1 Introduction

1.1 Motivation

Material behaviour at length scales greatly in excess of molecular dimensions (i.e., *macroscopic* behaviour) is usually modelled in terms of the continuum viewpoint. From such a perspective the matter associated with any physical system (or *body*) of interest is, at any instant, considered to be distributed continuously throughout some spatial region (deemed to be the region 'occupied' by the system at this instant). Reproducible macroscopic phenomena are modelled in terms of deterministic continuum theories. Such theories have been highly successful, particularly in engineering contexts, and include those of elasticity, fluid dynamics, and plasticity. The totality of such theories constitutes (deterministic) continuum mechanics. The link between actual material behaviour and relevant theory is provided by experimentation/observation. Specifically, it is necessary to relate local experimental measurements to continuum field values. However, the value of any local measurement made upon a physical system is the consequence of a local (both in space *and* time) interaction with this system. Further, local measurement values exhibit erratic features if the scale (in space-time) is sufficiently fine, and such features become increasingly evident with diminishing scale. Said differently, sufficiently sensitive instruments *always* yield measurement values which fluctuate chaotically in both space and time (i.e., these values change perceptibly, in random fashion, with both location and time), and the 'strength' of these fluctuations increases with instrument sensitivity (i.e., with increasingly fine-scale interaction between instrument and system). This *intrinsic* property of material behaviour can only be understood in terms of the essentially discrete nature of matter; that is, it proves necessary to adopt a microscopic viewpoint. Accordingly, such fundamental understanding requires that measurement values be related to local interactions with (or 'samplings' of) fundamental discrete entities (that is, molecules, atoms, or ions) of the system.

While the understanding of small-scale material behaviour requires a microscopic basis, the success of deterministic continuum mechanics might suggest that such considerations are of little relevance to engineering practice. There are two main reasons why this is not the case. Firstly, erratic material behaviour can be manifest at the macroscopic level, as evidenced by turbulent fluid motions. Recourse to stochastic continuum modelling is necessary in such cases. The natures of the fields

and balance relations of stochastic continuum mechanics can be fully understood only from the standpoint of microscopic considerations. Secondly, the macroscopic behaviour of any material system ultimately derives from its microscopic constitution, and in certain circumstances microstructural features may persist on a macroscopic scale and must be incorporated into continuum descriptions. (For example, in nematic phases of liquid crystals the co-operative effect of elongated molecules which tend to align with their neighbours is modelled in terms of a *director* field.) Further reasons for exploring the relationship between microscopic and macroscopic aspects of material behaviour are that it enhances the physical interpretation of continuum fields, clarifies basic continuum concepts, elucidates fundamental assumptions implicit in continuum modelling, and thereby improves awareness of the range of applicability of continuum mechanics. Such insight is essential in studies of nanoscale behaviour and in interpreting the results of molecular dynamical simulations.

The preceding remarks serve to motivate attempts to identify continuum field values with local space-time averages of microscopic quantities and to establish the balance relations satisfied by such fields. While these objectives constitute the main part of what follows, their consideration leads to natural implications for the modelling of fluid flow through porous media and for the manner in which observer consensus places restrictions upon constitutive relations. Elements of the probabilistic approach of classical statistical mechanics are outlined for comparison of viewpoints.

1.2 Contents

Basic elements of continuum mechanics are summarised in Chapter 2 for later reference. Included are discussions of the different physical interpretations to be placed on the notion of '*material point*' in solids and fluids, and the special case of rigid bodies.

Attention is drawn in Chapter 3 to conceptual problems associated with the continuum viewpoint. In particular, the manifest dependence of solid boundaries on scale is shown to imply similar sensitivity in mass density. Also discussed are the scale dependence of velocity, the inability to interpret the stress within a rarefied gas (i.e., its pressure) as a force per unit area, and the inappropriateness of deterministic continuum modelling at small length scales.

Local spatial averaging of the masses and momenta of fundamental discrete entities, modelled as point masses, is effected in Chapter 4 in terms of a weighting function w. The continuity equation is established for quite general, suitably normalised choices of w. A simple, physically distinguished choice w_ϵ, with associated length scale ϵ, is defined, and the corresponding boundary of any system of point masses at scale ϵ is thereby delineated. The physical interpretations of volume integrals of the mass and momentum densities appropriate to w_ϵ are obtained. Alternative choices of w are motivated and derived.

In Chapter 5 the velocity field (the ratio of the w-based density of momentum to that of mass) is employed to generate the corresponding motion map. (This is in contrast to the more standard derivation of velocity from a postulated motion map.) After discussing the subatomic origin of molecular interactions, a general *local* form of linear momentum balance is established directly (rather than being obtained

in conventional fashion as the localisation of an integral relation) on the basis of assumed pairwise interactions between point masses of quite general nature. (In particular, the interaction between a pair of point masses may depend upon other point masses adjacent to each of this pair.) The balance relation contains an interaction force density $\mathbf{f}_w$. The usual form of balance follows from determination of an interaction stress tensor $\mathbf{T}_w^-$ for which $\operatorname{div}\mathbf{T}_w^- = \mathbf{f}_w$. The corresponding Cauchy stress tensor is $\mathbf{T}_w := \mathbf{T}_w^- - \mathcal{D}_w$, where $\mathcal{D}_w$ is a symmetric tensor of thermal character. (Here *thermal* refers to any quantity which depends upon velocities of individual point masses relative to the local w-scale continuum velocity field values: such relative velocities have random character and are also termed *thermal*.) The non-uniqueness of $\mathbf{T}_w^-$ is explored, and three distinct classes of solutions for pairwise-balanced interactions are examined and compared.

Local forms of energy balance are obtained directly in Chapter 6. If $(eq)_i$ denotes the equation which governs the motion of point mass P_i in an inertial frame, and $\mathbf{v}_i$ denotes the velocity of $\mathbf{v}_i$, then such forms of balance follow by summing relations $(eq)_i \,.\, \mathbf{v}_i \, w$ over all point masses. [Linear momentum balance followed from a similar sum of relations $(eq)_i \, w$.] The distinction between fields of thermal and mechanical character depends upon the presence or otherwise of thermal velocities in their definitions. If interactions are governed by separation-dependent pair potentials, the standard form of balance is obtained in which the internal energy density is the sum of densities of energy of assembly and of heat content (a local density of kinetic energy associated with thermal velocities).

Fine-scale relations are obtained in Chapter 7 by taking suitable moments. Summation of weighted products of masses with displacements of point masses from a given location $\mathbf{x}$ yields a measure $\mathbf{d}_w$ of local inhomogeneity. The time evolution of $\mathbf{d}_w$ gives rise to a relation which expresses moment of mass conservation. Summation of tensorial products of the preceding displacements with $(eq)_i \, w$ yields a generalised local moment of momentum balance. The skew part of this balance constitutes the usual moment of momentum balance: skew tensors can simply be replaced by their equivalent axial vector counterparts. Couple stresses and body couples emerge naturally, together with an internal moment of momentum density. A corresponding fine-scale energy balance is derived, and relative magnitudes of relevant fields are discussed. In contrast with axiomatic approaches, in which (axial vector-valued) moment of momentum balance is considered to determine the symmetry or otherwise of the Cauchy stress tensor, the explicit forms $\mathbf{T}_w$ obtained in Chapter 5 yield this information directly. Moment of momentum balance constitutes an evolution equation for internal moment of momentum, with contributions from $\mathbf{T}_w$, body couple density, and the divergence of the couple stress (a third-order tensor field).

Time averaging is introduced in Chapter 8 with the aim of obtaining field values which reflect local space-time averages: it is such averages that are to be related to local measurements. Time-averaged versions of the continuity equation and balances of momentum and energy are derived. Systems with changing material content are studied in terms of a 'membership' function for the system in question. Global considerations are addressed (with examination of details specific to rocketry and jet propulsion) before the corresponding local forms of balance for mass, momentum and energy are established.

The methodology developed in preceding chapters is applied to mixtures in Chapter 9 which includes the resolution of a paradox associated with incorrect interpretation of the notion of *partial stress* and an introduction to the modelling of reacting constituents.

Fluid flow through porous media is analysed in Chapter 10 at two different scales, one at which pore structure is evident (here the scale-dependent notion of boundary, established in Chapter 3, proves indispensable) and the other at which pores are no longer distinguishable. The small-scale balance of momentum is averaged over so-called representative elementary volumes using an appropriate weighting function. For the case in which incompressible linearly viscous fluid saturates pores, a sequence of relations is obtained, each of which follows from a specific and transparent modelling assumption, culminating in the Brinkman equation and Darcy 'law'.

An alternative averaging procedure is outlined in Chapter 11. This addresses behaviour which is scale-insensitive over a range of length scales (a typical assumption in continuum modelling) and is implemented in terms of so-called ϵ-cells.

Although specific constitutive equations are not discussed (other than for fluid flows in porous media), the definitions of field values in terms of microscopic quantities have implications which are analysed in Chapter 12. These implications are imposed by the fundamental requirement that observers must be able to agree upon the physical interpretations of the fields employed in continuum modelling. Matters are subtle: time averaging must be effected, instant by instant, over the same sets of molecules for all observers if a consensus is to be established. In accomplishing such averaging a crucial role is played by inertial observers. Once field values are established for this class of observers, it is possible to envisage how these values appear to a general observer. The objective natures of time-averaged fields (of mass, momentum, interaction force and external body force densities, together with those of stress and heat flux) then follow. The nature of objectivity in a general scientific context is discussed, and its specific form in deterministic continuum mechanics is characterised in terms of five distinct aspects of consensus. Such consensus mandates restrictions upon response functions. For elastica, these restrictions are those universally accepted. The standard definition of a viscous fluid (as a material for which the stress depends upon the current local values of mass density and velocity gradient *computed in terms of the frame of a general observer*) is shown to simplify to its standard (spin-independent) form if the local measure $\mathbf{d}_w$ of inhomogeneity introduced in Chapter 7 vanishes. However, if the stress depends upon density and velocity gradient with respect to an (any) *inertial* observer, then objectivity does not exclude spin-dependence. Since the physical admissibility (or otherwise) of spin-dependent fluids has been the subject of controversy for forty years, remarks are made which concern the fundamental assumption in classical physics that in principle material behaviour is independent of its observation. Statements which are intended to formalise the consequences of this assumption are not equivalent, and are variously termed *material frame-indifference*, *invariance under superposed rigid body motions*, and *objectivity*. These are listed and compared. Only objectivity emerges as imposing no restriction upon Nature. Further, from the perspective offered by objectivity, there is no requirement that observers should choose the same response function(s) for a given material, restrictions upon response functions which

follow from objectivity involve only proper orthogonal tensors, and materials sensitive to spin relative to inertial frames are physically admissible. A personal history of involvement in the controversy is appended.

Chapter 13 examines two approaches to so-called non-local behaviour in the light of previous chapters: namely, the general viewpoint of Edelen, and the *peridynamics* introduced by Silling. Shortcomings in the physical basis of the long-range 'particle–particle' interactions of the latter theory are highlighted, and attention is drawn to the similarity of what is being attempted with the porous medium considerations of Chapter 10.

Elements of classical statistical mechanics are presented in Chapter 14. After introducing the concepts of dynamics in phase space, ensembles, and ensemble averaging in terms of probability density functions, strictly local forms of the continuity equation and linear momentum balance are obtained in the manner of Noll's revision of the pioneering work of Irving and Kirkwood. Two generalisations of this approach, due to Pitteri and to Admal and Tadmor, are discussed. A completely different perspective, due to Zwanzig, is outlined and applied to so-called continuously reproducible behaviour at prescribed scales of length and time. Key features are the selection of an appropriate projection operator coupled with postulates of local equilibrium and dynamic ergodicity. Semigroup formalism leads to a master equation and corresponding Fokker-Planck and fluctuation-dissipation equations. Attention is drawn to the need for a rigorous proof of a semigroup result central to projection operator methodology.

Remarks and suggestions are made in Chapter 15 which concern issues and topics not covered in this volume but which might benefit from the same approach and methodology. These relate to boundaries and interfacial regions, generalised continua, reacting mixtures, configurational forces, electromagnetic phenomena, and irreversibility. The question is raised of whether it might prove possible to derive, motivate, or otherwise gain insight into, the second law of thermodynamics on the basis of scale-dependent, corpuscular, and weighting function considerations.

Two extensive appendices introduce basic mathematical tools, results, and notation. While these will be familiar to many, the intention is to provide a comprehensive, readily accessible source of background material that might be required when studying the main text.

Appendix A is concerned with vectors and linear algebra. Starting from absolute basic, relevant concepts, definitions and results are developed both in direct (basis-free) and Cartesian tensor notation.

The geometry of Euclidean space is discussed in Appendix B, and isometries and homogeneous deformations are defined and characterised. Differentiation of scalar, vector, and linear transformation fields is treated in co-ordinate-free manner and related to equivalent Cartesian tensor formulations. Elements of integration over spatial regions are included, together with statements of divergence theorems and proofs of identities. Generalisations of differential and integral calculus to $\mathbb{R}^n$ are discussed in order to appreciate the phase-space analyses of Chapter 14.

Serious study of any work of this kind requires pencil and paper to hand for checking calculations and results. This is encouraged by the inclusion of many (usually simple and straightforward) exercises. The reader is also prompted on occasion by queries which are intended to help ensure that attention is paid to detail.

2 Some Elements of Continuum Mechanics

2.1 Preamble

In this chapter we address fundamental aspects of continuum modelling in respect of kinematics, mass conservation, balances of linear and rotational momentum, and balance of energy.

After considering the role of mass density in modelling the presence of 'matter', we discuss the manner in which the detailed macroscopic distortion of any material body can be monitored. This is markedly different for solids and fluids, but in both cases it is possible to motivate the notion of material point and thereby establish basic kinematic concepts such as deformation, motion, and velocity. The formal (axiomatic) approach to kinematics is outlined for comparison. Mass conservation is motivated for solids and postulated to hold in general. Dynamical considerations are first addressed for a body as a whole. In addition to tractions on boundaries, the possibility of surface and body couples is considered. Global balances of linear and rotational momentum are postulated and applied to rigid bodies both to emphasise their often-neglected status as a special case of material continua and to develop familiarity with notation, concepts and basic manipulations. Local forms of balance are derived in standard fashion by postulating balances for matter in arbitrary subregions of the region instantaneously occupied by the body, invoking a transport theorem, and then establishing the existence of stress and couple stress tensors and a heat flux vector. It is these local forms of balance that can be *derived* directly from molecular considerations using the weighting function methodology to be introduced in Chapter 4.

2.2 Matter and Its Distribution

Any specific material system of interest (e.g., a rubber tyre, brick, steel girder, liquid in a container, ocean current, atmospheric air, or water in an aquifer) is termed a *body*, B say. The presence of the matter which constitutes B is described in terms of its mass. Specifically, the measure of matter associated with a body is provided by a *mass density function* ρ of position and time which takes non-negative values. The function ρ for a given body has two physical *m*ass *d*ensity interpretations:

M.D.1. The spatial region considered to be occupied by the body at time t, B_t say, is that region in which ρ takes positive values at time t. That is,[1]

$$B_t := \{\mathbf{x} \in \mathcal{E} : \rho(\mathbf{x},t) > 0\}. \tag{2.2.1}$$

M.D.2. The mass, or amount of matter, of B material within any region R at time t is

$$m(R,t) := \int_R \rho(\mathbf{x},t)dV_{\mathbf{x}}. \tag{2.2.2}$$

Remark 2.2.1. In order for (2.2.2) to make sense, ρ must be spatially integrable at all times of interest. It is *assumed* that ρ has continuous partial derivatives with respect to both location $\mathbf{x}$ and time t. Accordingly, at any time of interest, ρ is a continuous function of position and is hence everywhere integrable.

2.3 Motion of Matter: Kinematics and Material Points

As time goes by, a given body B may change position and/or shape. Such time-dependent change is termed a *motion* of the body. To model physical behaviour associated with a motion, it is useful to define the *trajectory* of this body as

$$\mathcal{T}_B := \{(\mathbf{x},t) : \mathbf{x} \in B_t,\ t \in I\}. \tag{2.3.1}$$

Here I denotes the time interval over which the behaviour of B is being modelled. Functions of space and time defined on $\mathcal{T}_B$ are termed *fields*. In particular, ρ is the mass density field.

The detailed prescription of change of position and/or shape of B is modelled in terms of *material points*. Specifically, with each pair $(\mathbf{x},t) \in \mathcal{T}_B$ is associated a material point together with its *velocity* $\mathbf{v}(\mathbf{x},t)$. While the concept of material point is a primitive notion in formal continuum mechanics,[2] in order to link this with observation and experimentation it is necessary to be somewhat specific. (The next subsection contains an outline of the formal, axiomatic approach.)

For a *solid* body (in which any given molecule has near-neighbours which remain so as the body moves and/or changes shape), any group of neighbouring molecules can be 'doped' or, at least in principle, identified in some way. The motion of any such group can be monitored. If the group is localised at point $\hat{\mathbf{x}} \in B_{t_0}$ at time t_0, then at any subsequent time t it will be localised at some point $\mathbf{x} \in B_t$. Formalising this, we write

$$\mathbf{x} = \boldsymbol{\chi}_{t_0}(\hat{\mathbf{x}},t) \tag{2.3.2}$$

and term $\boldsymbol{\chi}_{t_0}$ the *motion map corresponding to the situation at time* t_0. Of course, the velocity at time t of that group localised at $\hat{\mathbf{x}}$ at time t_0 will be

[1] Here and henceforth $\mathcal{E}$ will denote Euclidean space; that is, 'space' as we perceive it. Any element $\mathbf{x}$ of $\mathcal{E}$ is a geometrical point. See Appendix B.1.

[2] Cf., e.g., Gurtin [1]. In the general continuum mechanics literature material points are also termed *particles* (cf., e.g., Truesdell & Noll [2] and Chadwick [3]) or, in fluid dynamics, *fluid particles* (cf., e.g., Landau & Lifschitz [4] and Paterson [5]). The term *material point* was introduced by Noll to avoid the common identification of *particle* with *point mass*. The latter has a definite mass, while, as will be seen, a material point has no associated mass but only, at any given time, a motion-dependent mass density.

$\dot{\boldsymbol{\chi}}_{t_0}(\hat{\mathbf{x}},t) := (\partial/\partial t)\{\boldsymbol{\chi}_{t_0}(\hat{\mathbf{x}},t)\}$. Thus the velocity of the doped group located at $\mathbf{x}$ at time t, written as $\mathbf{v}(\mathbf{x},t)$, is precisely $\dot{\boldsymbol{\chi}}_{t_0}(\hat{\mathbf{x}},t)$, via (2.3.2). That is, the *velocity field* $\mathbf{v}$ on $\mathcal{T}_B$ is given by

$$\mathbf{v}(\mathbf{x},t) := \dot{\boldsymbol{\chi}}_{t_0}(\hat{\mathbf{x}},t) \qquad \text{where} \quad \mathbf{x} = \boldsymbol{\chi}_{t_0}(\hat{\mathbf{x}},t). \tag{2.3.3}$$

Similarly, the acceleration field $\mathbf{a}$ on $\mathcal{T}_B$ is given by

$$\mathbf{a}(\mathbf{x},t) := \ddot{\boldsymbol{\chi}}_{t_0}(\hat{\mathbf{x}},t) \qquad \text{where} \quad \mathbf{x} = \boldsymbol{\chi}_{t_0}(\hat{\mathbf{x}},t). \tag{2.3.4}$$

In the case of *liquids* and *gases*, molecules close together at a given time do not remain so but diffuse rapidly.[3] An indication of *gross* molecular motion can be gained by the insertion and observation of small bubbles or suspended particles in liquids and smoke particles or balloons in gases. At any instant, such observations furnish velocity values of bubbles, particles or balloons which would seem, intuitively, to be representative of the instantaneous fluid velocity values at the locations of these 'foreign' objects. The modelling assumption made in fluid dynamics is that for a fluid body B there is a velocity field $\mathbf{v}$ defined on its trajectory $\mathcal{T}_B$. We can visualise an intuitive sense of fluid motion by looking at the situation at some time t_0 and then, on choosing any point $\hat{\mathbf{x}} \in B_{t_0}$, 'follow' the fluid by moving in such a way as always to have the same velocity as the local value of the fluid velocity. If, in such a motion, we arrive at point $\mathbf{x}$ at time t, then we can again write (2.3.2), where, by the foregoing, relation (2.3.3) [and, similarly, relation (2.3.4)] will also be satisfied. Further, with each $\hat{\mathbf{x}} \in B_{t_0}$ we can identify a *hypothetical* 'material point' which is to be regarded as located at $\mathbf{x}$ at time t.

Accordingly, *for both solid and fluid bodies* we have the concept of a motion (corresponding to the situation at some given time) which prescribes the distortion and movement of the relevant body in fine detail. This motion, given by (2.3.2), is related to the associated velocity and acceleration fields by (2.3.3) and (2.3.4).

For any $t \in I$ [see (2.3.1)], the motion map

$$\boldsymbol{\chi}_{t_0}(\cdot,t) : B_{t_0} \longrightarrow B_t \tag{2.3.5}$$

is assumed to be bijective. That is, if $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are any pair of distinct points in B_{t_0}, then, for any $t \in I$, $\boldsymbol{\chi}_{t_0}(\hat{\mathbf{x}},t)$ and $\boldsymbol{\chi}_{t_0}(\hat{\mathbf{y}},t)$ will not coincide, *and* for each $\mathbf{x} \in B_t$ there exists an $\hat{\mathbf{x}} \in B_{t_0}$ for which (2.3.2) holds.

Point to ponder 1. Consider how one might be led to the bijectivity hypothesis by recalling how a motion can be physically monitored (via doped molecular clusters for solids and immersed entities for fluids).

Point to ponder 2. Note the intrinsic difficulty of monitoring the internal deformation of *solids* and the necessity of remote sensing via a scanning procedure, and how in engineering practice one may only make measurements on the surface of a body (e.g., via attached strain gauges, transducers, or optical monitoring devices).

Point to ponder 3. Note that for fluids the flows of interest can involve very different length scales. For example, the velocity profile of flow down a pipe can only

[3] Typical molecular speeds for fluids (which may be macroscopically motionless) at *s*tandard *t*emperature and *p*ressure (STP) are, on average, of order 10^3 ms^{-1}. Further, individual molecular trajectories are highly erratic, much more so than the Brownian motion of small suspended particles (cf., e.g., Brush [6]).

be monitored at scales smaller than cross-sectional dimensions, while atmospheric wind velocity may be of interest at small scale (motion over an aerofoil), medium scale (motion around a skyscraper), or large scale (weather reporting). Accordingly, the notion of material point would appear to be context/scale-dependent.

Point to ponder 4. The question of scale dependence also arises with solids: consider deformations of small crystalline samples and motions of the Earth (namely terrestrial – solid – tides and seismic waves).

Summary. The notion of material point has been motivated quite differently for solid and fluid phases of matter. In a solid one can, roughly speaking, identify the position of a material point at a given time with the location of a small cluster of neighbouring molecules. The motion of this material point then can be tracked (at least in principle) by monitoring the motion of this cluster since any cluster of near-neighbouring molecules maintains its integrity. On the other hand, for fluids a material point can, loosely speaking, be thought of as a hypothetical immersed object whose motion is governed by the action thereon of fluid molecules with which it interacts/collides. Of course, the particular interacting/colliding molecules in question change rapidly with time. What should be clear is that

the key role played by the notion of material point, whether the body concerned is in solid, liquid, or gaseous state, is that of tracking the macroscopic distortion/flow of the body as time passes.

*2.4[4] The Formal (Axiomatic) Approach to Matter and Material Points

In formal continuum mechanics[5] the notion of material point is *primitive* (i.e., a formal concept which serves as a building block for subsequent development of the subject but is otherwise undefined). A body B is considered to be a set of material points. Any possible physical manifestation of the body is termed a *configuration*. More precisely, a configuration $\boldsymbol{\kappa}$ is a map

$$\boldsymbol{\kappa} : B \longrightarrow \mathcal{E}. \tag{2.4.1}$$

It is assumed that in no configuration can two distinct material points coincide. That is, if $\mathbf{X}, \mathbf{Y} \in B$ are distinct material points, then $\boldsymbol{\kappa}(\mathbf{X}) \neq \boldsymbol{\kappa}(\mathbf{Y})$ for any configuration $\boldsymbol{\kappa}$. Accordingly any configuration $\boldsymbol{\kappa}$ must be a bijection (i.e., one-to-one correspondence) as a map from B onto its range $\boldsymbol{\kappa}(B)$. For any pair of configurations $\boldsymbol{\kappa}$ and $\boldsymbol{\mu}$, it is assumed that the ranges $\boldsymbol{\kappa}(B)$ and $\boldsymbol{\mu}(B)$ are open subsets of $\mathcal{E}$ and that the bijection[6]

$$\mathbf{d} := \boldsymbol{\mu} \circ \boldsymbol{\kappa}^{-1} : \boldsymbol{\kappa}(B) \to \boldsymbol{\mu}(B) \tag{2.4.2}$$

is of class C^1. Any such map is termed a *deformation* of B.

A *motion* of B is a one-parameter family of configurations, parametrised by time, for some time interval I. If $\boldsymbol{\chi}(.,t)$ denotes the member of this family at time[7] $t \in I$

[4] Any starred section, subsection, or item may be skipped without affecting subsequent unstarred discussions.

[5] Cf., e.g., Gurtin [1], Truesdell & Noll [2], and Chadwick [3].

[6] Property (2.4.2) endows B with the structure of a C^1 differentiable manifold whose charts are configurations.

[7] Time t is usually regarded as present time, and $\boldsymbol{\chi}(.,t)$ is described as the *current configuration*.

and $\mathbf{X} \in B$, then $\boldsymbol{\chi}(\mathbf{X},t)$ is the location (a point in $\mathcal{E}$) of $\mathbf{X}$ at time t in this motion and $\dot{\boldsymbol{\chi}}(\mathbf{X},t)$ (where $\dot{\boldsymbol{\chi}} := \partial\boldsymbol{\chi}/\partial t$) is its *velocity* at this time. Given configurations $\boldsymbol{\kappa}$ and $\boldsymbol{\chi}(.,t)$, from (2.4.2) with $\boldsymbol{\mu} = \boldsymbol{\chi}(.,t)$, the deformation

$$\boldsymbol{\chi}_{\boldsymbol{\kappa}}(.,t) := \boldsymbol{\chi}(.,t) \circ \boldsymbol{\kappa}^{-1} \tag{2.4.3}$$

which maps $\boldsymbol{\kappa}(B)$ onto $\boldsymbol{\chi}(B,t) \subset \mathcal{E}$ is (spatially) of class C^1 (here t is considered fixed) and is termed the *deformation of B at time t with respect to configuration $\boldsymbol{\kappa}$*. Function

$$\boldsymbol{\chi}_{\boldsymbol{\kappa}} : \boldsymbol{\kappa}(B) \times I \longrightarrow \mathcal{E} \tag{2.4.4}$$

is termed the *motion relative to configuration $\boldsymbol{\kappa}$*. Region

$$_{\boldsymbol{\chi}}B_t := \boldsymbol{\chi}(B,t) \subset \mathcal{E} \tag{2.4.5}$$

is that *region occupied by the body at time t* in motion $\boldsymbol{\chi}$, and the *trajectory* associated with this motion is

$$_{\boldsymbol{\chi}}\mathcal{T}_B := \{(\mathbf{x},t) : \mathbf{x} \in {}_{\boldsymbol{\chi}}B_t \ \text{ with } \ t \in I\}. \tag{2.4.6}$$

[Cf. (2.2.1) and (2.3.1).] Since from (2.4.3)

$$\boldsymbol{\chi}(.,t) = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\cdot,t) \circ \boldsymbol{\kappa}, \tag{2.4.7}$$

the velocity of $\mathbf{X}$ at time t is

$$\dot{\boldsymbol{\chi}}(\mathbf{X},t) = \frac{\partial}{\partial t}\{\boldsymbol{\chi}_{\boldsymbol{\kappa}}(\boldsymbol{\kappa}(\mathbf{X}),t)\} =: \dot{\boldsymbol{\chi}}_{\boldsymbol{\kappa}}(\boldsymbol{\kappa}(\mathbf{X}),t). \tag{2.4.8}$$

The *velocity field* $\mathbf{v}$ on $_{\boldsymbol{\chi}}\mathcal{T}_B$ is defined by

$$\mathbf{v}(\mathbf{x},t) := \dot{\boldsymbol{\chi}}(\mathbf{X},t), \qquad \text{where } \ \mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t). \tag{2.4.9}$$

That is, the velocity at the geometrical point $\mathbf{x} \in {}_{\boldsymbol{\chi}}B_t$ at time t is the velocity of that material point which is located at $\mathbf{x}$ at time t. Similarly, the *acceleration field* $\mathbf{a}$ on $_{\boldsymbol{\chi}}\mathcal{T}_B$ is defined by

$$\mathbf{a}(\mathbf{x},t) := \ddot{\boldsymbol{\chi}}(\mathbf{X},t), \qquad \text{where } \ \mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t). \tag{2.4.10}$$

In view of the bijective nature of $\boldsymbol{\kappa}$, to each point $\hat{\mathbf{x}}$ in region $\boldsymbol{\kappa}(B)$ corresponds a unique material point and vice versa. Accordingly, points in $\boldsymbol{\kappa}(B)$ are identifiable with material points, and definitions (2.4.9) and (2.4.10) can be expressed in terms of physically accessible entities [namely, points $\hat{\mathbf{x}}$ in $\boldsymbol{\kappa}(B)$] via (2.4.8) as

$$\mathbf{v}(\mathbf{x},t) = \dot{\boldsymbol{\chi}}_{\boldsymbol{\kappa}}(\hat{\mathbf{x}},t) \qquad \text{and} \qquad \mathbf{a}(\mathbf{x},t) = \ddot{\boldsymbol{\chi}}_{\boldsymbol{\kappa}}(\hat{\mathbf{x}},t), \tag{2.4.11}$$

where

$$\mathbf{x} = \boldsymbol{\chi}_{\boldsymbol{\kappa}}(\hat{\mathbf{x}},t). \tag{2.4.12}$$

In this context $\boldsymbol{\kappa}$ is termed a *reference configuration*. Choosing $\boldsymbol{\kappa} = \boldsymbol{\chi}(.,t_0)$ and writing, for $\hat{\mathbf{x}} \in {}_{\boldsymbol{\chi}}B_{t_0}$,

$$\boldsymbol{\chi}_{t_0}(\hat{\mathbf{x}},t) := \boldsymbol{\chi}(\mathbf{X},t), \qquad \text{where } \ \hat{\mathbf{x}} = \boldsymbol{\chi}(\mathbf{X},t_0), \tag{2.4.13}$$