hardly any words at all in the lost notebook), he had discovered their generating functions. From the entries on ranks and cranks in this and the following two chapters, it is clear that Ramanujan placed considerable importance on these ideas, and it is regrettable indeed that we do not know Ramanujan's motivations and thoughts on these two fundamental concepts in the theory of partitions.

We finally record the last two results to be included in this chapter. Actually in each entry below, Ramanujan gives only the left-hand side or hints at it. However, the analogies with Entries 2.1.1 and 2.1.2 are so clear that we have filled in what was clearly intended for the right-hand sides. For Entry 2.1.4, Garvan has supplied the right-hand side in [146, p. 62].

Entry 2.1.4 (p. 19). *Let ζ_7 be a primitive seventh root of unity, and let*

$$F_7(q) := \frac{(q;q)_\infty}{(\zeta_7 q;q)_\infty (\zeta_7^{-1} q;q)_\infty}. \tag{2.1.28}$$

Then

$$\begin{aligned}F_7(q) = (q^7;q^7)_\infty \Big\{ & X^2(q^7) + \left(\zeta_7+\zeta_7^{-1}-1\right) qX(q^7)Y(q^7) \\ & + \left(\zeta_7^2+\zeta_7^{-2}\right) q^2Y^2(q^7) + \left(\zeta_7^3+\zeta_7^{-3}+1\right) q^3X(q^7)Z(q^7) \\ & - \left(\zeta_7+\zeta_7^{-1}\right) q^4Y(q^7)Z(q^7) - \left(\zeta_7^2+\zeta_7^{-2}+1\right) q^6Z^2(q^7)\Big\},\end{aligned} \tag{2.1.29}$$

where

$$X(q) := \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 3 \,(\mathrm{mod}\, 7)}}^\infty (1-q^n)^{-1}, \tag{2.1.30}$$

$$Y(q) := \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 2 \,(\mathrm{mod}\, 7)}}^\infty (1-q^n)^{-1}, \tag{2.1.31}$$

$$Z(q) := \prod_{\substack{n=1 \\ n\not\equiv 0,\pm 1 \,(\mathrm{mod}\, 7)}}^\infty (1-q^n)^{-1}. \tag{2.1.32}$$

There are series representations for $X(q)$, $Y(q)$, and $Z(q)$ that yield analogues of the Rogers-Ramanujan identities for $G(q)$ and $H(q)$ [12, p. 117, Exercise 10].

In order to state the last major entry of this chapter, we need considerable notation. First, introducing the notation of Atkin and Swinnerton-Dyer [28, p. 94], we let

$$\Sigma(z,\zeta,q) = \sum_{n=-\infty}^\infty \frac{(-1)^n\zeta^n q^{3n(n+1)/2}}{1-zq^n}. \tag{2.1.33}$$

Furthermore, to simplify future considerations, in particular to state and prove Entry 2.1.5 below, we make the conventions

$$P_7(a) := \left(q^{7a}, q^{49-7a}; q^{49}\right)_\infty \quad (a\neq 0), \tag{2.1.34}$$

$$P_7(0) := \left(q^{49}; q^{49}\right)_\infty, \tag{2.1.35}$$

$$\Sigma_7(a,b) := \Sigma\left(q^{7a}, q^{7b}, q^{49}\right) \quad (a\neq 0), \tag{2.1.36}$$

$$\Sigma_7(0,b) := \sum_{\substack{n=-\infty \\ n\neq 0}}^\infty \frac{(-1)^n q^{147n(n+1)/2+7bn}}{1-q^{49n}}. \tag{2.1.37}$$

We note in passing that by (2.1.30)–(2.1.32),

$$P_7(1) = \frac{(q^7;q^7)_\infty Z(q^7)}{(q^{49};q^{49})_\infty}, \tag{2.1.38}$$

$$P_7(2) = \frac{(q^7;q^7)_\infty Y(q^7)}{(q^{49};q^{49})_\infty}, \tag{2.1.39}$$

$$P_7(3) = \frac{(q^7;q^7)_\infty X(q^7)}{(q^{49};q^{49})_\infty}. \tag{2.1.40}$$

Finally, we are ready to supply the right-hand side for the analogue of Entry 2.1.2 for the modulus 7.

Entry 2.1.5 (p. 19). *Let ζ_7 be a primitive seventh root of unity, and let*

$$f_7(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n(\zeta_7^{-1}q;q)_n}. \tag{2.1.41}$$

Then

$$\begin{aligned} f_7(q) = &\left(2-\zeta_7-\zeta_7^{-1}\right)\left(1-A_7(q^7)+q^7Q_1(q^7)\right)+A_7(q^7) \\ &+qT_1(q^7)+q^2\left\{\left(\zeta_7+\zeta_7^{-1}\right)B_7(q^7)+q^{14}Q_3(q^7)(\zeta_7+\zeta_7^{-1}-\zeta_7^{-2}-\zeta_7^{2}\right\} \\ &+q^3T_2(q^7)\left(1+\zeta_7^2+\zeta_7^{-2}\right)-q^4\left(\zeta_7^2+\zeta_7^{-2}\right)T_3(q^7) \\ &+q^6\left\{q^7Q_2(q^7)\left(\zeta_7^2+\zeta_7^{-2}-\zeta_7^3-\zeta_7^{-3}\right)-C_7(q^7)\left(1+\zeta_7^3+\zeta_7^{-3}\right)\right\}, \end{aligned} \tag{2.1.42}$$

where

$$A_7(q) := \frac{(q^7,q^3,q^4;q^7)_\infty}{(q,q^2,q^5,q^6;q^7)_\infty}, \tag{2.1.43}$$

$$B_7(q) := \frac{(q^7,q^2,q^5;q^7)_\infty}{(q,q^3,q^4,q^6;q^7)_\infty}, \tag{2.1.44}$$

$$C_7(q) := \frac{(q^7,q,q^6;q^7)_\infty}{(q^2,q^3,q^4,q^5;q^7)_\infty}, \tag{2.1.45}$$

and for $m=1,2,3$,

$$Q_m(q^7) := \frac{\Sigma_7(m,0)}{P_7(0)} \tag{2.1.46}$$

and

$$T_m(q^7) := \frac{P_7(0)}{P_7(m)}. \tag{2.1.47}$$

We remark that the functions $Q_m(q^7)$ in (2.1.46) can be expressed in terms of the generating function for ranks. By a result of Garvan [146, p. 68, Lemma (7.9)], for $|q|<|z|<1/|q|$ and $z\neq 1$,

$$-1+\frac{1}{1-z}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n} = \frac{z}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{1-zq^n}.$$

Hence, after modest rearrangement, we find that

$$\Sigma_7(m,0) = \frac{(q^{49};q^{49})_\infty}{q^{7m}}\left\{-1+\sum_{n=0}^{\infty}\frac{q^{49n^2}}{(q^{7m};q^{49})_{n+1}(q^{49-7m};q^{49})_n}\right\}.$$

Throughout this chapter our work will follow closely the marvelous papers by Atkin and Swinnerton-Dyer [28] and Garvan [146].

2.2 Proof of Entry 2.1.1

Here we shall follow the elegant proof given by Garvan [146]. Throughout this section ζ_5 is a primitive fifth root of unity. We begin with the observation [146, p. 58, Lemma (3.9)]

$$\frac{1}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty}=G(q^5)+q(\zeta_5+\zeta_5^{-1})H(q^5), \tag{2.2.1}$$

where $G(q)$ is defined in (2.1.11) and $H(q)$ is defined in (2.1.12). We prove the identity (2.2.1). Using the Jacobi triple product identity (2.1.3) twice, we find that

$$\begin{aligned}
\frac{1}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty} &= \frac{(q,\zeta_5^2 q,\zeta_5^{-2};q)_\infty}{(q,\zeta_5 q,\zeta_5^{-1}q,\zeta_5^2 q,\zeta_5^{-2};q)_\infty}\\
&= \frac{(q,\zeta_5^2 q,\zeta_5^{-2};q)_\infty}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\\
&= \frac{1}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\zeta_5^{2n}q^{(n^2+n)/2}\\
&= \frac{1}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\sum_{\nu=-2}^{2}\sum_{m=-\infty}^{\infty}(-1)^{5m+\nu}\zeta_5^{10m+2\nu}q^{(5m+\nu)(5m+\nu+1)/2}\\
&= \frac{1}{(1-\zeta_5^{-2})(q^5;q^5)_\infty}\sum_{\nu=-2}^{2}(-1)^{\nu}\zeta_5^{2\nu}q^{\nu(\nu+1)/2}\sum_{m=-\infty}^{\infty}(-1)^m q^{\left(25m^2+(10\nu+5)m\right)/2}\\
&= \frac{1}{(1-\zeta_5^{-2})}\sum_{\nu=-2}^{2}(-1)^{\nu}\zeta_5^{2\nu}q^{\nu(\nu+1)/2}\frac{f(-q^{15+5\nu},-q^{10-5\nu})}{(q^5;q^5)_\infty}\\
&= \frac{1}{(1-\zeta_5^{-2})}\sum_{\nu=-2}^{2}(-1)^{\nu}\zeta_5^{2\nu}q^{\nu(\nu+1)/2}\frac{\left(q^{15+5\nu},q^{10-5\nu},q^{25};q^{25}\right)_\infty}{(q^5;q^5)_\infty}.
\end{aligned}$$

Now, by (2.1.11) and (2.1.12),

$$\frac{\left(q^{15+5\nu},q^{10-5\nu},q^{25};q^{25}\right)_\infty}{(q^5;q^5)_\infty}=\begin{cases}G(q^5), & \text{if } \nu=0,-1,\\ H(q^5), & \text{if } \nu=1,-2,\\ 0, & \text{if } \nu=2.\end{cases}$$

Hence,

$$\begin{aligned}
\frac{1}{(\zeta_5 q;q)_\infty(\zeta_5^{-1}q;q)_\infty} &= \frac{1}{(1-\zeta_5^{-2})}G(q^5)\left(1-\zeta_5^{-2}\right)\\
&\quad+\frac{1}{(1-\zeta_5^{-2})}H(q^5)\left(-\zeta_5^2 q+\zeta_5^{-4}q\right)\\
&= G(q^5)+q\left(\zeta_5+\zeta_5^{-1}\right)H(q^5),
\end{aligned}$$

which is (2.2.1).

Next, we continue to follow Garvan in [146, p. 60, Lemma 3.18] and so employ the identity

$$(q;q)_\infty = (q^{25};q^{25})_\infty \left(\frac{G(q^5)}{H(q^5)} - q - q^2 \frac{H(q^5)}{G(q^5)} \right), \tag{2.2.2}$$

which is one of the famous identities for the Rogers–Ramanujan continued fraction [15, p. 11, equation (1.1.10)]

$$\frac{1}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots = \frac{H(q)}{G(q)}.$$

We now multiply together (2.2.1) and (2.2.2) to obtain

$$\begin{aligned}
\frac{(q;q)_\infty}{(\zeta_5 q;q)_\infty (\zeta_5^{-1} q;q)_\infty} &= \left(G(q^5) + q(\zeta_5 + \zeta_5^{-1}) H(q^5) \right) \\
&\quad \times (q^{25};q^{25})_\infty \left(\frac{G(q^5)}{H(q^5)} - q - q^2 \frac{H(q^5)}{G(q^5)} \right) \\
&= (q^{25};q^{25})_\infty \left\{ \frac{G^2(q^5)}{H(q^5)} + q \left(-1 + \zeta_5 + \zeta_5^{-1} \right) G(q^5) \right. \\
&\qquad \left. + q^2 \left(-1 - (\zeta_5 + \zeta_5^{-1}) \right) H(q^5) + q^3 \left(-(\zeta_5 + \zeta_5^{-1}) \right) \frac{H^2(q^5)}{G(q^5)} \right\} \\
&= A(q^5) - q(\zeta_5 + \zeta_5^{-1})^2 B(q^5) \\
&\quad + q^2 \left(\zeta_5^2 + \zeta_5^{-2} \right) C(q^5) - \left(\zeta_5 + \zeta_5^{-1} \right) q^3 D(q^5),
\end{aligned}$$

and Entry 2.1.1 is proved.

2.3 Background for Entries 2.1.2 and 2.1.4

As was mentioned in Section 2.1, Atkin and Swinnerton-Dyer [28] proved the conjectures of Dyson [127]. Garvan [146] proved that their work for the modulus 5 was in fact equivalent to Entry 2.1.2. Our proof here relies completely on Garvan's observation. We will modify the work of Atkin and Swinnerton-Dyer to the extent that we will eschew using their Lemma 2, which we state below.

Lemma 2.3.1. *Let $f(z)$ be a single-valued analytic function of z, except possibly for a finite number of poles, in every region $0 \le z_1 \le |z| \le z_2$; and suppose that for some constants A and w with $0 < |w| < 1$, and some (positive, zero, or negative) integer n, we have*

$$f(zw) = Az^n f(z)$$

identically in z. Then either $f(z)$ has exactly n more poles than zeros in

$$|w| \le |z| \le 1,$$

or $f(z)$ vanishes identically.

While this is a beautiful, powerful, and useful result, it is unlikely to have been the type of result that Ramanujan would have utilized.

The principal idea is to transform (2.1.16), (2.1.18), and (2.1.19) into certain bilateral series, which are called higher-level Appell series [355]. In particular, see Lemma 2.4.1 and the functions (2.1.33) and (2.3.11), which we define and develop in the next several pages.

The next identity does not appear in the lost notebook. However, it is effectively a generalization of Entries 12.4.4 (as restated in (12.4.15)) and 12.5.3 (as restated in (12.5.14)) in our first book [15, pp. 276, 283]. Consequently, it is a partial fraction decomposition of precisely the sort that Ramanujan often considered.

Lemma 2.3.2. [28, p. 94, Lemma 7] *For $\Sigma(z,\zeta,q)$ defined by* (2.1.33),

$$\begin{aligned}\zeta^3\Sigma(z\zeta,\zeta^3,q)+\Sigma(z\zeta^{-1},\zeta^{-3},q)-\zeta\frac{(\zeta^2,q/\zeta^2;q)_\infty}{(\zeta,q/\zeta;q)_\infty}\Sigma(z,1,q)&\\ =\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_\infty}.&\qquad(2.3.1)\end{aligned}$$

This formula was first proved by G.N. Watson [335], and we shall follow his proof. M. Jackson [185] has given a third proof from the theory of q-hypergeometric series, and S.H. Chan [105] has established a considerable generalization of Lemma 2.3.2.

Proof. Let us fix a positive integer N and consider the partial fraction decomposition with respect to z of the rational function

$$F_N(z):=\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_N}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_N}.\qquad(2.3.2)$$

This function has simple poles at $z=\zeta q^m$, q^m, and $\zeta^{-1}q^m$ for $-(N-1)\le m\le N$. Hence, we see that

$$F_N(z):=\sum_{m=-N}^{N-1}\frac{A_m(N)}{1-z\zeta q^m}+\sum_{m=-N}^{N-1}\frac{B_m(N)}{1-zq^m/\zeta}+\sum_{m=-N}^{N-1}\frac{C_m(N)}{1-zq^m}.\qquad(2.3.3)$$

Now for any integer m, algebraic simplification reveals that

$$(xq^{-m},q^{1+m}/x;q)_N=(-1)^m q^{-m(m+1)/2}x^m(q/x;q)_{N+m}(x;q)_{N-m}.\qquad(2.3.4)$$

First, after three applications of (2.3.4), with $x=\zeta^{-2},\zeta^{-1},1$, respectively, we find that

$$\begin{aligned}A_m(N)=\lim_{z\to\zeta^{-1}q^{-m}}(1-z\zeta q^m)F_N(z)&=\frac{(-1)^m q^{3m(m+1)/2}\zeta^{3m+3}}{(q/\zeta^2;q)_{N-m-1}(\zeta^2;q)_{N+m+1}}\\ &\times\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_N}{(q/\zeta;q)_{N-m-1}(\zeta;q)_{N+m+1}(q;q)_{N-m-1}(q;q)_{N+m}},\qquad(2.3.5)\end{aligned}$$

and

$$\lim_{N\to\infty} A_m(N) = (-1)^m q^{3m(m+1)/2}\zeta^{3m+3}. \tag{2.3.6}$$

Second, applying (2.3.4) three times once again, but now with $x = 1, \zeta, \zeta^2$, respectively, we find that

$$\begin{aligned} B_m(N) &= \lim_{z\to\zeta q^{-m}} (1 - z\zeta^{-1}q^m)F_N(z) \qquad (2.3.7)\\ &= \frac{(-1)^m q^{3m(m+1)/2}\zeta^{-3m}(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_N}{(\zeta; q)_{N-m}(q/\zeta; q)_{N+m}(\zeta^2; q)_{N-m}(q/\zeta^2; q)_{N+m}(q; q)_{N-m-1}(q; q)_{N+m}}, \end{aligned}$$

and

$$\lim_{N\to\infty} B_m(N) = (-1)^m q^{3m(m+1)/2}\zeta^{-3m}. \tag{2.3.8}$$

Third, applying (2.3.4) with $x = \zeta^{-1}, 1, \zeta$, respectively, we find that

$$\begin{aligned} C_m(N) &= \lim_{z\to q^{-m}} (1 - zq^m)F_N(z) \qquad (2.3.9)\\ &= \frac{-\zeta(-1)^m q^{3m(m+1)/2}(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_N}{(\zeta; q)_{N-m}(q/\zeta; q)_{N+m}(\zeta; q)_{N+m+1}(q/\zeta; q)_{N-m-1}(q; q)_{N-m-1}(q; q)_{N+m}}, \end{aligned}$$

and

$$\lim_{N\to\infty} C_m(N) = \frac{-\zeta(\zeta^2, q/\zeta^2; q)_\infty(-1)^m q^{3m(m+1)/2}}{(\zeta, q/\zeta; q)_\infty}. \tag{2.3.10}$$

We can now easily deduce (2.3.1). Clearly $F_N(z)$ converges uniformly to the right-hand side of (2.3.1) as $N \to \infty$.

Equations (2.3.6), (2.3.8), and (2.3.10) when applied to (2.3.3) yield the left-hand side of (2.3.1), provided we are allowed to take the limit $N \to \infty$ inside the summation signs, and indeed this interchange of limit and summation is legitimate because the convergence is uniformly independent of m, and the resulting series, after letting $N \to \infty$, is convergent as long as $|q| < 1$ and z is restricted away from the poles. Thus (2.3.1) is proved. □

Following Atkin and Swinnerton-Dyer [28, p. 96], we now define

$$\begin{aligned} g(z, q) &:= z\frac{(z^2, q/z^2; q)_\infty}{(z, q/z; q)_\infty}\Sigma(z, 1, q) - z^3\Sigma(z^2, z^3, q) \qquad (2.3.11)\\ &\quad - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^n z^{-3n} q^{3n(n+1)/2}}{1 - q^n}. \qquad (2.3.12) \end{aligned}$$

Now the definition of $g(z, q)$ is motivated as follows. We would like to set $\zeta = z$ in (2.3.1); however, this would produce an undefined term at $n = 0$ in $\sum(1, z^{-3}, q)$ in (2.3.1). Note that $g(z, q)$ is the negative of the left-hand side of (2.3.1), with $\zeta = z$ and the one offending term at $n = 0$ in $\sum(1, z^{-3}, q)$ removed. Thus,

$$g(z,q)=\lim_{\zeta\to z}\left(\frac{1}{1-z/\zeta}-\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_\infty}\right). \tag{2.3.13}$$

It is now a straightforward exercise to prove the next lemma, which is the second half of Lemma 8 in [28, p. 96].

Lemma 2.3.3. *We have*

$$g(z,q)+g(q/z,q)=1. \tag{2.3.14}$$

Proof. We proceed as follows:

$$\begin{aligned}
g(z,q)+g(q/z,q)&=\lim_{\zeta\to z}\left(\frac{1}{1-z/\zeta}-\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_\infty}\right)\\
&\quad+\lim_{\zeta\to q/z}\left(\frac{1}{1-q/(z\zeta)}-\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(q/(z\zeta),z\zeta,q/z,z,q\zeta/z,z/\zeta;q)_\infty}\right)\\
&=\lim_{\zeta\to z}\left(\frac{1}{1-z/\zeta}-\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_\infty}\right.\\
&\quad\left.+\frac{1}{1-\zeta/z}+\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_\infty}\right)\\
&=\lim_{\zeta\to z}\left(\frac{1}{1-z/\zeta}+\frac{1}{1-\zeta/z}\right)\\
&=1,
\end{aligned}$$

where in the antepenultimate line we replaced ζ by q/ζ in the second limit and algebraically simplified the second infinite product into the first product with opposite sign. This then completes the proof of (2.3.14). □

Our next objective is to establish a second component of Lemma 8 of Atkin and Swinnerton-Dyer [28, p. 96].

Lemma 2.3.4. *We have*

$$g(z,q)+g(z^{-1},q)=-2. \tag{2.3.15}$$

Proof. Replacing ζ by $1/\zeta$ in the second equality below, we find that

$$\begin{aligned}
&g(z,q)+g(z^{-1},q)\\
&=\lim_{\zeta\to z}\left(\frac{1}{1-z/\zeta}+\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_\infty}\right)\\
&\quad+\lim_{\zeta\to 1/z}\left(\frac{1}{1-1/(z\zeta)}-\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(1/(z\zeta),q\zeta z,1/z,qz,\zeta/z,qz/\zeta;q)_\infty}\right)\\
&=\lim_{\zeta\to z}\left(\frac{1}{1-z/\zeta}-\frac{(\zeta,q/\zeta,\zeta^2,q/\zeta^2,q,q;q)_\infty}{(z/\zeta,q\zeta/z,z,q/z,z\zeta,q/(z\zeta);q)_\infty}\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1}{1-\zeta/z} - \frac{(1/\zeta, \zeta q, 1/\zeta^2, \zeta^2 q, q, q; q)_\infty}{(\zeta/z, qz/\zeta, 1/z, qz, 1/(\zeta z), qz\zeta; q)_\infty}\Bigg) \\
&= \lim_{\zeta\to z}\Bigg(1 - \frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty} \\
&\quad + \frac{z^3}{\zeta^3}\frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(z/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty}\Bigg) \\
&= \lim_{\zeta\to z}\left(1 - \frac{(1-z^3/\zeta^3)}{(1-z/\zeta)}\frac{(\zeta, q/\zeta, \zeta^2, q/\zeta^2, q, q; q)_\infty}{(qz/\zeta, q\zeta/z, z, q/z, z\zeta, q/(z\zeta); q)_\infty}\right) \\
&= 1 - \lim_{\zeta\to z}\frac{(1-z^3/\zeta^3)}{(1-z/\zeta)} \\
&= 1 - \lim_{\zeta\to z}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2}\right) \\
&= 1 - 3 = -2,
\end{aligned}
$$

as desired to prove (2.3.15). □

As an immediate consequence of (2.3.15), we deduce the next corollary.

Corollary 2.3.1. *With $g(z, q)$ defined by (2.3.11),*

$$g(z, q) - g(zq, q) = -3.$$

Proof. By (2.3.15) and (2.3.14),

$$
\begin{aligned}
g(z, q) - g(zq, q) &= \big(g(z, q) + g(z^{-1}, q)\big) - \big(g(z^{-1}, q) + g(zq, q)\big) \\
&= -2 - 1 = -3, \qquad (2.3.16)
\end{aligned}
$$

as desired to prove (2.3.15). □

We shall now prove the other identity that occurs in Lemma 8 of [28].

Lemma 2.3.5. *If $g(z, q)$ is defined by (2.3.11), then*

$$2g(z, q) - g(z^2, q) - \frac{\left(z^3, \frac{q}{z^3}, q; q\right)_\infty^2}{\left(z, \frac{q}{z}; q\right)_\infty^2 \left(z^4, \frac{q}{z^4}; q\right)_\infty} + 1 = 0. \qquad (2.3.17)$$

Proof. To prove (2.3.17), we denote the left-hand side of (2.3.17) by $f(z)$. Then, by Corollary 2.3.1,

$$
\begin{aligned}
f(z) - f(zq) &= 2\big(g(z, q) - g(zq, q)\big) \\
&\quad - \Big\{\big(g(z^2, q) - g(z^2q, q)\big) + \big(g(z^2q, q) - g(z^2q^2, q)\big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\left(z^3, \frac{q}{z^3}, q; q\right)_\infty^2}{\left(z, \frac{q}{z}; q\right)_\infty^2 \left(z^4, \frac{q}{z^4}; q\right)_\infty} + \frac{\left(z^3q^3, \frac{1}{z^3q^2}, q; q\right)_\infty^2}{\left(zq, \frac{1}{z}; q\right)_\infty^2 \left(z^4q^4, \frac{1}{z^4q^3}; q\right)_\infty} \\
&= 2(-3) - (-3-3) \\
&- \frac{\left(z^3, \frac{q}{z^3}, q; q\right)_\infty^2}{\left(z, \frac{q}{z}; q\right)_\infty^2 \left(z^4, \frac{q}{z^4}; q\right)_\infty} + \frac{\left(z^3, \frac{q}{z^3}, q; q\right)_\infty^2}{\left(z, \frac{q}{z}; q\right)_\infty^2 \left(z^4, \frac{q}{z^4}; q\right)_\infty} \\
&= 0. \qquad (2.3.18)
\end{aligned}
$$

Next, we show that if $\omega = e^{2\pi i/3}$, then

$$
g(\omega, q) = -1. \qquad (2.3.19)
$$

By (2.3.11), we see that

$$
\begin{aligned}
g(\omega, q) &= -\sum(\omega, 1, q) - \sum(\omega^2, 1, q) - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1-q^n} \\
&= -\frac{1}{1-\omega} - \frac{1}{1-\omega^2} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n q^{3n(n+1)/2} \\
&\quad \times \left(\frac{1}{1-\omega q^n} + \frac{1}{1-\omega^2 q^n} + \frac{1}{1-q^n} \right) \\
&= -1 - 3 \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1-q^{3n}} \\
&= -1,
\end{aligned}
$$

because the nth and $(-n)th$ terms of the sum cancel. This proves (2.3.19).

Substituting $z = \omega$ on the left-hand side of (2.3.17), invoking (2.3.19), and observing that $g(\omega^2, q) = -1$ as well, we see that

$$
f(\omega) = -2 + 1 - 0 + 1 = 0. \qquad (2.3.20)
$$

Observe that (2.3.18) and (2.3.20) imply that

$$
f(\omega q^n) = 0 \quad \text{for} \quad n \geq 0.
$$

Therefore we need to prove that $f(z)$ is analytic except possibly at $z = 0, \infty$. However, the functional equation (2.3.18), namely

$$
f(z) = f(qz),
$$

means that we need to examine the possible poles only in the annulus $|q| < |z| \leq 1$. Potential poles occur at $z = \pm 1, \pm i, \pm q^{1/4}, \pm iq^{1/4}$ and are at worst simple poles. However, when we return to the definition of g in (2.3.11) to calculate the residue at each possible pole, we find that it is 0. Consequently, $f(z)$ is analytic except possibly at 0 and at ∞. However, $f(z)$ must, in fact, be analytic at $z = 0$ also, because all values of $f(z)$ in a deleted neighborhood of 0 are bounded by the maximum value of $|f(z)|$ in the annulus $|q| < |z| \leq 1$, owing to the functional equation above, and if $f(z)$ had a singularity at $z = 0$ (either a pole or an essential singularity), then it would have to be unbounded in a neighborhood of $z = 0$.

So we have shown that $f(z)$ is analytic for $|z| < 1$ and that $f(z)$ is identically 0 on a sequence of points ωq^n that converge in the interior of $|z| < 1$. We conclude that

$$f(z) \equiv 0,$$

and (2.3.17) is established. □

2.4 Proof of Entry 2.1.2

Let

$$S_5(b) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{bn+n(3n+1)/2}}{1-q^{5n}}. \tag{2.4.1}$$

Replacing n by $-n$, we see that

$$S_5(b) = -S_5(4-b), \tag{2.4.2}$$

from which it readily follows that

$$S_5(2) = 0. \tag{2.4.3}$$

Furthermore, either applying the Jacobi triple product identity (2.1.3) and algebraic simplification or applying (2.1.4) with $n = b/3$, $(b-1)/3$, and $(b+1)/3$, respectively, we find that

$$\begin{aligned} S_5(b) - S_5(b+5) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{bn+n(3n+1)/2} - 1 = f(-q^{2+b}, -q^{1-b}) - 1 \\ &= \begin{cases} (-1)^b q^{-b(b+1)/6}(q;q)_\infty - 1, & \text{if } b \equiv 0 \pmod 3, \\ -1, & \text{if } b \equiv 1 \pmod 3, \\ (-1)^{b-1} q^{-b(b+1)/6}(q;q)_\infty - 1, & \text{if } b \equiv 2 \pmod 3. \end{cases} \end{aligned} \tag{2.4.4}$$

We now establish the relationship between $S_5(b)$ and Entry 2.1.2.

Lemma 2.4.1. *We have*

$$(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(q/z)_n} = (1-z) \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n}. \tag{2.4.5}$$

Proof. Recall that Entry 4.2.16 of [16, p. 89] is given by

$$\begin{aligned}(abq)_\infty &\sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2}}{(-aq)_n(-bq)_n} \\ &= 1 + (1+a)(1+b) \sum_{n=1}^{\infty} \frac{(-1)^n (abq)_{n-1}(1-abq^{2n})a^n b^n q^{n(3n+1)/2}}{(q)_n(1+aq^n)(1+bq^n)}.\end{aligned}$$

Setting $a = 1/b = -z$ above, we find that

$$\begin{aligned}(q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq)_n(q/z)_n} &= 1 + (1-z)(1-1/z) \sum_{n=1}^{\infty} \frac{(-1)^n(1+q^n)q^{n(3n+1)/2}}{(1-zq^n)(1-q^n/z)} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}\{(1-z)(1-q^n/z)+(1-1/z)(1-zq^n)\}}{(1-zq^n)(1-q^n/z)} \\
&= 1 + (1-z)\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n} - \frac{1-z}{z}\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-q^n/z} \\
&= 1 + (1-z)\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n} + (1-z)\sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n-1)/2}}{1-zq^{-n}} \\
&= (1-z)\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-zq^n},\end{aligned}$$

which establishes (2.4.5). □

Lemma 2.4.1 gives a connection with a variant of the rank-generating function (2.1.41), namely

$$R_b(q) := \sum_{n=0}^{\infty} \frac{q^{7n^2}}{(q^b;q^7)_{n+1}(q^{7-b};q^7)_n}, \tag{2.4.6}$$

and the functions (2.1.33) and (2.1.36). By (2.4.5) and (2.4.6), with $z = q^{7b}$,

$$\begin{aligned}R_b(q^7) &= \sum_{n=0}^{\infty} \frac{q^{49n^2}}{(q^{7b};q^{49})_{n+1}(q^{49-7b};q^{49})_n} \\
&= \frac{1}{(q^{49};q^{49})_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{49n(3n+1)/2}}{1-q^{49n+7b}} \\
&= \frac{\Sigma_7(b,-7)}{P_7(0)},\end{aligned}$$

by (2.1.33) and (2.1.36).

Lemma 2.4.2. *Let ζ_5 be a primitive fifth root of unity. Then*

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n(\zeta_5^{-1}q;q)_n} = 1 + \bigl(S_5(1) - 2S_5(4)\bigr) + (\zeta_5+\zeta_5^{-1})\bigl(2S_5(1)+S_5(4)\bigr). \quad (2.4.7)$$

Proof. By (2.4.5), and then by (2.4.2) and (2.4.3),

$$\begin{aligned}
&(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n(\zeta_5^{-1}q;q)_n}\\
&= 1 + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1-\zeta_5)(1-q^n)(1-\zeta_5^2q^n)(1-\zeta_5^3q^n)(1-\zeta_5^4q^n)}{1-q^{5n}}\\
&= 1 + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-q^{5n}}\Bigl((1-q^{4n}) + (q^n-1)\zeta_5 + (q^{2n}-q^n)\zeta_5^2\\
&\quad + (q^{3n}-q^{2n})\zeta_5^3 + (q^{4n}-q^{3n})\zeta_5^4\Bigr)\\
&= 1 + S_5(0)(1-\zeta_5) + S_5(1)(\zeta_5-\zeta_5^2) + S_5(2)(\zeta_5^2-\zeta_5^3) + S_5(3)(\zeta_5^3-\zeta_5^4)\\
&\quad + S_5(4)(\zeta_5^4-1)\\
&= 1 + (\zeta_5+\zeta_5^{-1}-2)S_5(4) + (\zeta_5+\zeta_5^{-1}-\zeta_5^2-\zeta_5^{-2})S_5(1)\\
&= 1 - 2S_5(4) + (\zeta_5+\zeta_5^{-1})\bigl(S_5(4)+S_5(1)\bigr) - (\zeta_5^2+\zeta_5^{-2})S_5(1)\\
&= 1 + \bigl(S_5(1)-2S_5(4)\bigr) + (\zeta_5+\zeta_5^{-1})\bigl(2S_5(1)+S_5(4)\bigr).
\end{aligned}$$

□

Lemma 2.4.3. *Recall that $\phi(q)$ and $\psi(q)$ are defined in (2.1.18) and (2.1.19), respectively, and that $\sum(z,\zeta,q)$ is defined in (2.1.33). Then*

$$\phi(q) = \frac{q}{(q^5;q^5)_\infty}\Sigma(q,1,q^5), \quad (2.4.8)$$

$$\frac{\psi(q)}{q} = \frac{q}{(q^5;q^5)_\infty}\Sigma(q^2,1,q^5). \quad (2.4.9)$$

Proof. To prove (2.4.8), we apply (2.4.5) with q replaced by q^5 and then $z=q$. Then dividing both sides by $1-q$ and using (2.1.18), we find that

$$(q^5;q^5)_\infty\{\phi(q)+1\} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(3n+1)/2}}{1-q^{5n+1}}.$$

Subtracting $(q^5;q^5)_\infty$ from both sides above and using the pentagonal number theorem, we find that

$$
\begin{aligned}
(q^5;q^5)_\infty \phi(q) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(3n+1)/2}}{1-q^{5n+1}} - \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(3n+1)/2} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{5n(3n+1)/2} \left(\frac{1}{1-q^{5n+1}} - 1 \right) \\
&= q \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{15n(n+1)/2}}{1-q^{5n+1}} \\
&= q\Sigma(q,1,q^5),
\end{aligned}
$$

by (2.1.33).

To obtain (2.4.9), we apply (2.4.5) with q replaced by q^5 and then $z = q^2$. Now proceed with the same steps as in the foregoing proof, but now using (2.1.19) instead of (2.1.18), and we deduce (2.4.9). ☐

To simplify further considerations, we make the following conventions:

$$P_5(a) := (q^{5a}, q^{25-5a}; q^{25})_\infty \qquad (a \neq 0), \tag{2.4.10}$$

$$P_5(0) := (q^{25}; q^{25})_\infty, \tag{2.4.11}$$

$$\Sigma_5(a,b) := \Sigma(q^{5a}, q^{5b}, q^{25}) \qquad (a \neq 0), \tag{2.4.12}$$

$$\Sigma_5(0,b) := \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{75n(n+1)/2+5bn}}{1-q^{25n}}, \tag{2.4.13}$$

$$
\begin{aligned}
g_5(a) &:= g(q^{5a}, q^{25}) \qquad (2.4.14)\\
&= \frac{q^{5a} P_5(2a)}{P_5(a)} \Sigma_5(a,0) - q^{15a}\Sigma_5(2a,3a) - \Sigma_5(0,-3a),
\end{aligned}
$$

by (2.3.11). We note in passing that the Rogers–Ramanujan identities (2.1.11) and (2.1.12) can be written in the forms

$$P_5(1) = \frac{1}{G(q^5)}, \tag{2.4.15}$$

$$P_5(2) = \frac{1}{H(q^5)}. \tag{2.4.16}$$

Lemma 2.4.4. *If $S_5(b)$ is defined by (2.4.1), then*

$$S_5(1) = -g_5(2) - q^8 \frac{\Sigma_5(2,0)}{P_5(0)} (q;q)_\infty - q^3 \frac{P_5^2(0)}{P_5(2)}. \tag{2.4.17}$$

Proof. We begin by dissecting the series for $S_5(1)$ modulo 5. By (2.4.1),

$$S_5(1) = \sum_{b=0}^{4} \sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty} \frac{(-1)^{m+b} q^{(5m+b)(15m+3b+1)/2+5m+b}}{1-q^{25m+5b}} \tag{2.4.18}$$

$$
\begin{aligned}
&= \sum_{b=0}^{4}(-1)^b q^{3b(b+1)/2} \sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty} \frac{(-1)^m q^{75m(m+1)/2+5m(3b-6)}}{1-q^{25m+5b}} \\
&= \Sigma_5(0,-6) - q^3\Sigma_5(1,-3) + q^9\Sigma_5(2,0) - q^{18}\Sigma_5(3,3) + q^{30}\Sigma_5(4,6) \\
&= q^9\Sigma_5(2,0) + \Sigma_5(0,-6) + q^{30}\Sigma_5(4,6) - q^3\big(q^{15}\Sigma_5(3,3) + \Sigma_5(1,-3)\big).
\end{aligned}
$$

Now, by (2.4.14),

$$
\begin{aligned}
g_5(2) &= g_5(q^{10}, q^{25}) \\
&= \frac{q^{10}P_5(4)}{P_5(2)}\Sigma_5(2,0) - q^{30}\Sigma_5(4,6) - \Sigma_5(0,-6),
\end{aligned}
$$

and by Lemma 2.3.2 with q replaced by q^{25}, $\zeta = q^5$, and $z = q^{10}$, we find that

$$
q^{15}\Sigma_5(3,3) + \Sigma_5(1,-3) = q^5\frac{P_5(2)}{P_5(1)}\Sigma_5(2,0) + \frac{P_5^2(0)P_5(1)P_5(2)}{P_5(3)P_5(2)P_5(1)}.
$$

Therefore, by (2.4.18), the last two equalities, (2.4.15), and (2.4.16),

$$
\begin{aligned}
S_5(1) &= -g_5(2) + \frac{q^{10}P_5(4)}{P_5(2)}\Sigma_5(2,0) + q^9\Sigma_5(2,0) \\
&\quad - q^3\left(\frac{q^5P_5(2)}{P_5(1)}\Sigma_5(2,0) + \frac{P_5^2(0)}{P_5(3)}\right) \\
&= -g_5(2) - q^3\Sigma_5(2,0)\left(\frac{G(q^5)}{H(q^5)}q^5 - q^6 - q^7\frac{H(q^5)}{G(q^5)}\right) - q^3\frac{P_5^2(0)}{P_5(2)} \\
&= -g_5(2) - q^8\frac{\Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - q^3\frac{P_5^2(0)}{P_5(2)},
\end{aligned}
$$

by (2.2.2). □

Lemma 2.4.5. *For $S_5(b)$ defined by (2.4.1), we have*

$$
S_5(4) = -g_5(1) + \frac{q^5\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} + q^2\frac{P_5^2(0)}{P_5(1)}. \tag{2.4.19}
$$

Proof. By (2.4.1),

$$
\begin{aligned}
S_5(4) &= \sum_{b=0}^{4} \sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty} \frac{(-1)^{m+b}q^{(5m+b)(15m+3b+1)/2+(20m+4b)}}{1-q^{25m+5b}} \\
&= \sum_{b=0}^{4}(-1)^b q^{3b(b+3)/2} \sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty} \frac{(-1)^m q^{75m(m+1)/2+5(3b-3)m}}{1-q^{25m+5b}} \\
&= \Sigma_5(0,-3) - q^6\Sigma_5(1,0) + q^{15}\Sigma_5(2,3) - q^{27}\Sigma_5(3,6) + q^{42}\Sigma_5(4,9)
\end{aligned}
$$

$$
\begin{aligned}
&= \Sigma_5(0,-3) + q^{15}\Sigma_5(2,3) - q^6\Sigma_5(1,0) \\
&\quad - q^{-3}(q^{30}\Sigma_5(3,6) + \Sigma_5(-1,-6)),
\end{aligned} \tag{2.4.20}
$$

where we have replaced m by $m-1$ in the sum for $\Sigma_5(4,9)$. Now, by (2.4.14),

$$
g_5(1) = \frac{q^5 P_5(2)}{P_5(1)}\Sigma_5(1,0) - q^{15}\Sigma_5(2,3) - \Sigma_5(0,-3), \tag{2.4.21}
$$

and by Lemma 2.3.2 with q replaced by q^{25}, $\zeta = q^{10}$, and $z = q^5$,

$$
q^{30}\Sigma_5(3,6) + \Sigma_5(-1,-6) - q^{10}\frac{P_5(4)}{P_5(2)}\Sigma_5(1,0) - \frac{P_5^2(0)P_5(2)P_5(4)}{P_5(3)P_5(1)P_5(-1)} = 0. \tag{2.4.22}
$$

Therefore, by (2.4.20), (2.4.21), (2.4.22), (2.4.15), and (2.4.16),

$$
\begin{aligned}
S_5(4) &= -g_5(1) + q^5\frac{P_5(2)}{P_5(1)}\Sigma_5(1,0) - q^6\Sigma_5(1,0) \\
&\quad - q^{-3}\left(\frac{q^{10}P_5(4)}{P_5(2)}\Sigma_5(1,0) + \frac{P_5^2(0)P_5(4)}{P_5(1)P_5(-1)}\right) \\
&= -g_5(1) + q^5\Sigma_5(1,0)\left(\frac{G(q^5)}{H(q^5)} - q - q^2\frac{H(q^5)}{G(q^5)}\right) + q^2\frac{P_5^2(0)}{P_5(1)} \\
&= -g_5(1) + q^5\frac{\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} + q^2\frac{P_5^2(0)}{P_5(1)},
\end{aligned}
$$

by (2.2.2). ∎

Lemma 2.4.6. *Recall that $S_5(b)$ is defined by (2.4.1). Then*

$$
\begin{aligned}
&1 + S_5(1) - 2S_5(4) \\
&= \left\{\frac{P_5(0)P_5(2)}{P_5^2(1)} - 2q^5\frac{\Sigma_5(1,0)}{P_5(0)} + q\frac{P_5(0)}{P_5(1)} - q^8\frac{\Sigma_5(2,0)}{P_5(0)}\right\}(q;q)_\infty.
\end{aligned} \tag{2.4.23}
$$

Proof. By Lemmas 2.4.4 and 2.4.5,

$$
\begin{aligned}
1 + S_5(1) - 2S_5(4) &= 1 - g_5(2) - \frac{q^8\Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - \frac{q^3P_5^2(0)}{P_5(2)} \\
&\quad + 2g_5(1) - \frac{2q^5\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} - \frac{2q^2P_5^2(0)}{P_5(1)} \\
&= \frac{P_5^2(0)P_5^2(2)}{P_5^3(1)} - \frac{q^8\Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - \frac{q^3P_5^2(0)}{P_5(2)} \\
&\quad - \frac{2q^5\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} - \frac{2q^2P_5^2(0)}{P_5(1)},
\end{aligned}
$$

by (2.3.17). Thus in order to establish (2.4.23), we need to show that

$$\frac{P_5^2(0)P_5^2(2)}{P_5^3(1)} - \frac{q^3P_5^2(0)}{P_5(2)} - \frac{2q^2P_5^2(0)}{P_5(1)} = \left\{\frac{P_5(0)P_5(2)}{P_5^2(1)} + q\frac{P_5(0)}{P_5(1)}\right\}(q;q)_\infty. \tag{2.4.24}$$

But by (2.4.15), (2.4.16), and (2.2.2), we see that

$$(q;q)_\infty = P_5(0)\left(\frac{P_5(2)}{P_5(1)} - q - \frac{q^2P_5(1)}{P_5(2)}\right). \tag{2.4.25}$$

Consequently, the right-hand side of (2.4.24) is equal to

$$\begin{aligned}
&\left\{\frac{P_5(0)P_5(2)}{P_5^2(1)} + q\frac{P_5(0)}{P_5(1)}\right\}P_5(0)\left\{\frac{P_5(2)}{P_5(1)} - q - \frac{q^2P_5(1)}{P_5(2)}\right\}\\
&\qquad = \frac{P_5^2(0)P_5^2(2)}{P_5^3(1)} + \frac{qP_5^2(0)P_5(2)}{P_5^2(1)} - \frac{qP_5^2(0)P_5(2)}{P_5^2(1)}\\
&\qquad\quad - \frac{q^2P_5^2(0)}{P_5(1)} - \frac{q^2P_5^2(0)}{P_5(1)} - \frac{q^3P_5^2(0)}{P_5(2)}\\
&\qquad = \frac{P_5^2(0)P_5^2(2)}{P_5^3(1)} - \frac{q^3P_5^2(0)}{P_5(2)} - \frac{2q^2P_5^2(0)}{P_5(1)}.
\end{aligned}$$

Thus (2.4.24) has been proved, and therefore (2.4.23) has also been proved. □

Lemma 2.4.7. *With $S_5(b)$ as given in the previous lemmas,*

$$\begin{aligned}
&2S_5(1) + S_5(4)\\
&= \left\{\frac{q^5\Sigma_5(1,0)}{P_5(0)} + \frac{q^2P_5(0)}{P_5(2)} - \frac{2q^8\Sigma_5(2,0)}{P_5(0)} - \frac{q^3P_5(0)P_5(1)}{P_5^2(2)}\right\}(q;q)_\infty.
\end{aligned} \tag{2.4.26}$$

Proof. By Lemmas 2.3.3 and 2.3.5, we find that

$$2g_5(2) + g_5(1) = 2g_5(2) - g_5(4) + 1 = \frac{P_5^2(6)P_5^2(0)}{P_5^2(2)P_5(8)} = -\frac{q^5P_5^2(1)P_5^2(0)}{P_5^3(2)}. \tag{2.4.27}$$

Next, by Lemmas 2.4.4 and 2.4.5 and (2.4.27),

$$\begin{aligned}
2S_5(1) + S_5(4) &= -2g_5(2) - \frac{2q^8\Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - \frac{2q^3P_5^2(0)}{P_5(2)}\\
&\quad - g_5(1) + \frac{q^5\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} + \frac{q^2P_5^2(0)}{P_5(1)}\\
&= \frac{q^5P_5^2(0)P_5^2(1)}{P_5^3(2)} - \frac{2q^8\Sigma_5(2,0)(q;q)_\infty}{P_5(0)} - \frac{2q^3P_5^2(0)}{P_5(2)}\\
&\quad + \frac{q^5\Sigma_5(1,0)(q;q)_\infty}{P_5(0)} + \frac{q^2P_5^2(0)}{P_5(1)}.
\end{aligned}$$

Thus, in order to establish (2.4.26), we need to show that

$$\frac{q^5P_5^2(0)P_5^2(1)}{P_5^3(2)} - \frac{2q^3P_5^2(0)}{P_5(2)} + \frac{q^2P_5^2(0)}{P_5(1)} = \left\{\frac{q^2P_5(0)}{P_5(2)} - \frac{q^3P_5(0)P_5(1)}{P_5^2(2)}\right\}(q;q)_\infty. \tag{2.4.28}$$

Invoking (2.4.25), which is a restatement of (2.2.2), we see that the right-hand side of (2.4.28) is equal to

$$\begin{aligned}
&\left\{\frac{q^2P_5(0)}{P_5(2)} - \frac{q^3P_5(0)P_5(1)}{P_5^2(2)}\right\}P_5(0)\left\{\frac{P_5(2)}{P_5(1)} - q - q^2\frac{P_5(1)}{P_5(2)}\right\}\\
&\qquad = \frac{q^2P_5^2(0)}{P_5(1)} - \frac{q^3P_5^2(0)}{P_5(2)} - \frac{q^3P_5^2(0)}{P_5(2)} + \frac{q^4P_5^2(0)P_5(1)}{P_5^2(2)}\\
&\qquad\quad - \frac{q^4P_5^2(0)P_5(1)}{P_5^2(2)} + \frac{q^5P_5^2(0)P_5^2(1)}{P_5^3(2)}\\
&\qquad = \frac{q^2P_5^2(0)}{P_5(1)} - \frac{2q^3P_5^2(0)}{P_5(2)} + \frac{q^5P_5^2(0)P_5^2(1)}{P_5^3(2)}.
\end{aligned}$$

Thus (2.4.28) and therefore (2.4.26) have been proved. □

We are finally ready to put all this together.

Proof of Entry 2.1.2. We first note that by Lemma 2.4.3,

$$\phi(q^5) = \frac{q^5}{P_5(0)}\Sigma_5(1,0) \tag{2.4.29}$$

and

$$\frac{\psi(q^5)}{q^5} = \frac{q^5}{P_5(0)}\Sigma_5(2,0). \tag{2.4.30}$$

Now by Lemma 2.4.2,

$$\begin{aligned}
&(q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(\zeta_5q;q)_n(\zeta_5^{-1}q;q)_n}\\
&= 1 + S_5(1) - 2S_5(4) + (\zeta_5+\zeta_5^{-1})\big(2S_5(1)+S_5(4)\big)\\
&= (q;q)_\infty\left\{\frac{P_5(0)P_5(2)}{P_5^2(1)} - \frac{2q^5\Sigma_5(1,0)}{P_5(0)} + \frac{qP_5(0)}{P_5(1)} - \frac{q^8\Sigma_5(2,0)}{P_5(0)}\right.\\
&\quad\left. + (\zeta_5+\zeta_5^{-1})\left(\frac{q^5\Sigma_5(1,0)}{P_5(0)} + \frac{q^2P_5(0)}{P_5(2)} - \frac{2q^8\Sigma_5(2,0)}{P_5(0)} - \frac{q^3P_5(0)P_5(1)}{P_5^2(2)}\right)\right\},
\end{aligned}$$

by Lemmas 2.4.6 and 2.4.7. So, by (2.4.29), (2.4.30), (2.4.15), (2.4.16), and (2.1.7)–(2.1.10),

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_5 q;q)_n(\zeta_5^{-1}q;q)_n} \\
&= \frac{(q^{25};q^{25})_\infty G^2(q^5)}{H(q^5)} - 2\phi(q^5) + q\left(q^{25};q^{25}\right)_\infty G(q^5) - q^3\frac{\psi(q^5)}{q^5} + \left(\zeta_5+\zeta_5^{-1}\right) \\
&\quad \times \left(\phi(q^5) + q^2\left(q^{25};q^{25}\right)_\infty H(q^5) - 2q^3\frac{\psi(q^5)}{q^5} - \frac{q^3(q^{25};q^{25})_\infty H^2(q^5)}{G(q^5)}\right) \\
&= A(q^5) + \left(\zeta_5+\zeta_5^{-1}-2\right)\phi(q^5) + qB(q^5) + \left(\zeta_5+\zeta_5^{-1}\right)q^2C(q^5) \\
&\quad - q^3\left(\zeta_5+\zeta_5^{-1}\right)D(q^5) + \frac{\psi(q^5)}{q^5}q^3\left(-1-2\left(\zeta_5+\zeta_5^{-1}\right)\right) \\
&= A(q^5) + \left(\zeta_5+\zeta_5^{-1}-2\right)\phi(q^5) + qB(q^5) + \left(\zeta_5+\zeta_5^{-1}\right)q^2C(q^5) \\
&\quad - q^3\left(\zeta_5+\zeta_5^{-1}\right)D(q^5) + q^3\frac{\psi(q^5)}{q^5}\left(\zeta_5+\zeta_5^{-1}\right)\left(\zeta_5^2+\zeta_5^{-2}-2\right),
\end{aligned}$$

and this is the assertion made by Entry 2.1.2. □

The argument above is not in the Ramanujan tradition; however, we are unable to replace it with something more appropriate.

2.5 Proof of Entry 2.1.4

The proof here is more direct than that for Entry 2.1.1 in that we do not require an analogue of (2.2.2). Recalling that $F_7(q)$ is defined by (2.1.28), we find that

$$\begin{aligned}
F_7(q) &= \frac{(q;q)_\infty}{(\zeta_7 q;q)_\infty(\zeta_7^{-1}q;q)_\infty} \\
&= \frac{(q;q)_\infty^2(\zeta_7^2 q;q)_\infty(\zeta_7^{-2}q;q)_\infty(\zeta_7^3 q;q)_\infty(\zeta_7^{-3}q;q)_\infty}{(q;q)_\infty(\zeta_7 q;q)_\infty(\zeta_7^{-1};q)_\infty(\zeta_7^2 q;q)_\infty(\zeta_7^{-2}q,q)_\infty(\zeta_7^3 q;q)_\infty(\zeta_7^{-3}q;q)_\infty} \\
&= \frac{\displaystyle\sum_{n=-\infty}^{\infty}(-1)^n\zeta_7^{2n}q^{n(n-1)/2}\sum_{m=-\infty}^{\infty}(-1)^m\zeta_7^{3m}q^{m(m-1)/2}}{\left(1-\zeta_7^2\right)\left(1-\zeta_7^3\right)\left(q^7;q^7\right)_\infty},
\end{aligned} \tag{2.5.1}$$

by Jacobi's triple product identity (2.1.3). Now for any primitive seventh root of unity (as are each of ζ_7, ζ_7^2, and ζ_7^3),

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n\zeta_7^n q^{n(n-1)/2} \\
&= \sum_{\nu=-3}^{3}\sum_{m=-\infty}^{\infty}(-1)^{7m+\nu}\zeta_7^{7m+\nu}q^{(7m+\nu)(7m+\nu-1)/2}
\end{aligned}$$

$$= \sum_{\nu=-3}^{3} (-1)^\nu q^{\nu(\nu-1)/2} \zeta_7^\nu \sum_{m=-\infty}^{\infty} (-1)^m q^{7m\left(7m+(2\nu-1)\right)/2}$$
$$= \sum_{\nu=-3}^{3} (-1)^\nu q^{\nu(\nu-1)/2} \zeta_7^\nu f(-q^{21+7\nu}, -q^{28-7\nu}). \tag{2.5.2}$$

Now, by the Jacobi triple product identity (2.1.3) and the definitions (2.1.30)–(2.1.32),

$$\frac{f(-q^{21+7\nu}, -q^{28-7\nu})}{(q^7;q^7)_\infty} = \frac{\left(q^{7\nu+21}, q^{28-7\nu}, q^{49}; q^{49}\right)_\infty}{(q^7;q^7)_\infty} = \begin{cases} X(q^7), & \text{if } \nu = 0, 1, \\ Y(q^7), & \text{if } \nu = -1, 2, \\ Z(q^7), & \text{if } \nu = -2, 3, \\ 0, & \text{if } \nu = -3. \end{cases} \tag{2.5.3}$$

Therefore, by (2.5.2) and (2.5.3),

$$\sum_{n=-\infty}^{\infty} (-1)^n \zeta_7^n q^{n(n-1)/2}$$
$$= (1-\zeta_7)\, X(q^7) + q\left(-\zeta_7^{-1} + \zeta_7^2\right) Y(q^7) + q^3\left(-\zeta_7^3 + \zeta_7^{-2}\right) Z(q^7). \tag{2.5.4}$$

We now use this evaluation for both series on the far right side of (2.5.1). Hence,

$$\begin{aligned} F_7(q) &= \left\{(1-\zeta_7^2)X(q^7) + q(-\zeta_7^{-2} + \zeta_7^4)Y(q^7) + q^3\left(-\zeta_7^6 + \zeta_7^{-4}\right) Z(q^7)\right\} \\ &\quad \times \left\{(1-\zeta_7^3)X(q^7) + q\left(-\zeta_7^{-3} + \zeta_7^6\right) Y(q^7) + q^3\left(-\zeta_7^2 + \zeta_7\right) Z(q^7)\right\} \\ &\quad \times \frac{(q^7;q^7)_\infty}{(1-\zeta_7^2)(1-\zeta_7^3)} \\ &= (q^7;q^7)\Big\{X^2(q^7) + \left(\zeta_7 + \zeta_7^{-1} - 1\right) qX(q^7)Y(q^7) \\ &\quad + \left(\zeta_7^2 + \zeta_7^{-2}\right) q^2 Y^2(q^7) + \left(\zeta_7^3 + \zeta_7^{-3} + 1\right) q^3 X(q^7)Z(q^7) \\ &\quad - \left(\zeta_7 + \zeta_7^{-1}\right) q^4 Y(q^7)Z(q^7) - \left(\zeta_7^2 + \zeta_7^{-2} + 1\right) q^6 Z^2(q^7)\Big\}, \end{aligned}$$

which proves Entry 2.1.4.

2.6 Proof of Entry 2.1.5

Let

$$S_7(b) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{bn+n(3n+1)/2}}{1-q^{7n}}. \tag{2.6.1}$$

Replacing n by $-n$, we see that

$$S_7(b) = -S_7(6-b), \tag{2.6.2}$$

from which an immediate consequence is

$$S_7(3) = 0. \tag{2.6.3}$$

Furthermore,

$$\begin{aligned} S_7(b) - S_7(b+7) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{bn+n(3n+1)/2} - 1 = f(-q^{2+b}, -q^{1-b}) - 1 \\ &= \begin{cases} (-1)^b q^{-b(b+1)/6}(q;q)_\infty - 1, & \text{if } b \equiv 0 \pmod 3, \\ -1, & \text{if } b \equiv 1 \pmod 3, \\ (-1)^{b-1} q^{-b(b+1)/6}(q;q)_\infty - 1, & \text{if } b \equiv 2 \pmod 3, \end{cases} \end{aligned} \tag{2.6.4}$$

as we have previously observed in (2.4.4).

Referring to (2.4.5), we are able to prove the following.

Lemma 2.6.1. *If ζ_7 be a primitive seventh root of unity, then*

$$\begin{aligned} (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n(\zeta_7^{-1}q;q)_n} &= 1 - S_7(4) + 2S_7(0) + \left(\zeta_7 + \zeta_7^{-1}\right)(S_7(1) - S_7(4) - S_7(0)) \\ &\quad + \left(\zeta_7^2 + \zeta_7^{-2}\right)(-S_7(1) - 2S_7(4)). \end{aligned} \tag{2.6.5}$$

Proof. Invoking (2.4.5), (2.6.2), and (2.6.3), we find that

$$\begin{aligned} &(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n(\zeta_7^{-1}q;q)_n} \\ &= 1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^n q^{n(3n+1)/2} \\ &\quad \times \frac{(1-\zeta_7)(1-q^n)(1-\zeta_7^2q^n)(1-\zeta_7^3q^n)(1-\zeta_7^4q^n)(1-\zeta_7^5q^n)(1-\zeta_7^6q^n)}{1-q^{7n}} \\ &= 1 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-q^{7n}} \Big((1-q^{6n}) + (q^n-1)\zeta_7 + q^n(q^n-1)\zeta_7^2 \\ &\quad + q^{2n}(q^n-1)\zeta_7^3 + q^{3n}(q^n-1)\zeta_7^4 + q^{4n}(q^n-1)\zeta_7^5 + q^{5n}(q^n-1)\zeta_7^6 \Big) \\ &= 1 + (1-\zeta_7)S_7(0) + (\zeta_7-\zeta_7^2)S_7(1) + (\zeta_7^2-\zeta_7^3)S_7(2) + (\zeta_7^3-\zeta_7^4)S_7(3) \\ &\quad + (\zeta_7^4-\zeta_7^5)S_7(4) + (\zeta_7^5-\zeta_7^6)S_7(5) + (\zeta_7^6-1)S_7(6) \\ &= 1 + S_7(0)\left(2-\zeta_7-\zeta_7^{-1}\right) + \left(\zeta_7+\zeta_7^{-1}-\zeta_7^2-\zeta_7^{-2}\right)S_7(1) \end{aligned}$$

$$
\begin{aligned}
&+\left(\zeta_7^3+\zeta_7^{-3}-\zeta_7^2-\zeta_7^{-2}\right)S_7(4)\\
&=1-S_7(4)+2S_7(0)+\left(\zeta_7+\zeta_7^{-1}\right)\left(S_7(1)-S_7(4)-S_7(0)\right)\\
&\quad+\left(\zeta_7^2+\zeta_7^{-2}\right)\left(-S_7(1)-2S_7(4)\right).
\end{aligned}
$$

□

Recalling the notation (2.1.34)–(2.1.37), we define

$$
\begin{aligned}
g_7(a) &:= g(q^{7a},q^{49}) \qquad (2.6.6)\\
&= \frac{q^{7a}P_7(2a)}{P_7(a)}\Sigma_7(a,0)-q^{21a}\Sigma_7(2a,3a)-\Sigma_7(0,-3a),
\end{aligned}
$$

by (2.3.11).

Lemma 2.6.2. *With $S_7(b)$ defined by* (2.6.1),

$$
S_7(1)=-g_7(3)+q^{16}\frac{\Sigma_7(3,0)}{P_7(0)}(q;q)_\infty+\frac{q^9P_7^2(0)P_7(1)}{P_7^2(3)}-\frac{q^3P_7^2(0)}{P_7(1)}. \qquad (2.6.7)
$$

Proof. We begin by dissecting the series for $S_7(1)$ modulo 7. To that end,

$$
\begin{aligned}
S_7(1)&=\sum_{b=0}^{6}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty}\frac{(-1)^{m+b}q^{(7m+b)(21m+3b+1)/2+7m+b}}{1-q^{49m+7b}}\\
&=\sum_{b=0}^{6}(-1)^bq^{3b(b+1)/2}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty}\frac{(-1)^mq^{147m(m+1)/2+7m(3b-9)}}{1-q^{49m+7b}}\\
&=\Sigma_7(0,-9)-q^3\Sigma_7(1,-6)+q^9\Sigma_7(2,-3)-q^{18}\Sigma_7(3,0)+q^{30}\Sigma_7(4,3)\\
&\quad-q^{45}\Sigma_7(5,6)+q^{63}\Sigma_7(6,9). \qquad (2.6.8)
\end{aligned}
$$

Now, by (2.6.6),

$$
g_7(3)-\frac{q^{21}P_7(6)}{P_7(3)}\Sigma_7(3,0)=-q^{63}\Sigma_7(6,9)-\Sigma_7(0,-9). \qquad (2.6.9)
$$

In addition, by Lemma 2.3.2, with q replaced by q^{49}, $\zeta=q^{14}$, and $z=q^{21}$,

$$
q^{42}\Sigma_7(5,6)+\Sigma_7(1,-6)=q^{14}\frac{P_7(4)}{P_7(2)}\Sigma_7(3,0)+\frac{P_7^2(0)P_7(2)P_7(4)}{P_7(5)P_7(3)P_7(1)}. \qquad (2.6.10)
$$

By Lemma 2.3.2 with q replaced by q^{49}, $\zeta=q^7$, and $z=q^{21}$,

$$
q^{21}\Sigma_7(4,3)+\Sigma_7(2,-3)=q^7\frac{P_7(2)}{P_7(1)}\Sigma_7(3,0)+\frac{P_7^2(0)P_7(1)P_7(2)}{P_7(4)P_7(3)P_7(2)}. \qquad (2.6.11)
$$

We now substitute the right-hand sides of (2.6.9), (2.6.10), and (2.6.11) for the appearances of their respective left-hand sides in (2.6.8). Hence,

$$\begin{aligned}S_7(1) &= -g_7(3) + \Sigma_7(3,0)\left(\frac{q^{21}P_7(6)}{P_7(3)} - q^{17}\frac{P_7(4)}{P_7(2)} + q^{16}\frac{P_7(2)}{P_7(1)} - q^{18}\right) \\ &\quad - q^3\frac{P_7^2(0)}{P_7(1)} + q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)},\end{aligned}$$

by the fact that $P_7(a) = P_7(7-a)$. We now invoke Ramanujan's identity [55, p. 303, Entry 17(v)]

$$\frac{(q;q)_\infty}{P_7(0)} = \frac{P_7(2)}{P_7(1)} - q\frac{P_7(4)}{P_7(2)} - q^2 + q^5\frac{P_7(6)}{P_7(3)} \tag{2.6.12}$$

to conclude that

$$S_7(1) = -g_7(3) + \frac{q^{16}\Sigma_7(3,0)(q;q)_\infty}{P_7(0)} + \frac{q^9P_7^2(0)P_7(1)}{P_7^2(3)} - \frac{q^3P_7^2(0)}{P_7(1)},$$

as desired. □

Lemma 2.6.3. *We have*

$$S_7(4) = -g_7(2) - \frac{q^{13}\Sigma_7(2,0)}{P_7(0)}(q;q)_\infty - \frac{q^6P_7^2(0)}{P_7(3)} + \frac{q^4P_7^2(0)P_7(3)}{P_7^2(2)}. \tag{2.6.13}$$

Proof. We dissect the series for $S_7(4)$ modulo 7 to deduce that

$$\begin{aligned}S_7(4) &= \sum_{b=-1}^{5}\sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty}\frac{(-1)^{m+b}q^{(7m+b)(21m+3b+1)/2+28m+4b}}{1-q^{49m+7b}} \\ &= \sum_{b=-1}^{5}(-1)^b q^{3b(b+3)/2}\sum_{\substack{m=-\infty \\ (b,m)\neq(0,0)}}^{\infty}\frac{(-1)^m q^{147m(m+1)/2+7m(3b-6)}}{1-q^{49m+7b}} \\ &= -q^{-3}\Sigma_7(-1,-9) + \Sigma_7(0,-6) - q^6\Sigma_7(1,-3) + q^{15}\Sigma_7(2,0) \\ &\quad - q^{27}\Sigma_7(3,3) + q^{42}\Sigma_7(4,6) - q^{60}\Sigma_7(5,9).\end{aligned} \tag{2.6.14}$$

Now, by (2.6.6),

$$-g_7(2) + \frac{q^{14}P_7(4)}{P_7(2)}\Sigma_7(2,0) = q^{42}\Sigma_7(4,6) + \Sigma_7(0,-6). \tag{2.6.15}$$

By Lemma 2.3.2, with q replaced by q^{49}, $\zeta = q^{21}$, and $z = q^{14}$,

$$q^{63}\Sigma_7(5,9) + \Sigma_7(-1,-9) = \frac{q^{21}P_7(6)}{P_7(3)}\Sigma_7(2,0) + \frac{P_7^2(0)P_7(3)P_7(6)}{P_7(5)P_7(2)P_7(-1)}, \tag{2.6.16}$$

and by Lemma 2.3.2 with q replaced by q^{49}, $\zeta = q^7$, and $z = q^{14}$,

$$q^{21}\Sigma_7(3,3)+\Sigma_7(1,-3)=\frac{q^7P_7(2)}{P_7(1)}\Sigma_7(2,0)+\frac{P_7^2(0)P_7(1)P_7(2)}{P_7(3)P_7(2)P_7(1)}. \tag{2.6.17}$$

We now substitute the right-hand sides of (2.6.15), (2.6.16), and (2.6.17) for the appearances of their respective left-hand sides in (2.6.14). Hence,

$$\begin{aligned}S_7(4)&=-g_7(2)+\frac{q^{14}P_7(4)}{P_7(2)}\Sigma_7(2,0)-q^{-3}\left(\frac{q^{21}P_7(6)}{P_7(3)}\Sigma_7(2,0)\right.\\&\quad\left.-\frac{q^7P_7^2(0)P_7(3)}{P_7^2(2)}\right)-q^6\left(\frac{q^7P_7(2)}{P_7(1)}\Sigma_7(2,0)+\frac{P_7^2(0)}{P_7(3)}\right)+q^{15}\Sigma_7(2,0),\end{aligned}$$

since $P_7(-1)=-q^{-7}P_7(1)$. Hence,

$$\begin{aligned}S_7(4)&=-g_7(2)+\Sigma_7(2,0)\left(\frac{q^{14}P_7(4)}{P_7(2)}+q^{15}-\frac{q^{18}P_7(6)}{P_7(3)}-\frac{q^{13}P_7(2)}{P_7(1)}\right)\\&\quad+\frac{q^4P_7^2(0)P_7(3)}{P_7^2(2)}-\frac{q^6P_7^2(0)}{P_7(3)}\\&=-g_7(2)-q^{13}\Sigma_7(2,0)\left(\frac{P_7(2)}{P_7(1)}-\frac{qP_7(4)}{P_7(2)}-q^2+\frac{q^5P_7(6)}{P_7(3)}\right)\\&\quad-\frac{q^6P_7^2(0)}{P_7(3)}+\frac{q^4P_7^2(0)P_7(3)}{P_7^2(2)},\end{aligned}$$

and by (2.6.12),

$$S_7(4)=-g_7(2)-\frac{q^{13}\Sigma_7(2,0)(q;q)_\infty}{P_7(0)}-\frac{q^6P_7^2(0)}{P_7(3)}+\frac{q^4P_7^2(0)P_7(3)}{P_7^2(2)},$$

as desired. □

Lemma 2.6.4. *Recalling that $S_7(b)$ is defined by (2.6.1), we have*

$$S_7(7)=-g_7(1)+\frac{q^7\Sigma_7(1,0)}{P_7(0)}(q;q)_\infty+\frac{qP_7^2(0)P_7(2)}{P_7^2(1)}-\frac{q^5P_7^2(0)}{P_7(2)}. \tag{2.6.18}$$

Proof. As before, we begin by dissecting the series for $S_7(7)$ modulo 7 to arrive at

$$\begin{aligned}S_7(7)&=\sum_{b=-2}^{4}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty}\frac{(-1)^{m+b}q^{(7m+b)(21m+3b+1)/2+49m+7b}}{1-q^{49m+7b}}\\&=\sum_{b=-2}^{4}(-1)^bq^{3b(b+5)/2}\sum_{\substack{m=-\infty\\(b,m)\neq(0,0)}}^{\infty}\frac{(-1)^mq^{147m(m+1)/2+7m(3b-3)}}{1-q^{49m+7b}}\\&=q^{-9}\Sigma_7(-2,-9)-q^{-6}\Sigma_7(-1,-6)+\Sigma_7(0,-3)\\&\quad-q^9\Sigma_7(1,0)+q^{21}\Sigma_7(2,3)-q^{36}\Sigma_7(3,6)+q^{54}\Sigma_7(4,9).\end{aligned} \tag{2.6.19}$$

Now, by (2.6.6),

$$-g_7(1) + \frac{q^7 P_7(2)}{P_7(1)} \Sigma_7(1,0) = q^{21} \Sigma_7(2,3) + \Sigma_7(0,-3). \tag{2.6.20}$$

By Lemma 2.3.2 with q replaced by q^{49}, $\zeta = q^{14}$, and $z = q^7$,

$$q^{42} \Sigma_7(3,6) + \Sigma_7(-1,-6) = \frac{q^{14} P_7(4)}{P_7(2)} \Sigma_7(1,0) + \frac{P_7^2(0) P_7(2) P_7(4)}{P_7(3) P_7(1) P_7(-1)}, \tag{2.6.21}$$

and by Lemma 2.3.2 with q replaced by q^{49}, $\zeta = q^{21}$, and $z = q^7$,

$$q^{63} \Sigma_7(4,9) + \Sigma_7(-2,-9) = \frac{q^{21} P_7(6) \Sigma_7(1,0)}{P_7(3)} + \frac{P_7^2(0) P_7(3) P_7(6)}{P_7(4) P_7(1) P_7(-2)}. \tag{2.6.22}$$

We now substitute the right-hand sides of (2.6.20), (2.6.21), and (2.6.22) for the appearances of their respective left-hand sides in (2.6.19). Hence,

$$\begin{aligned}
S_7(7) &= -g_7(1) + \frac{q^7 P_7(2)}{P_7(1)} \Sigma_7(1,0) - q^9 \Sigma_7(1,0) \\
&\quad - q^{-6} \left(\frac{q^{14} P_7(4)}{P_7(2)} \Sigma_7(1,0) + \frac{P_7^2(0) P_7(2) P_7(4)}{P_7(3) P_7(1) P_7(-1)} \right) \\
&\quad + q^{-9} \left(\frac{q^{21} P_7(6)}{P_7(3)} \Sigma_7(1,0) + \frac{P_7^2(0) P_7(3) P_7(6)}{P_7(4) P_7(1) P_7(-2)} \right) \\
&= -g_7(1) + q^7 \Sigma_7(1,0) \left(\frac{P_7(2)}{P_7(1)} - \frac{q P_7(4)}{P_7(2)} - q^2 + \frac{q^5 P_7(6)}{P_7(3)} \right) \\
&\quad + \frac{q P_7^2(0) P_7(2)}{P_7^2(1)} - \frac{q^5 P_7^2(0)}{P_7(2)},
\end{aligned}$$

by the facts that $P_7(a) = P_7(7-a)$ and $P_7(-a) = -q^{-7a} P_7(a)$. We now invoke (2.6.12) to conclude that

$$S_7(7) = -g_7(1) + \frac{q^7 \Sigma_7(1,0)(q;q)_\infty}{P_7(0)} + \frac{q P_7^2(0) P_7(2)}{P_7^2(1)} - \frac{q^5 P_7^2(0)}{P_7(2)},$$

as desired. □

Lemma 2.6.5. *We have*

$$P_7(1) P_7^3(3) - P_7(3) P_7^3(2) + q^7 P_7^3(1) P_7(2) = 0.$$

Proof. This identity can be found in Ramanujan's second notebook [282, p. 300], where it is given as an identity involving quotients of theta functions. A proof can be found in Berndt's book [57], with the statement of Ramanujan's identity being given in Entry 32(ii) of Chapter 25 [57, p. 176]. (Unfortunately, there is a misprint in the definition of w in Entry 32; read 25/56 instead of 25/26.) Lemma 2.6.5 is also equivalent to a specialization of the three-term relation for the Weierstrass sigma function [339, p. 451, Exercise 5]. □

Lemma 2.6.6. *With $S_7(b)$ defined by (2.6.1),*

$$-2g_7(1)+g_7(2)-1+2q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}-q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}-2q^5\frac{P_7^2(0)}{P_7(2)}+q^6\frac{P_7^2(0)}{P_7(3)}$$
$$=\left\{-\frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}+q\frac{P_7(0)}{P_7(1)}+q^3\frac{P_7(0)}{P_7(2)}\right\}(q;q)_\infty. \tag{2.6.23}$$

Proof. By Lemma 2.3.5 with q replaced by q^{49} and $z=q^7$, we see that

$$-2g_7(1)+g_7(2)-1=-\frac{P_7^2(0)P_7^2(3)}{P_7^2(1)P_7(4)}=-\frac{P_7^2(0)P_7(3)}{P_7^2(1)}.$$

Hence (2.6.23) is, with the use of (2.6.12), equivalent to the assertion

$$\frac{-P_7^2(0)P_7(3)}{P_7^2(1)}+2q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}-q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}-2q^5\frac{P_7^2(0)}{P_7(2)}+q^6\frac{P_7^2(0)}{P_7(3)}$$
$$=\left\{-\frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}+q\frac{P_7(0)}{P_7(1)}+q^3\frac{P_7(0)}{P_7(2)}\right\}$$
$$\times P_7(0)\left\{\frac{P_7(2)}{P_7(1)}-q\frac{P_7(4)}{P_7(2)}-q^2+q^5\frac{P_7(6)}{P_7(3)}\right\},$$

and if we collect terms on the left-hand side according to powers of q, we see that this last identity is equivalent (after cancellation of like terms) to

$$q\left\{\frac{P_7^2(0)P_7(2)}{P_7^2(1)}-\frac{P_7^2(0)P_7(3)P_7(4)}{P_7(1)P_7^2(2)}-q^7\frac{P_7^2(0)P_7(6)}{P_7(3)P_7(2)}\right\}=0.$$

This last identity can be written in the form

$$\frac{qP_7^2(0)}{P_7^2(1)P_7^2(2)P_7(3)}\left\{P_7^3(2)P_7(3)-P_7^3(3)P_7(1)-q^7P_7^3(1)P_7(2)\right\}=0,$$

by Lemma 2.6.5. Thus (2.6.23) is proved. ☐

Lemma 2.6.7. *We have*

$$2g_7(2)+g_7(3)-q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)}+q^3\frac{P_7^2(0)}{P_7(1)}-2q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}+2q^6\frac{P_7^2(0)}{P_7(3)}$$
$$=\left\{q^3\frac{P_7(0)}{P_7(2)}-q^4\frac{P_7(0)}{P_7(3)}+q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\right\}(q;q)_\infty. \tag{2.6.24}$$

Proof. By Lemma 2.3.3, with $z=q^{21}$, Lemma 2.3.3, with $z=q^{14}$, and the fact that $P_7(8)=-q^{-7}P_7(1)$,

$$2g_7(2)+g_7(3)=-q^7\frac{P_7^2(0)P_7(1)}{P_7^2(2)}.$$

Therefore (2.6.24) is, by (2.6.12), equivalent to the assertion

$$\begin{aligned}-q^7\frac{P_7^2(0)P_7(1)}{P_7^2(2)}-q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)}+q^3\frac{P_7^2(0)}{P_7(1)}-2q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}+2q^6\frac{P_7^2(0)}{P_7(3)}\\
=\left\{q^3\frac{P_7(0)}{P_7(2)}-q^4\frac{P_7(0)}{P_7(3)}+q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\right\}\\
\times P_7(0)\left\{\frac{P_7(2)}{P_7(1)}-q\frac{P_7(4)}{P_7(2)}-q^2+q^5\frac{P_7(6)}{P_7(3)}\right\}.\end{aligned}$$

Multiply out the right-hand side above and cancel like terms. It then remains to show that

$$\begin{aligned}0&=q^4\left\{-\frac{P_7^2(0)P_7(3)}{P_7^2(2)}+\frac{P_7^2(0)P_7(2)}{P_7(1)P_7(3)}-q^7\frac{P_7^2(0)P_7^2(1)}{P_7(2)P_7^2(3)}\right\}\\
&=\frac{q^4P_7^2(0)}{P_7^2(2)P_7^2(3)P_7(1)}\left\{-P_7^3(3)P_7(1)+P_7^3(2)P_7(3)-q^7P_7^3(1)P_7(2)\right\}.\end{aligned}$$

By Lemma 2.6.5, the equality above is indeed true. Thus (2.6.24) is proved. □

Lemma 2.6.8. *We have*

$$\begin{aligned}-g_7(1)-2g_7(3)+q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}+2q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)}-2q^3\frac{P_7^2(0)}{P_7(1)}-q^5\frac{P_7^2(0)}{P_7(2)}\\
=\left\{q\frac{P_7(0)}{P_7(1)}+q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)}+q^4\frac{P_7(0)}{P_7(3)}\right\}(q;q)_\infty.\end{aligned}\tag{2.6.25}$$

Proof. By Lemma 2.3.3, with $z=q^7$, Lemma 2.3.5, with $z=q^{21}$, and the facts that $P_7(9)=-q^{-14}P_7(2)$ and $P_7(12)=-q^{-35}P_7(2)$,

$$-g_7(1)-2g_7(3)=q^7\frac{P_7^2(0)P_7(2)}{P_7^2(3)}.$$

Thus, by (2.6.12), (2.6.25) is equivalent to the assertion

$$\begin{aligned}q^7\frac{P_7^2(0)P_7(2)}{P_7^2(3)}+q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}+2q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)}-2q^3\frac{P_7^2(0)}{P_7(1)}-q^5\frac{P_7^2(0)}{P_7(2)}\\
=\left\{q\frac{P_7(0)}{P_7(1)}+q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)}+q^4\frac{P_7(0)}{P_7(3)}\right\}\\
\times P_7(0)\left\{\frac{P_7(2)}{P_7(1)}-q\frac{P_7(4)}{P_7(2)}-q^2+q^5\frac{P_7(6)}{P_7(3)}\right\}.\end{aligned}$$

As before, we collect terms on the left-hand side according to powers of q. This reduces the last identity to the equivalent assertion

$$
\begin{aligned}
&q^2\left\{\frac{q^7P_7^2(0)P_7(1)}{P_7^2(3)}+\frac{P_7^2(0)P_7(4)}{P_7(1)P_7(2)}-\frac{P_7^2(0)P_7^2(2)}{P_7^2(1)P_7(3)}\right\}\\
&\quad=\frac{q^2P_7^2(0)}{P_7^2(3)P_7^2(1)P_7(2)}\left\{q^7P_7^3(1)P_7(2)+P_7^3(3)P_7(1)-P_7^3(2)P_7(3)\right\}\\
&\quad=0,
\end{aligned}
$$

by Lemma 2.6.5. Thus (2.6.25) is proved. □

Lemma 2.6.9. *We have*

$$
\begin{aligned}
-1+(q;q)_\infty-S_7(4)+2S_7(7)=&\left\{2q^7\frac{\Sigma_7(1,0)}{P_7(0)}-\frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}\right.\\
&\left.+1+q\frac{P_7(0)}{P_7(1)}+q^3\frac{P_7(0)}{P_7(2)}+q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}\right\}(q;q)_\infty. \qquad (2.6.26)
\end{aligned}
$$

Proof. By Lemmas 2.6.3 and 2.6.4,

$$
\begin{aligned}
&-1+(q;q)_\infty-S_7(4)+2S_7(7)\\
&\qquad=-1+(q;q)_\infty+g_7(2)+q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}(q;q)_\infty+q^6\frac{P_7^2(0)}{P_7(3)}\\
&\qquad\quad-q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}-2g_7(1)+2q^7\frac{\Sigma_7(1,0)}{P_7(0)}(q;q)_\infty\\
&\qquad\quad+2q\frac{P_7^2(0)P_7(2)}{P_7^2(1)}-2q^5\frac{P_7^2(0)}{P_7(2)}.
\end{aligned}
$$

We now note that the entire left-hand side of (2.6.23) appears on the right-hand side of the preceding expression. Hence, by (2.6.23),

$$
\begin{aligned}
-1+(q;q)_\infty-S_7(4)+2S_7(7)=&\left\{2q^7\frac{\Sigma_7(1,0)}{P_7(0)}+q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}\right.\\
&\left.+1-\frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}+q\frac{P_7(0)}{P_7(1)}+q^3\frac{P_7(0)}{P_7(2)}\right\}(q;q)_\infty,
\end{aligned}
$$

which is equivalent to (2.6.26). □

Lemma 2.6.10. *We have*

$$
\begin{aligned}
-S_7(1)-2S_7(4)=&\left\{-q^{16}\frac{\Sigma_7(3,0)}{P_7(0)}+q^3\frac{P_7(0)}{P_7(2)}-q^4\frac{P_7(0)}{P_7(3)}\right.\\
&\left.+2q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}+q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\right\}(q;q)_\infty. \qquad (2.6.27)
\end{aligned}
$$

Proof. By Lemmas 2.6.2 and 2.6.3,

$$
\begin{aligned}
-S_7(1)-2S_7(4) &= g_7(3) - q^{16}\frac{\Sigma_7(3,0)}{P_7(0)}(q;q)_\infty - q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)} + q^3\frac{P_7^2(0)}{P_7(1)} \\
&\quad + 2g_7(2) + 2q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}(q;q)_\infty + 2q^6\frac{P_7^2(0)}{P_7(3)} - 2q^4\frac{P_7^2(0)P_7(3)}{P_7^2(2)}.
\end{aligned}
$$

We observe that the entire left-hand side of (2.6.24) appears in the preceding expression's right-hand side. Therefore, by (2.6.24),

$$
\begin{aligned}
-S_7(1)-2S_7(4) = \Bigg\{ &-q^{16}\frac{\Sigma_7(3,0)}{P_7(0)} + 2q^{13}\frac{\Sigma_7(2,0)}{P_7(0)} + q^3\frac{P_7(0)}{P_7(2)} - q^4\frac{P_7(0)}{P_7(3)} \\
&+ q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\Bigg\}(q;q)_\infty,
\end{aligned}
$$

which is equivalent to (2.6.27). □

Lemma 2.6.11. *We have*

$$
\begin{aligned}
2S_7(1)+S_7(7) = \Bigg\{ &q^7\frac{\Sigma_7(1,0)}{P_7(0)} + q\frac{P_7(0)}{P_7(1)} + q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)} \\
&+ 2q^{16}\frac{\Sigma_7(3,0)}{P_7(0)} + q^4\frac{P_7(0)}{P_7(3)}\Bigg\}(q;q)_\infty.
\end{aligned} \tag{2.6.28}
$$

Proof. By Lemmas 2.6.2 and 2.6.4,

$$
\begin{aligned}
2S_7(1)+S_7(7) &= -2g_7(3) + 2q^{16}\frac{\Sigma_7(3,0)}{P_7(0)}(q;q)_\infty + 2q^9\frac{P_7^2(0)P_7(1)}{P_7^2(3)} \\
&\quad - 2q^3\frac{P_7^2(0)}{P_7(1)} - g_7(1) + q^7\frac{\Sigma_7(1,0)}{P_7(0)}(q;q)_\infty + q\frac{P_7^2(0)P_7(2)}{P_7^2(1)} - q^5\frac{P_7^2(0)}{P_7(2)}.
\end{aligned}
$$

As before, we see that the entire left-hand side of (2.6.25) appears on the right-hand side of the preceding expression. So, by (2.6.25),

$$
\begin{aligned}
2S_7(1)+S_7(7) = \Bigg\{ &2q^{16}\frac{\Sigma_7(3,0)}{P_7(0)} + q^7\frac{\Sigma_7(1,0)}{P_7(0)} + q\frac{P_7(0)}{P_7(1)} \\
&+ q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)} + q^4\frac{P_7(0)}{P_7(3)}\Bigg\}(q;q)_\infty,
\end{aligned}
$$

which is equivalent to (2.6.28). □

Lemma 2.6.12. *We have*

$$
\begin{aligned}
&1-(q;q)_\infty + S_7(1) - S_7(4) - S_7(7) \\
&\quad = \Bigg\{ \frac{P_7(0)P_7(3)}{P_7(1)P_7(2)} - 1 - q^7\frac{\Sigma_7(1,0)}{P_7(0)} + q^2\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)} + q^{16}\frac{\Sigma_7(3,0)}{P_7(0)} \\
&\qquad + q^6\frac{P_7(0)P_7(1)}{P_7(2)P_7(3)} + q^{13}\frac{\Sigma_7(2,0)}{P_7(0)}\Bigg\}(q;q)_\infty.
\end{aligned} \tag{2.6.29}
$$

Proof. If we add together the left-hand sides of (2.6.27) and (2.6.28) and subtract the left-hand side of (2.6.26), we obtain the left-hand side of (2.6.29). The same combination of right-hand sides produces the right-hand side of (2.6.29). □

Lemma 2.6.13. *We have*

$$S_7(0) = S_7(7) + (q;q)_\infty - 1.$$

Proof. This is (2.6.4) with $b = 0$. □

Finally we are ready to prove Entry 2.1.5.

Proof of Entry 2.1.5. By Lemma 2.6.1,

$$\begin{aligned}
(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n(\zeta_7^{-1}q;q)_n}
&= 1 - S_7(4) + 2S_7(0) + (\zeta_7+\zeta_7^{-1})\big(S_7(1) - S_7(4) - S_7(0)\big) \\
&\quad + (\zeta_7^2+\zeta_7^{-2})\big(-S_7(1) - 2S_7(4)\big) \\
&= \big(-1 + (q;q)_\infty - S_7(4) + 2S_7(7)\big) + (q;q)_\infty \\
&\quad + (\zeta_7+\zeta_7^{-1})\big(S_7(1) - S_7(4) - S_7(7) - (q;q)_\infty + 1\big) \\
&\quad + (\zeta_7^2+\zeta_7^{-2})\big(-S_7(1) - 2S_7(4)\big),
\end{aligned}$$

by Lemma 2.6.13.

We now apply Lemmas 2.6.9, 2.6.12, and 2.6.10 to the combinations of S_7's contained in parentheses. This yields a large expression multiplied by $(q;q)_\infty$. We cancel $(q;q)_\infty$ from each side, use (2.4.5), and recall the notation (2.1.43)–(2.1.47) to obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta_7 q;q)_n(\zeta_7^{-1}q;q)_n} \\
&= 2 - \zeta_7 - \zeta_7^{-1} + \frac{P_7(0)P_7(3)}{P_7(1)P_7(2)}\left(\zeta_7+\zeta_7^{-1}-1\right) \\
&\quad + q^7\frac{\Sigma_7(1,0)}{P_7(0)}\left(2-\zeta_7-\zeta_7^{-1}\right) + q\frac{P_7(0)}{P_7(1)} \\
&\quad + q^2\left\{\left(\zeta_7+\zeta_7^{-1}\right)\left(\frac{P_7(0)P_7(2)}{P_7(1)P_7(3)} + q^{14}\frac{\Sigma_7(3,0)}{P_7(0)}\right) - \left(\zeta_7^2+\zeta_7^{-2}\right)q^{14}\frac{\Sigma_7(3,0)}{P_7(0)}\right\} \\
&\quad + q^3\frac{P_7(0)}{P_7(2)}\left(1+\zeta_7^2+\zeta_7^{-2}\right) - q^4\frac{P_7(0)}{P_7(3)}\left(\zeta_7^2+\zeta_7^{-2}\right) \\
&\quad + q^6\left\{q^7\frac{\Sigma_7(2,0)}{P_7(0)}\left(1+\zeta_7+\zeta_7^{-1}+2\zeta_7^2+2\zeta_7^{-2}\right)\right. \\
&\qquad\qquad \left. + \frac{P_7(0)P_7(1)}{P_7(2)P_7(3)}\left(\zeta_7+\zeta_7^{-1}+\zeta_7^2+\zeta_7^{-2}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left(2 - \zeta_7 - \zeta_7^{-1}\right)\left(1 - A_7(q^7) + q^7 Q_1(q^7)\right) + qT_1(q^7) + A_7(q^7) \\
&\quad + q^2 \left\{\left(\zeta_7 + \zeta_7^{-1}\right) B_7(q^7) + q^{14} Q_3(q^7)\left(\zeta_7 + \zeta_7^{-1} - \zeta_7^2 - \zeta_7^{-2}\right)\right\} \\
&\quad + q^3 T_2(q^7)\left(1 + \zeta_7^2 + \zeta_7^{-2}\right) - q^4\left(\zeta_7^2 + \zeta_7^{-2}\right) T_3(q^7) \\
&\quad + q^6 \left\{q^7 Q_2(q^7)\left(\zeta_7^2 + \zeta_7^{-2} - \zeta_7^3 - \zeta_7^{-3}\right) - C_7(q^7)\left(1 + \zeta_7^3 + \zeta_7^{-3}\right)\right\},
\end{aligned}
$$

as desired. $\square$