Chapter 2
Mechanisms

Abstract This chapter begins with a description of the different types of mechanisms that are generally used, especially in industrial robots. The parameters and variables of the mechanisms are defined and the degrees of freedom are calculated. Two methods to model a mechanism are presented. We show that in the Denavit-Hartenberg method, the attachment of local coordinate frames to the links is precisely specified, and relative to these frames a minimum number of translational and rotational parameters that describe the relative pose of two neighboring links are defined. In the so-called method of Vector Parameters, link and joint vectors are used to determine the geometry of the mechanism. As the reference position of a mechanism is a free choice, this method enables us to select the most appropriate reference position with respect to the requirements of the robot task.

In this chapter we shall consider a system of rigid bodies interconnected by joints. Joints allow particular types of relative motions between the connected bodies. For example, a rotational joint acts as a hinge and allows only a relative rotation between the connected bodies about the axis of the joint. The relative movements allowed by a joint are referred to as the joint variables or the internal coordinates. The rotational joint has only one joint variable and that is the relative rotation between the connected bodies.

A system of rigid bodies interconnected by joints is called a kinematic chain. Individual rigid bodies within the kinematic chain are called links. A kinematic chain can be serial, parallel, or serial and parallel combined together, i.e. the kinematic chain can be open, closed, or branched.

A mechanism results when a kinematic chain has one of its links fixed to ground, that link then being made immobile. The layout of the links and joints within a kinematic chain determines the motion properties of the mechanism that results from that kinematic chain.

In this chapter we first determine how the motion of a mechanism can be determined by considering the number of links in its kinematic chain and the constraints imposed by the joints connecting the links. We then describe several characteristic types of joints and pay special attention to the kinematic pair, which is the simplest kinematic chain, consisting of only two links connected by one joint. We then introduce the parameters of the kinematic pair and use them in modeling of mechanisms.

J. Lenarčič et al., *Robot Mechanisms*,
Intelligent Systems, Control and Automation: Science and Engineering 60,
DOI 10.1007/978-94-007-4522-3_2,

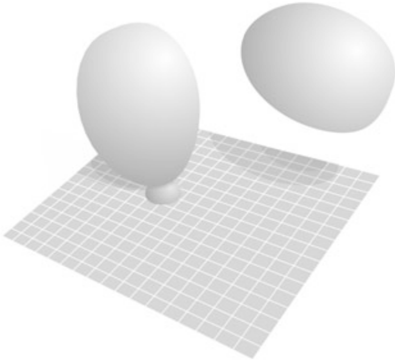

Fig. 2.1 Free body and body attached to the base by the help of a joint

We also become acquainted with the layout of links and joints which are frequently used in robotics.

2.1 Joints and Degrees of Freedom

A basic property of a mechanism is its ability to move. A mechanism results from a kinematic chain when, one of the links in the chain becomes fixed to ground. Thus a mechanism has a fixed base link. This fixed link connects to one or more links by one or more joints, and these links are then connected by joints to the remaining links. The moving links in a mechanism are accomplishing various tasks, such as grasping, picking and placing of objects, drilling, or grinding.

From a mathematical point of view, the degrees of freedom within a mechanism equals the minimum number of the mechanism's independent joint variables which must be specified in order to uniquely determine the spatial pose of all bodies belonging to the mechanism. This is also referred to as the number of degrees of freedom in the mechanism [21].

2.1.1 Types of Joints

A rigid body which is free to move in a 3-dimensional space has six degrees of freedom. The pose of a body is determined by $\lambda = 6$ parameters, where there are three positional and three orientational coordinates. A body free to move in a plane has $\lambda = 3$ degrees of freedom, two describing its position and one belonging to the orientation of a body.

When a body is attached to the base by virtue of a joint, as shown in Fig. 2.1, its number of degrees of freedom will be less than or equal to the number of degrees of freedom of a free body. Suppose there are f independent joint variables associated with a joint. We would say that the joint allows f degrees of freedom.

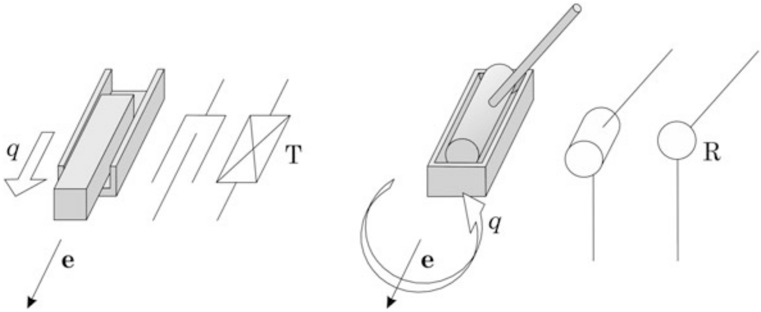

Fig. 2.2 Translational and rotational joint

The number of degrees of freedom of a body rigidly connected to the base is zero. When connected to the base by a joint, the body has at most as many degrees of freedom as allowed by the joint, which is always less then λ. Typically a joint is considered as a connection that allows certain motion of the body. Inversely, a joint can be considered as a connection which limits the motion of a body by virtue of constraints imposed by the joint. The difference between the possible degrees of freedom (λ) and the number of degrees of freedom allowed by a joint (f) is called the number of constraints

$$c = \lambda - f. \tag{2.1}$$

For example, in a rotational joint we have seen that $f = 1$ and hence in spatial motion ($\lambda = 6$), $c = 6 - 1 = 5$. That is to say that a rotational joint eliminates 5 degrees of freedom of relative motion between the connected bodies.

The number of degrees of freedom allowed by a joint, and the nature of those degrees of freedom, are determined by the shape of the contact areas between the two bodies that are associated with the particular type of joint. The two simplest joints allow $f = 1$ degrees of freedom and impose $c = 5$ constraints in spatial motion, or $c = 2$ constraints in planar motion. These are the translational and rotational joints, shown in Fig. 2.2, often denoted in the literature by the letters T and R. Both of these fundamental joints can be described by a unit vector $\mathbf{e}$, which defines the axis of either the linear displacement or rotation. The joint variable is the coordinate q describing either the distance of translation or the angle of rotation. In Fig. 2.2 the graphical representations of translational and rotational joints are shown. These representations will be adopted in the remainder of the text.

All other joints can be modeled as combinations of these two fundamental joints. In some cases there is an interdependence between the rotation(s) and translation(s). Let us examine some of the basic joints.

In addition to the translational and rotational joints there are the two degree of freedom cylindrical joint and universal joint, which have $f = 2$, and the three degree of freedom spherical joint, which has $f = 3$. The variety of joints in mechanical engineering is much larger than these, but these are the joints fundamental to robotics.

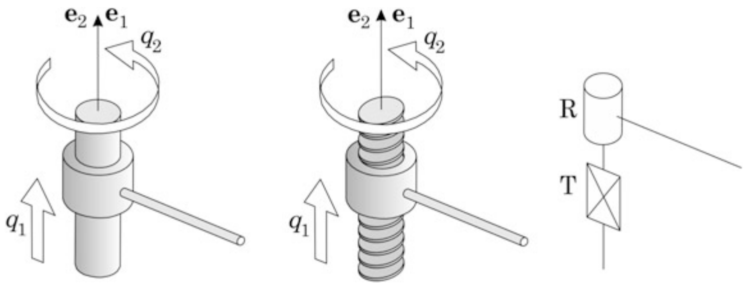

Fig. 2.3 Cylindrical and screw joints

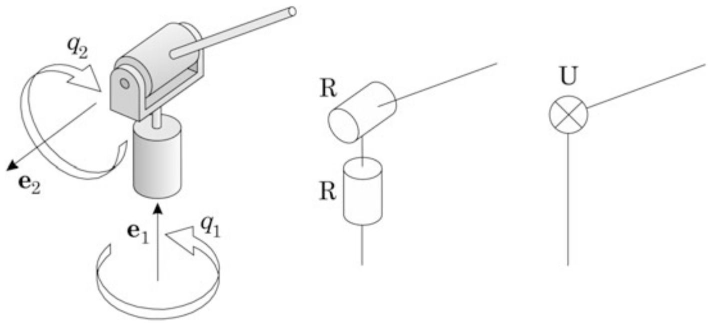

Fig. 2.4 Universal joint

Figure 2.3 presents two joints types of joints found in spatial mechanisms which can be represented by combination of a translational joint and a rotational joint, shown on the right hand side of the figure. The first joint on the left is called a cylindrical joint. Translation takes place in the direction of $\mathbf{e}_1$, rotation is about the vector $\mathbf{e}_2$, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are coincident. The joint variables are the distance q_1 and the angle of rotation q_2, and they are independent. Hence $f = 2$ and $c = 4$. The second joint is called a screw joint. This joint is also represented by a translation in the direction of the vector $\mathbf{e}_1$ and rotation about the axis $\mathbf{e}_2$ and again $\mathbf{e}_1$ and $\mathbf{e}_2$ are coincident. However, in the screw joint there is an interdependence between rotation q_2 and translation displacement q_1. This interdependence can be described by the relation $\Delta q_1 = \xi \Delta q_2$, where Δq_1 and Δq_2 represent changes in the joint variables q_1 and q_2 and ξ is a known constant called the pitch of the screw. Thus although the screw joint has two joint variables, it has only one independent joint variable, either q_1 or q_2, and therefore it has $f = 1$ and $c = 5$.

Figure 2.4 shows a universal joint, or so called U-joint. This type of joint is also found in spatial mechanisms and it is equivalent to two rotational joints whose axes are intersecting. Kinematically it is represented by the rotational axes $\mathbf{e}_1$ and $\mathbf{e}_2$ and the angles q_1 and q_2 which are independent. Hence the U-joint has $f = 2$ and $c = 4$. A possible realization is shown in Fig. 2.4.

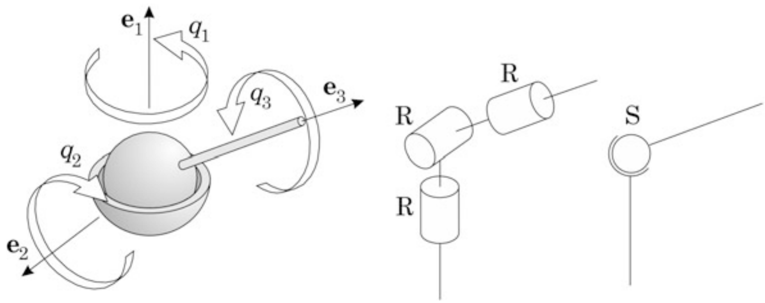

Fig. 2.5 Spherical joint

Among the three degree of freedom joints, the spherical joint is encountered most often. It is usually presented as the ball joint shown in Fig. 2.5. It is denoted by the letter S. For this joint we have $f = 3$ and $c = 3$. The joint allows independent rotations about three axes and is modeled by three rotations whose axes are intersecting at the same point. In Fig. 2.5 these are the axes $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ with corresponding rotations q_1, q_2, and q_3. We must be aware that the spherical joint with three sequential rotations is not completely equivalent to the ball joint. The joint with three sequential rotations looses one degree of freedom when the third axis is collinear with the first axis. Such a situation does not occur with the ball joint. However, due to a constraint of force closure, the socket which encapsulates the ball must wrap over more than a hemisphere of the ball, hence the spherical joint has a workspace which is limited to significantly less than a hemisphere, and this is effectively as limited a workspace as the joint with three sequential rotations.

As in robotics, the mechanics of joints is important when studying human movements [93]. Among the so called synovial joints, where two bones are in contact, we have ellipsoidal and saddle joints. These resemble rotational joints, but are not completely equivalent. There also exist joints which do not have firm contact areas. Here, the two bones are connected through elastic bands (ligaments). A unique joint is that between the shoulder blade and the trunk, the scapulothoracic joint. In this non-synovial joint, the shoulder blade actually slides and rotates between layers of muscles on the back.

2.1.2 Types of Mechanisms

Links connected by joints create kinematic chains. These can be open, closed, or branched [86] kinematic chains. Figure 2.6 shows some mechanisms coming from different kinematic chains. The mechanism on the left results from a six link open kinematic chain where one of the links is the fixed base. The five remaining links of the chain move in space. The mechanism in the back results an eleven link closed loop kinematic chain which has one of it links fixed to ground. The mechanism in front and on the right results from a branched open kinematic chain.

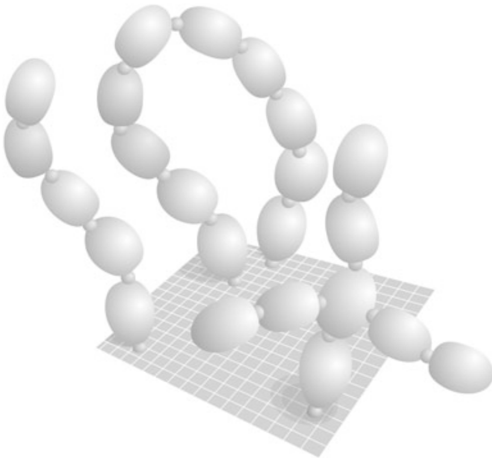

Fig. 2.6 Mechanisms from open, closed and hybrid open/closed kinematic chains

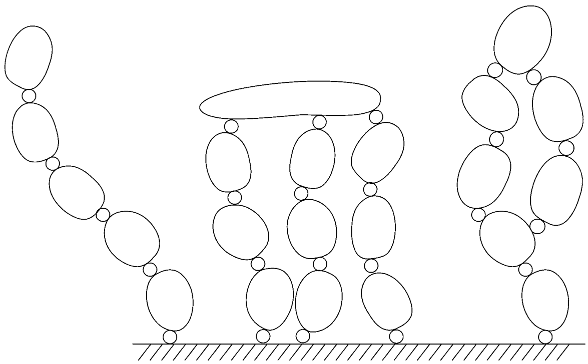

Fig. 2.7 Serial, parallel, and hybrid kinematic chains

Figure 2.7 shows additional examples of mechanisms that result from open, closed and hybrid open/closed kinematic chains. Mechanisms that result from open kinematic chains, such as that on the left, are referred to as serial mechanisms. Serial mechanisms typically have larger reachable workspace, but are less rigid. Mechanisms that result from closed kinematic chains, or groups of closed kinematic chains, such as that in the center, are referred to as parallel mechanisms. Parallel mechanisms typically are more rigid and exhibit larger load capacity and improved accuracy, but have smaller workspace. In parallel mechanisms, many times the multiple closed loops carry a single common link, referred to as the moving platform. In these cases, typically the fixed link is also a common element and is referred to as the fixed platform, or base. The links which connect between the moving and fixed platforms are referred to as legs. The example of a parallel mechanism in Fig. 2.7 has an identifiable moving platform, fixed platform (base) and three legs. Mechanisms that result from hybrid open/closed kinematic chains are referred as hybrid serial/parallel mechanisms. Hybrid serial/parallel mechanisms exhibit a combina-

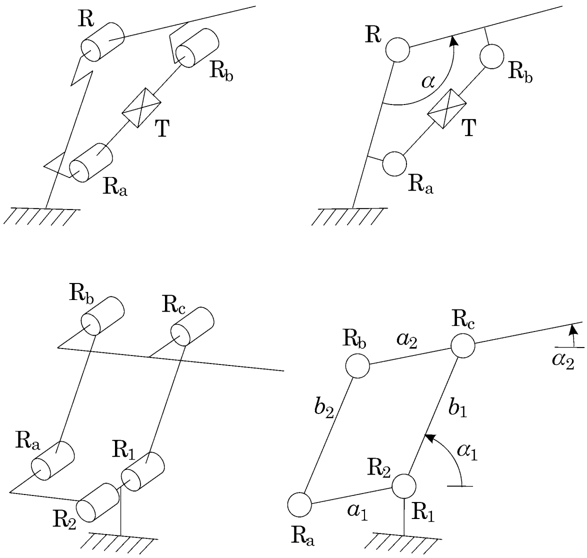

Fig. 2.8 Remotely actuated joints in a mechanism

tion of the traits of serial and parallel mechanisms. Both in robotics and with living organisms we can find examples of hybrid parallel/serial mechanisms.

Industrial robots are usually serial mechanisms. It was only recently that parallel robots (i.e. robots which are parallel mechanisms) were introduced into the industrial arena. Within serial mechanisms all joints are actuated by the motors and the motors closer to the base carry the motors farther from the base. This causes the motors closer to the base to be larger which has negative effects of increasing weight and cost. Many times in parallel manipulators, the driving motors can all be located on the base. The motors are smaller, less expensive and the machine is lighter.

Two simple examples of mechanisms often used in practice are shown in Fig. 2.8. The first mechanism represents a situation where actuation of the rotation joint R is accomplished through the translation T. We can think of R as being remotely actuated by T. This is accomplished by connecting two additional rotational joints R_a and R_b at both sides of translational joint. All together the mechanism now has four joints, however, only one is actuated. As will be shown, this mechanism has only one degree of freedom. Remote actuation of the rotation in a rotational joint is often used in mechanical engineering. In robots with hydraulic and pneumatic actuation, the T joint is an actuated hydraulic or pneumatic cylinder. In electrically driven robots, the T joint is a lead screw driven by an electric motor. The joints of living organisms are similarly remotely actuated by muscles, and in some cases several joints are actuated by the same muscle. This causes coupling in human joint motions. For example, when you curl your finger, your distal two knuckle joints are actuated simultaneously. This is because the muscle actuating this distal pair of knuckle joints spans across the two joints.

The second mechanism, shown in Fig. 2.8 is a parallelogram or pantograph mechanism. The pantograph is a device originally intended to copy drawings in

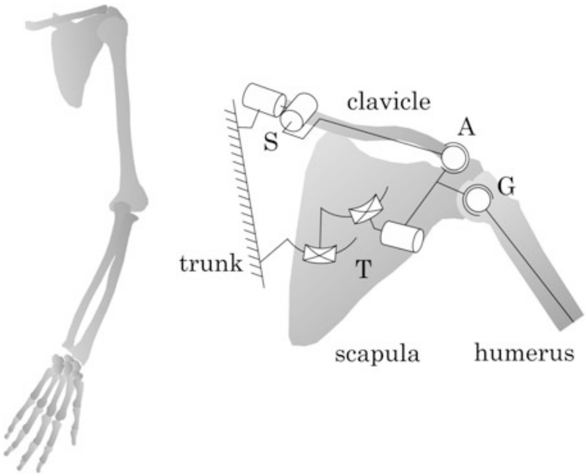

Fig. 2.9 Skeleton of human arm and the shoulder girdle

either enlarged or reduced scale. Office lamps commonly make use of a pantograph mechanism. The mechanism has five rotational joints, but only two degrees of freedom. It has interesting properties when $a_1 = a_2$ and $b_1 = b_2$. In that case, links a_1 and a_2 are parallel throughout the motion of the mechanism. The same is also true for links b_1 and b_2. Assume that the joints R_1 and R_2 are actuated. When the rotation R_1 is held constant, the angle α_1 is constant. Then by rotating the joint R_2, only the angle α_2 is changing. When the rotation R_2 is held constant, the angle α_2 is constant. Then by rotating R_1, only the angle α_1 varies. Despite rotation R_1, the orientation of link a_2 remains constant and only angle α_1 varies. This alternate type of remote actuation is used in many industrial robots, particularly those with high payloads. The mechanism also has the advantage that the actuators for R_1 and R_2 do not translate.

The skeleton of a human arm consists of a large number of bones, see Fig. 2.9, creating serial and parallel chains. The human arm is attached to the trunk through the shoulder girdle, this being a parallel mechanism. The clavicle is connected to the breastbone (a.k.a. sternum) via the sternoclavicular joint (S) from one side, and from the other side to the scapula via the acromioclavicular joint (A). The scapula glides along the back of the rib cage via the scapulothoracic joint (T) and at the lateral end it is connected through the glenohumeral joint (G) to the upper arm. A simplified model of a mechanism emulating the shoulder girdle is shown on the right hand side of Fig. 2.9. The joints of the model all together have eleven degrees of freedom. Later in the text we shall show however that this shoulder mechanism has only five independent degrees of freedom.

The upper arm (humeral bone) can be considered as part of a serial mechanism. In the elbow joint the humeral bone connects to radial bone and ulna, creating a parallel mechanism. Every finger together with the corresponding metacarpal bone represents a serial mechanism. In this way the palm is an example of the branched kinematic chain. When the palm and the fingers grasp an object, the kinematic chains of the fingers are closed and the hand is transformed into a parallel mechanism.

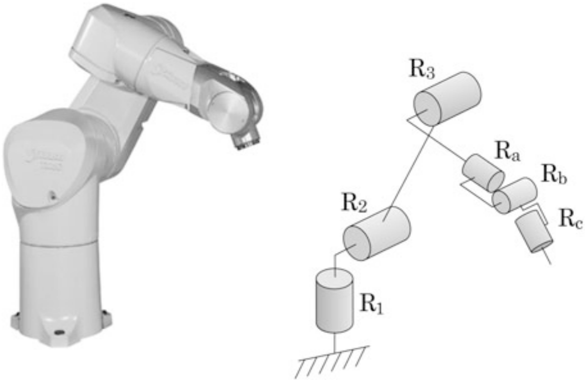

Fig. 2.10 Mechanism of industrial manipulator (courtesy of Stäubli)

The kinematic structure of a human arm is adapted to three principal functions. The primary function of the shoulder and elbow joints is to bring the hand into a desired position in space. The function of the wrist is to rotate the hand into a desired orientation, while grasping is the function of the palm and fingers.

The great majority of today's robots are industrial manipulators. An example is shown in Fig. 2.10. These are mechanical arms, which, through the shoulder joint, are attached to a fixed base. A robot gripper or another tool is attached to the distal end of the manipulator. The mechanisms of most industrial robots can be divided into two mechanisms, one for positioning and another for orientating. The positional part of the mechanism in Fig. 2.10 includes the rotations R_1, R_2, and R_3 and ranges from the shoulder to the lower arm. Its main task is to position the robot end-point into the desired position. The orientational part of the mechanism, which is in Fig. 2.10 denoted by rotations R_a, R_b, and R_c, represents the robot wrist, whose main task is to bring the gripper or end-point tool into the desired orientation. Each of these two parts requires three degrees of freedom.

Let us examine first the positional part of the mechanism. In this positional part of the mechanism, both translational and rotational can be used, since both types of joints can contribute to the positioning a point belonging to the mechanism. The axes of these joints can be directed arbitrarily in space. However, we shall limit ourselves to those structures where the successive translational or rotational axes are either parallel or perpendicular and at times intersecting, which is the case in industrial manipulators. Such positional mechanisms can always be placed in a pose such that all its joint axes are parallel to one of the axes of the fixed coordinate frame **x**, **y**, **z**. Therefore we can select among translational joints T_x, T_y, and T_z in the **x**, **y**, **z** directions, and rotational joints R_x, R_y, and R_z about the **x**, **y**, **z** directions, for the three joints of the mechanism. The number of all possible combinations of three of these six joints is $6^3 = 216$. All of these combinations do not result in a spatial mechanism. For example, structures, such as $T_xT_xT_x$ and $T_xT_yR_z$, do not permit motion in at least one of the directions of the coordinate frame **x**, **y**, **z**. If we wish to have a spatial mechanism, its joint variables must enable motion in all three directions. Thus each joint variable must provides for a component of motion

Table 2.1 Spatial positional mechanisms

$R_xR_xR_y$	$R_xR_xR_x$	$R_xT_xR_y$	$R_xT_yT_x$	$T_xR_xT_z$
$R_xR_xR_z$	$R_xR_yT_x$	$R_xT_xR_z$	$R_xT_zT_x$	$T_xR_yT_y$
$R_xR_yR_z$	$R_xR_yT_y$	$R_xT_yR_y$	$T_xR_xR_x$	$T_xT_yR_x$
$R_xR_yR_y$	$R_xR_yT_z$	$R_xT_yR_z$	$T_xR_xR_y$	$T_xT_yR_y$
$R_xR_yR_z$	$R_xR_zT_x$	$R_xT_zR_y$	$T_xR_xR_x$	$T_xT_yT_z$
$R_xR_zR_x$	$R_xR_zT_y$	$R_xT_zR_z$	$T_xR_yR_x$	
$R_xR_zR_y$	$R_xR_zT_z$	$R_xT_xT_y$	$T_xR_yR_z$	
$R_xR_zR_z$	$R_xT_xR_x$	$R_xT_xT_z$	$T_xR_xT_y$	

in one direction which is independent of the motions caused by the other two joint variables.

The letter in parentheses denotes the direction of the action of each degree of freedom. In this way we have $T_x \rightarrow (x)$, $T_y \rightarrow (y)$, $T_z \rightarrow (z)$, $R_x \rightarrow (y, z)$, $R_y \rightarrow (x, z)$ and $R_z \rightarrow (x, y)$. The mechanism $R_zR_xT_y \rightarrow (x, y)(y, z)(y)$ is a spatial mechanism, as each degree of freedom enables displacement of the mechanism in a unique direction. With such analysis of all 216 mechanisms, 129 spatial mechanism are selected. Some of them appear twice in the analysis and some differ only in the orientation with respect to the coordinate frame. Such examples are $R_zR_xR_x$ and $R_zR_yR_y$. When excluding all repeated variations, 37 different positional spatial mechanism remain [46], as shown in Table 2.1.

Only some of these mechanism are used as industrial robots. The International Federation of Robotics classifies in its statistical reports five types of positioning mechanisms that are found in industrial robots. They are shown in Fig. 2.11. Together with five serial mechanisms, we are presenting also a parallel mechanism, whose structure is not included in the classification. The parallel mechanism in Fig. 2.11 has three degrees of freedom. In general, parallel mechanisms differ considerably from serial mechanisms, in that the positioning and orienting functions in parallel mechanisms are typically completely coupled whereas in serial mechanisms they are typically only partially decoupled.

The orientational part of the robot mechanism needs to include at least three degrees of freedom in order to be able to bring the end-point tool into the desired orientation. By combining the rotations R_x, R_y, and R_z, 27 different wrist structures can be created. However, only the structures with successively perpendicular axes are considered. If the first rotation has the **x** direction, the next must be in the **y** or **z** direction. If the second rotation goes along **y** axis, the third must be about **z** or **x** and when the direction of the second joint axis is **z**, the third must be aligned with **y** or **x** axes. We have 12 such structures, which are all kinematically equivalent. They differ only with respect to the orientation of their attachment to the terminal link of the positioning mechanism.

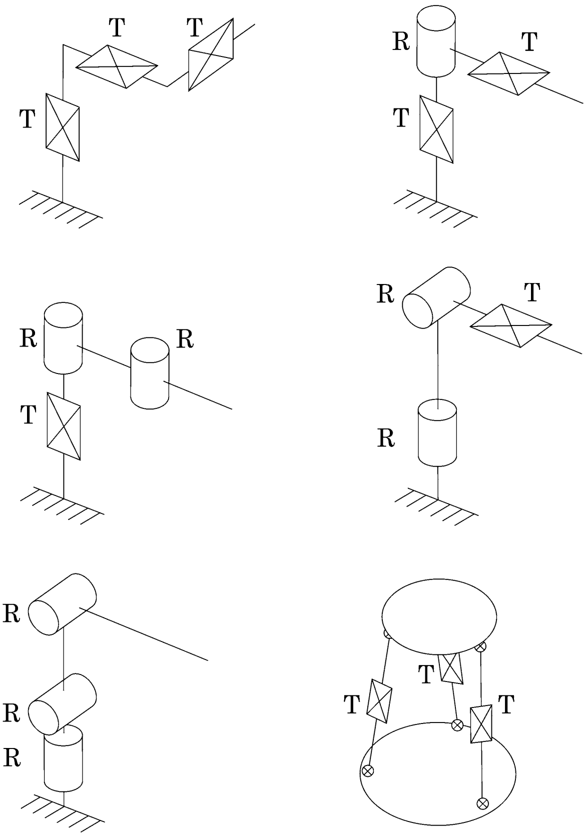

Fig. 2.11 Positional mechanisms of industrial manipulators according to the classification of International Federation of Robotics include TTT Cartesian manipulator, TRT cylindrical manipulator, TRR Scara manipulator, RRT spherical manipulator, RRR articulated manipulator and parallel manipulator

2.1.3 Degrees of Freedom in Mechanisms

The links of a mechanism are interconnected with various types of joints, either single or multiple points. Some examples of mechanisms are shown in Fig. 2.12. The number of independent degrees of freedom allowed by a joint i will be denoted as f_i, while the number of constraints imposed by the joints by c_i. All together the mechanism has $i = 1, 2, \ldots, n$ joints and $i = 0, 1, 2, \ldots, N$ links. The number 0 belongs to the reference link, i.e. the base. This is the fixed link in the kinematic chain from which the mechanism was derived. The base link does not move, while the remaining links $i = 1, 2, \ldots, N$ are in motion.

The number of degrees of freedom F belonging to a mechanism is defined as the number of independent joint variables which need to be specified in order to uniquely define the spatial configuration of the mechanism, i.e. in order to uniquely define the pose of every link in the mechanism. The number of degrees of freedom in a mechanism is obtained by first summing up the degrees of freedom made available by all mobile links of a mechanism, which would be λN. This would be the number of degrees of freedom in the mechanism if there were no joints. From this we must

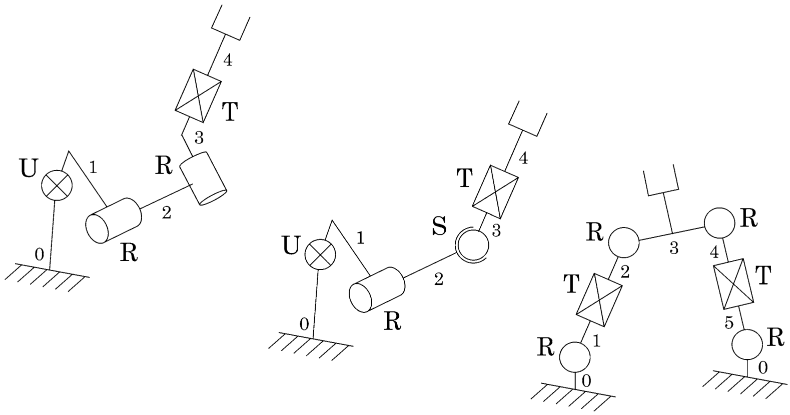

Fig. 2.12 Two spatial and one planar mechanism

deduct the number of constraints introduced into the mechanism by each of the joints. This is expressed by the following equation

$$F = \lambda N - \sum_{i=1}^{n} c_i .$$

We shall insert $c_i = \lambda - f_i$ from (2.1) into the above equation. It follows

$$F = \lambda(N - n) + \sum_{i=1}^{n} f_i . \tag{2.2}$$

The expression above is well known as Grübler's formula [84].

Let us now calculate the number of degrees of freedom of the mechanisms shown in Fig. 2.12. On the far left we have a serial mechanism in space ($\lambda = 6$) with four joints ($n = 4$), a two degree of freedom universal joint ($f_1 = 2$), two rotational joints ($f_2 = f_3 = 1$), and a translational joint ($f_4 = 1$). The mechanism has five links (including the base), four of them are mobile ($N = 4$). According to Grübler's formula the number of degrees of freedom in the mechanism is

$$F = 6(4 - 4) + 5 = 5.$$

Therefore, the gripper of the robot end-point has five degrees of freedom. We can also observe, that in serial mechanisms the number of mobile links N always equals the number of the joints n

$$n = N. \tag{2.3}$$

Therefore, the expression in parentheses in Grübler's formula, $(N - n)$, is always zero for serial mechanisms. The number of degrees of freedom of a serial mechanism is equal to the sum of the independent joint variables associated with each joint

$$F = \sum_{i=1}^{n} f_i \tag{2.4}$$

for both planar and spatial mechanisms.

Let us now examine the degrees of freedom of the mechanism shown in the middle of Fig. 2.12. This mechanism is also serial and spatial ($\lambda = 6$) and has four joints ($n = 4$), a two degree of freedom universal joint ($f_1 = 2$), a rotational joint ($f_2 = 1$), a spherical joint ($f_3 = 3$) and a translational joint ($f_4 = 1$). In accordance with (2.4) the mechanism has

$$F = 2 + 1 + 3 + 1 = 7$$

degrees of freedom. The robot gripper cannot have more than $\lambda = 6$ degrees of freedom, the mechanism is therefore kinematically redundant. A redundant mechanism can hold the gripper in a desired position and orientation, while the rest of the mechanism is still movable. This is referred to as a self-motion of the mechanism and it is a characteristic of a redundant mechanism.

In the right side of Fig. 2.12 is shown a planar parallel mechanism, hence $\lambda = 3$. The mechanism has six joints ($n = 6$), all of them are either rotational or translational, and six links, where five are mobile ($N = 5$). In accordance with Grübler's formula, the number of degrees of freedom is

$$F = 3(5 - 6) + 6 = 3.$$

This is also the number of the degrees of freedom of the gripper. In parallel mechanisms we are dealing with the following inequality

$$n > N, \tag{2.5}$$

and the simplification (2.4) does not hold.

By using Grübler's formula we shall calculate the number of degrees of freedom for the mechanisms from the previous section. The mechanisms from Fig. 2.8 are both planar, therefore $\lambda = 3$. With the upper mechanism we have $n = 4$ and $f_1 = f_2 = f_3 = f_4 = 1$. The mechanism has four links one of them is selected as a reference link, therefore $N = 3$. It follows

$$F = 3(3 - 4) + 4 = 1,$$

which further proves the statement that this mechanism has only a single degree of freedom. In the lower mechanism there is $n = 5$ and $f_1 = f_2 = f_3 = f_4 = f_5 = 1$. This mechanism has five links, from which one is a reference one, therefore $N = 4$. It follows

$$F = 3(4 - 5) + 5 = 2.$$

The pantographic mechanism has two degrees of freedom and can be used to position a point on its terminal link.

The mechanism of the shoulder girdle from Fig. 2.9 is a spatial mechanism, hence $\lambda = 6$. The mechanism has five one degree o freedom joints $f_1 = f_2 = f_3 = f_4 = f_5 = 1$ and two three degree of freedom joints $f_6 = f_7 = 3$, therefore $n = 7$. After subtracting the reference link, we have all together $N = 6$ links. There is

$$F = 6(6-7) + 11 = 5.$$

Thus the upper arm has five degrees of freedom. The shoulder girdle, the mechanism without the spherical glenohumeral joint G, has only two degrees of freedom. This can be calculated by subtracting one spherical joint and one link in the preceding equation, giving

$$F = 6(5-6) + 8 = 2.$$

The calculation of the number of degrees of freedom of the industrial manipulator from Fig. 2.10 is even simpler as the expression (2.4) can be used. This spatial serial mechanism has only rotations and $f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 1$, therefore

$$F = 6.$$

The gripper at the end of the robot has six degrees of freedom and can be placed in an arbitrary position and orientation inside its workspace.

2.2 Parameters and Variables of a Kinematic Pair

A kinematic pair is the basic element of a kinematic chain. It consists of two links connected by a joint with a translational or rotational degree of freedom. In the literature there exist two approaches to the mathematical description of a kinematic pair. The difference between them is in the attachment of the coordinate frames to both links. The so called Denavit and Hartenberg method [17] is based on an adapted homogeneous transformation matrix. This method makes use of four scalars, which will be called the Denavit and Hartenberg parameters of a kinematic pair. Four is the minimum number of parameters required for describing the link geometry and the relative joint displacement of the links in a kinematic pair. This leads to a minimum number of arithmetic operations in computations, which is an advantage of the method. Denavit and Hartenberg establish precise rules on how to position and orient the two coordinate frames and this first step in modeling of a mechanism can be cumbersome. If the rules are not followed precisely, the resulting kinematic equations are incorrect.

In the second approach, the method of Vector Parameters, vectors are used to describe a kinematic pair [46]. This method is based on the general rotational matrix and is therefore computationally more complex. The benefit of the method is in a simpler determination of the parameters of a kinematic pair. A similar vectorial method was introduced by [79] in modeling robot dynamics.

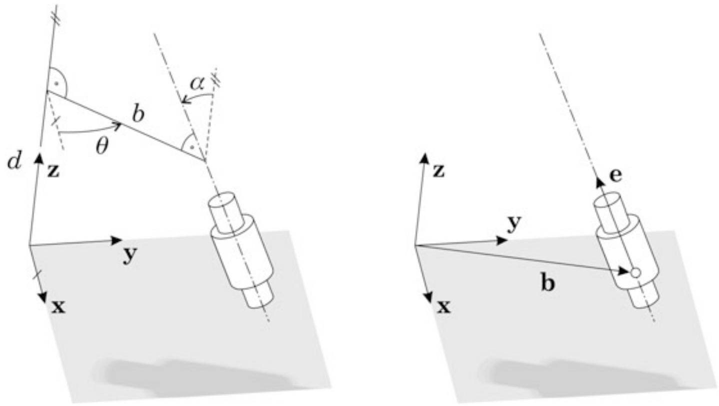

Fig. 2.13 Description of the position and direction of a cylindrical joint in a Cartesian coordinate frame

2.2.1 Cylindrical Joint in a Cartesian Space

The motion characteristics of the joints described in the beginning of this chapter can be illustrated with the aid of various combinations of rotational or translational joints, whose coordinates are appropriately mathematically related. As the basis for many of these joints, we use the cylindrical joint, wherein the rotation and translation between connected links takes place about and along a single axis, see Fig. 2.3. In order to describe this joint mathematically, we must know the position and direction of this axis with respect to the selected coordinate frame, and the values of the translational and rotational joint variables.

In the left side of Fig. 2.13 the position and direction of the joint axis is determined with respect to the Cartesian frame using the minimum possible number of parameters. To do this we introduce the common normal between the joint axis and the **z** axis of the coordinate frame. We know the common normal between two arbitrary non-parallel lines is unique. The location of the common normal can be determined by assessing the length, d, measured along **z** axis from the origin to the foot of the common normal on the **z** axis. The angle θ, represents the rotation about the **z** axis and is measured from the **x** axis to the direction of the common normal. The length of the common normal is b. The angle α represents the rotation of the joint axis about the common normal. These four scalar parameters d, θ, b, and α are the minimum number of parameters required to describe the location of the axis of a cylindrical joint in a Cartesian coordinate system. Denavit and Hartenberg's scalar parameters of a kinematic pair is based on these four parameters. Denavit and Hartenberg notation will be explained later in the text.

The right side of Fig. 2.13 the location of a joint axis is defined by the position of an arbitrary point on the axis. This is represented by the vector $\mathbf{b} = (b_x, b_y, b_z)^T$. The direction of the axis is defined by the unit vector **e**, which points in a direction along the axis. Here, the position and direction of the line is determined by the two

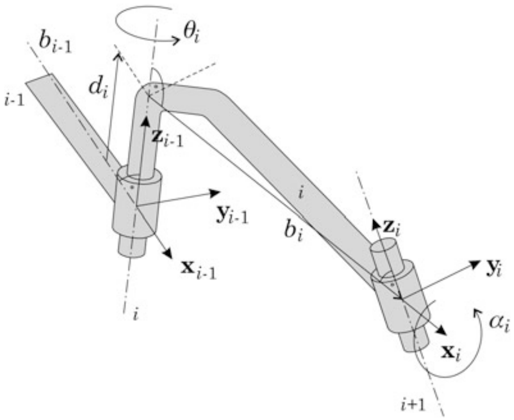

Fig. 2.14 Denavit-Hartenberg parameters of a kinematic pair

vectors, $\mathbf{b}$ and $\mathbf{e}$, where $\mathbf{e}^{\mathrm{T}}\mathbf{e} = 1$ and the component of $\mathbf{b}$ along $\mathbf{e}$ is arbitrary. The method of Vector Parameters is based on these two vectors and a general rotation matrix. Note that vectors $\mathbf{b}$ and $\mathbf{e}$ include information about the direction of the joint axis, which is not directly described by the four scalar parameters d, θ, b, and α.

2.2.2 Scalar Parameters of a Kinematic Pair

Let us examine first the characteristic properties of the Denavit-Hartenberg method in modeling a kinematic pair. The links and joints will be enumerated in a way which is widely accepted in robotics [67]. The links and joints can be enumerated in a second way [15] without changing the important properties of the approach. Here we follow [67].

Figure 2.14 shows a kinematic pair consisting of links $i - 1$ and i connected by cylindrical joint i. At the end of the link i we have the joint $i + 1$. The coordinate frame, which is attached to link i, is oriented in such a way, that its $\mathbf{z}_i$ axis is aligned with the joint axis $i + 1$, while the $\mathbf{x}_i$ axis goes along the common normal between the joint axes i and $i + 1$. The origin of this frame is positioned at the intersection of this common normal and the $i + 1$ joint axis. The third axis is represented by the $\mathbf{y}_i = \mathbf{z}_i \times \mathbf{x}_i$.

The Denavit-Hartenberg parameters, describing the geometry and the relative displacement between the bodies of a kinematic pair, are the following:

d_i: translational coordinate—the distance between the origin of the coordinate frame $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$ or the end-point of the normal b_{i-1} and the starting point of the normal b_i. The distance d_i is positive in direction of the axis $\mathbf{z}_{i-1}$;

θ_i: rotational coordinate—the angle of rotation between the links i and $i - 1$, which is measured from b_{i-1} to b_i, i.e. from $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$, about the $\mathbf{z}_{i-1}$ vector;

b_i: length of the link i—the length of the common normal of joint axes i and $i + 1$, representing the length of the i-th link, measured positively in the direction from the joint axis i to the joint axis $i + 1$;

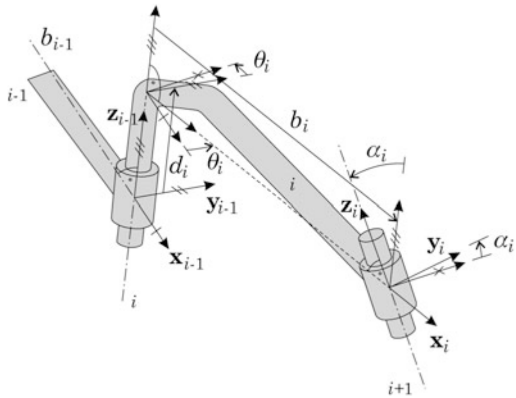

Fig. 2.15 Denavit-Hartenberg parameters shown as sequence of two translations and two rotations

α_i: angle of inclination between two consecutive axes—the angle measured about the $\mathbf{x}_i$ axis between the vectors $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$.

The scalars d_i, θ_i, b_i, and α_i represent the minimal number of the parameters required for a complete description of an arbitrary kinematic pair with the cylindric joint. When a joint is rotational, the joint variable of the kinematic pair is θ_i, while the rest of the parameters are constant. When a joint is translational, the joint variable of the kinematic pair is d_i, while the other three parameters are constant.

While using the Denavit-Hartenberg parameters, the kinematic pair is considered as a sequence of two translations and two rotations. As shown in Fig. 2.15, the coordinate frame $\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1}$ is superimposed over the frame $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i$ through translation by the distance d_i along the i-th joint axis and rotation by the angle θ_i about the joint axis i This is followed by a translation b_i, which is perpendicular to the joint axes i and $i+1$, and rotation by the angle α_i about the common normal between the joint axes i and $i+1$.

The pose between the coordinate frames $\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1}$ and $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i$ is given by the homogeneous transformation matrix $\mathbf{H}_{i-1,i}$, which always has the same form. It is a product of a homogeneous transformation matrix $\mathbf{Q}_{i-1,i}$, describing the joint translation d_i and rotation θ_i, and the homogeneous transformation matrix $\mathbf{S}_{i-1,i}$, describing the length b_i and the link inclination α_i. We have

$$\mathbf{H}_{i-1,i} = \mathbf{Q}_{i-1,i}\mathbf{S}_{i-1,i}, \tag{2.6}$$

with the matrices

$$\mathbf{Q}_{i-1,i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.7}$$

and

$$\mathbf{S}_{i-1,i} = \begin{bmatrix} 1 & 0 & 0 & b_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.8}$$

After the multiplication it follows

$$\mathbf{H}_{i-1,i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & b_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & b_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.9}$$

This matrix can be adapted to any kinematic pair which consists of one rotation and one translation. In robot systems the axes of consecutive joints i and $i+1$ are often either parallel or orthogonal. In these cases the transformation matrix has a simpler form. When the axes are parallel, the inclination angle between the joint axes is $\alpha_i = 0$ and we have

$$\mathbf{H}_{i-1,i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & b_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i & 0 & b_i\sin\theta_i \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.10}$$

When the axes are orthogonal, we have $\alpha_i = \pm\pi/2$. The simplified homogeneous transformation matrix has the following form

$$\mathbf{H}_{i-1,i} = \begin{bmatrix} \cos\theta_i & 0 & \pm\sin\theta_i & b_i\cos\theta_i \\ \sin\theta_i & 0 & \mp\cos\theta_i & b_i\sin\theta_i \\ 0 & \pm 1 & 0 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.11}$$

With the Denavit-Hartenberg method it is important to draw attention to the case when axes i and $i+1$ are orthogonal and intersecting. In this case parameter b_i is zero and the common normal $\mathbf{x}_i$ is the normal to the plane defined by the intersecting axes. There are two oppositely directed possibilities for the direction of $\mathbf{x}_i$. Either is acceptable. As well, in the case when neighboring axes i and $i+1$ are parallel, the direction $\mathbf{x}_i$ is defined, but its position is not. In this case parameter b_i is arbitrary and is usually selected so as to reduce the total number of parameters describing the mechanism.

2.2.3 Vector Parameters of a Kinematic Pair

Consider the method of Vector Parameters, which is primarily used in the remainder of this book, and the procedure for determining the parameters of a kinematic pair.

Fig. 2.16 Vector parameters of a kinematic pair

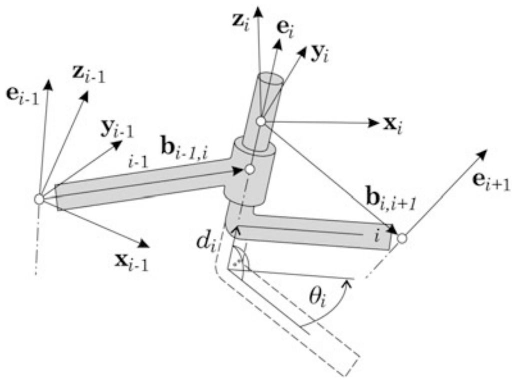

Figure 2.16 shows links $i-1$ and i connected by a cylindrical joint i. The position and direction of the joint axis is determined by the link vector $\mathbf{b}_{i-1,1}$ and the unit joint vector $\mathbf{e}_i$. Link i can be translated by a distance d_i in the direction $\mathbf{e}_i$, and rotated through an angle θ_i about $\mathbf{e}_i$, relative to link $i-1$. The coordinate frame $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ is attached to link i and frame $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$ is attached to link $i-1$.

In the reference position of the kinematic pair, both joint variables of the kinematic pair are assigned zero values, i.e. $\theta_i = 0$ and $d_i = 0$, and the frame is taken to be parallel to the preceding frame $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$.

With this method it is not necessary that one of the axes $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ is parallel to the joint vector $\mathbf{e}_i$, as is the case with the Denavit-Hartenberg method. In the method of Vector Parameters the geometry of a kinematic pair and the relative displacement between the link are defined by the following parameters:

- $\mathbf{e}_i$: joint vector—the unit vector defining the rotational or translational axis of the i-th joint;
- $\mathbf{b}_{i-1,i}$: link vector—the vector describing the length and direction of the link $i-1$ (the length and the direction of the link i is given by the vector $\mathbf{b}_{i,i+1}$);
- θ_i: rotational coordinate—the angle measured about the $\mathbf{e}_i$ axis in the plane perpendicular to the vector $\mathbf{e}_i$ (the rotation coordinate is zero when the kinematic pair is in its reference position);
- d_i: translation coordinate—the distance measured in the direction $\mathbf{e}_i$ (the translation coordinate is zero when the kinematic pair is in its reference position).

The cylindrical joint in Fig. 2.17 can be reduced to either a rotational joint, or a translational joint. When the kinematic pair is rotational (upper example in Fig. 2.17), the joint variable is the rotational coordinate θ_i, while $d_i = 0$. When the mechanism is in its reference position, then $\theta_i = 0$ and coordinate frames $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ and $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$ are parallel. When the kinematic pair is translational (lower example in Fig. 2.17), the joint variable is translational coordinate d_i, while $\theta_i = 0$. When the kinematic pair is in the reference position, we have $d_i = 0$. The coordinate frames $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ and $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$ are now parallel irrespective of the value of translational coordinate d_i.

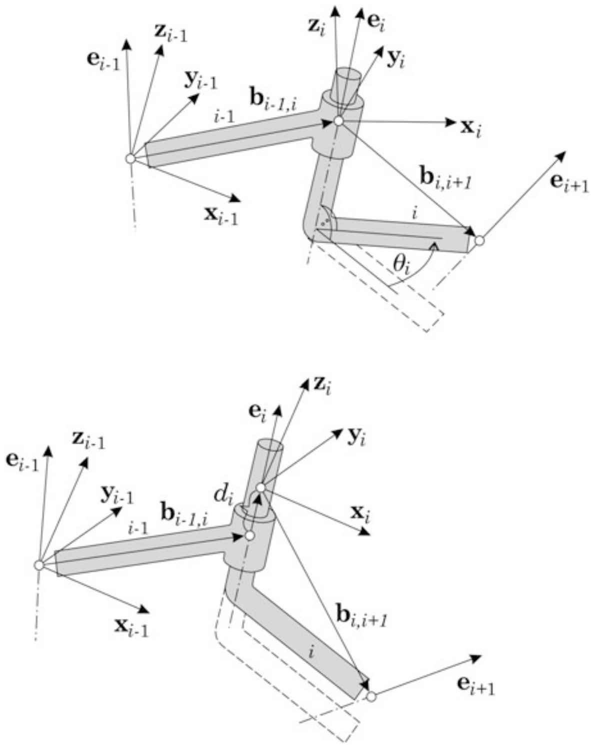

Fig. 2.17 Vector parameters of a kinematic pair

By changing the values of the rotational coordinate θ_i, the coordinate frame $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ is rotating together with the i-th link with respect to the preceding link $i-1$ i.e. with respect to the preceding frame $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$. When changing the values of the translational coordinate d_i, the displacement is translational and only the distance between the origins of both frames is changing.

Let us mathematically define the transformation between coordinate frame $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$ and coordinates frame $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$. Frame $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ is translated relative to frame $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$ along the vector $\mathbf{b}_{i-1,i}$ and along the vector $d_i\mathbf{e}_i$. Then it is rotated through an angle θ_i about the unit vector $\mathbf{e}_i$

$$\mathbf{H}_{i-1,i} = \begin{bmatrix} & \mathbf{A}_{i-1,i} & & d_i\mathbf{e}_i^{(i-1)} + \mathbf{b}_{i-1,i}^{(i-1)} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.12}$$

The rotation matrix $\mathbf{A}_{i-1,i}$, defining the transformation between the vector spaces $\Re_{i-1}^3$ and $\Re_i^3$, is obtained by the use of general formula (1.48), where the rotational vector is the joint vector $\mathbf{e}_i^{(i-1)}$ and the rotation coordinate is θ_i. In practical applications the kinematic pair can be placed in a reference position where joint vector $\mathbf{e}_i$ is parallel to one of the axes of the frame $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$. In this case the rotation matrix $\mathbf{A}_{i-1,i}$ can be calculated with one of the simplified expressions (1.51), (1.52) or (1.53).

In the reference position, coordinate frames $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$ and $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ are parallel ($\theta_i = 0$ in $d_i = 0$), and we have

$$\mathbf{H}_{i-1,i} = \begin{bmatrix} & \mathbf{I} & & \mathbf{b}_{i-1,i}^{(i-1)} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.13}$$

When a joint is only rotational ($d_i = 0$), then

$$\mathbf{H}_{i-1,i} = \begin{bmatrix} & \mathbf{A}_{i-1,i} & & \mathbf{b}_{i-1,i}^{(i-1)} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.14}$$

and when a joint is only translation ($\theta_i = 0$), we have

$$\mathbf{H}_{i-1,i} = \begin{bmatrix} & \mathbf{I} & & d_i \mathbf{e}_i^{(i-1)} + \mathbf{b}_{i-1,i}^{(i-1)} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.15}$$

In general, all the components of the vector parameters $\mathbf{e}_i$ and $\mathbf{b}_{i-1,i}$ are nonzero. In special cases, depending on the selection of a joint center and the reference position of a kinematic pair, some of the components of these vector parameters can be zero. The scalar Denavit-Hartenberg parameters of a kinematic pair represent a special case of vector parameters. The advantages of using the method of Vector Parameters is in the simplicity of the coordinate frame assignments and in the freedom of choosing any relative position of the two links as the position where the joint variables are zero.

When comparing the Vector Parameters of a kinematic pair with scalar Denavit-Hartenberg parameters, the following important difference can be noticed. With Vector Parameters the length of the translation d_i is measured from the selected reference position in the direction of vector $\mathbf{e}_i$. With Denavit and Hartenberg parameters the length of the translation d_i is the distance between the intersection of vector b_{i-1} with joint axis i and the intersection of vector b_i with joint axis i. With Vector Parameters the angle of rotation θ_i is assessed from the selected reference position about vector $\mathbf{e}_i$. With Denavit and Hartenberg parameters the angle of rotation θ_i is defined as the angle between vectors $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$.

2.3 Parameters and Variables of a Mechanism

In this section we extend the description of a kinematic pair to the description of an entire mechanism which consists of several links and joints. The extension is relatively straightforward. First we consider this description in terms of the Denavit-Hartenberg parameters and then by the Vector Parameters. The two methods are compared via the example robot mechanism shown in Fig. 2.18. Examples of mechanisms with various kinematic arrangements can be found in [6, 67, 77].

The serial spatial mechanism shown in Fig. 2.18 has $n = 4$ degrees of freedom. At one end it is attached to the base, while a gripper is mounted to the other end.

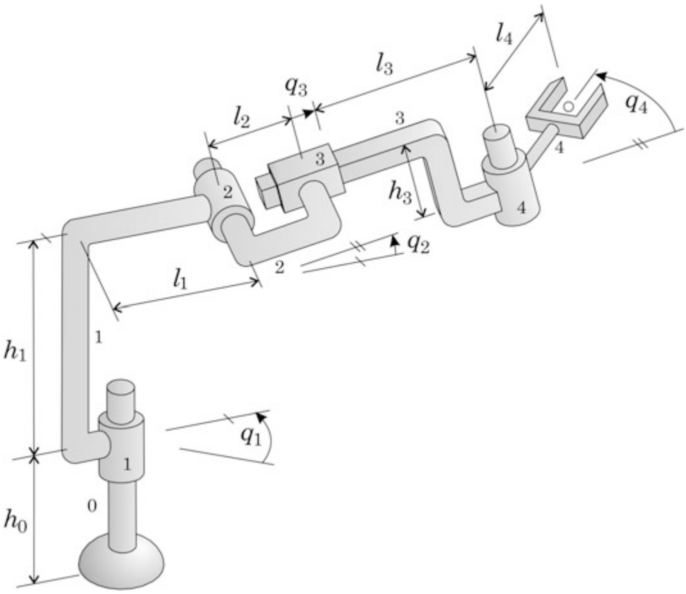

Fig. 2.18 Example of mechanism with four degrees of freedom

The mechanism has three rotations with joint variables q_1, q_2, and q_4 and one translation with joint variable q_3. The links are denoted as $0, 1, \ldots, 4$, and the joints are numbered $1, 2, 3, 4$. The mechanism has $N = n = 4$ of mobile links. Their lengths are given by distances h_0, h_1, l_1, l_2, l_3, h_3, and l_4. The link 0 is the fixed base.

2.3.1 Denavit and Hartenberg Parameters of a Mechanism

In the Denavit and Hartenberg method, a local coordinate frame is attached to each mobile link of a mechanism. The motion of this frame is observed relative to a fixed reference frame $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$ which is attached to the fixed base. The i-th joint connects the links $i - 1$ and i, $i = 1, 2, \ldots, n$. Recall, with serial mechanisms the number of joints n is equal to the number of mobile links N. The Denavit-Hartenberg parameters of a mechanism are determined in the five following steps [77, 84]:

Step 1 In order to reduce the number of parameters, the fixed frame $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$ is typically attached to body 0 in such a way that the $\mathbf{z}_0$ axis is coincident with the axis of joint 1. Axis $\mathbf{x}_0$ is arbitrarily directed in the plane perpendicular to the axis of joint 1. Typically this is a direction considered to be the forward reaching direction of the robot. The origin of the frame is positioned anywhere along the axis of joint 1. The choice of origin is typically made to try and eliminate additional parameters. Being a right hand coordinate system, the third axis of the frame, $\mathbf{y}_0$, is known from $\mathbf{x}_0$ and $\mathbf{z}_0$, $\mathbf{y}_0 = \mathbf{z}_0 \times \mathbf{x}_0$;

Step 2 Attach to each subsequent joint axis $i = 2, 3, \ldots, n$ the axis $\mathbf{z}_{i-1}$, which belongs to coordinate frame $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$. The origin of this frame is located on the axis of i-th joint at the foot of the common normal from the axis of joint $i-1$ to the axis of joint i. When the axes of the joints $i-1$ and i are parallel and the i-th joint is rotational, the origin of the frame is positioned so that $d_i = 0$, reducing the number of non-zero parameters. When the joint is translational, the origin of the frame can be positioned anywhere along the i-th joint axis. When joint axes $i-1$ and i intersect, the origin of the frame is positioned at the point of intersection;

Step 3 For each moving link $i-1$ we define axis $\mathbf{x}_{i-1}$, $i = 2, 3, \ldots, n$, in the direction of the common normal between joint axes $i-1$ and i, in the direction from joint axis $i-1$ towards joint axis i. When the axes of joints $i-1$ and i intersect, axis $\mathbf{x}_{i-1}$ is perpendicular to the plane containing the intersecting axes, and can be directed in either direction perpendicular to the plane. Be aware that which normal direction is chosen influences the value of angle θ_i, which is measured between the vectors $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ about axis $\mathbf{z}_{i-1}$. Being a right hand coordinate system, the third axis of the frame is determined as $\mathbf{y}_{i-1} = \mathbf{z}_{i-1} \times \mathbf{x}_{i-1}$;

Step 4 The robot end effector coordinate frame $\mathbf{x}_n$, $\mathbf{y}_n$, $\mathbf{z}_n$ has its origin placed at a reference point on the gripper. The axis $\mathbf{z}_n$ lies anywhere in the plane perpendicular to $\mathbf{x}_n$, where in Step 3, $\mathbf{x}_n$ was aligned along the common normal between $\mathbf{z}_{n-1}$ and $\mathbf{z}_n$. When the last joint n is rotational, axis $\mathbf{z}_n$ is taken to be parallel to the n-th joint axis. The $\mathbf{y}_n$ axis is found as, $\mathbf{y}_n = \mathbf{z}_n \times \mathbf{x}_n$;

Step 5 With coordinate frames for all links $i = 1, 2, \ldots, n$ assigned in the manner described above, we can determine the values of the Denavit-Hartenberg parameters, which are usually presented in a tabular form.

Following the above rules, we place coordinate frames onto the links of the mechanism shown in Fig. 2.18, resulting in Fig. 2.19. To make determination of the kinematic parameters easier, we position the mechanism in such a way that the consecutive joint axes are either perpendicular or parallel. We begin with the fixed body 0, and frame $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$. Axis $\mathbf{z}_0$ is placed along the axis of joint 1. The origin of the frame can be selected anywhere along this axis, most conveniently at $h_0 + h_1$. In this case h_0 and h_1 will not appear in the kinematic equations. It may be more convenient however, to place the frame's origin on the robot base. The axis $\mathbf{x}_0$ is taken perpendicular to $\mathbf{z}_0$, in what we consider to be the forward facing direction of the robot.

Axis $\mathbf{z}_1$ is placed along the axis of joint 2. Its origin is at the intersection of $\mathbf{z}_1$ with the common normal between the axes of joints 1 and 2. The length of the common normal is $b_1 = l_1$. The offset distance between the frames in the $\mathbf{z}_0$ direction is $d_1 = h_0 + h_1$ and is constant. Axis $\mathbf{x}_1$ lies in the direction of the common normal and the angle measured between the axes $\mathbf{z}_0$ and $\mathbf{z}_1$ about the axis $\mathbf{x}_1$ is the inclination angle $\alpha_1 = \pi/2$. Consistent with Fig. 2.18, the variable of the first joint is $\theta_1 = q_1 + \pi/2$. It is measured from axis $\mathbf{x}_0$ to axis $\mathbf{x}_1$ about $\mathbf{z}_0$.

Axis $\mathbf{z}_2$ is directed along joint axis 3. Its origin is at the intersection of the axes of joints 2 and 3. The length of the common normal is $b_2 = 0$ and the offset distance between the frames in the $\mathbf{z}_1$ direction is zero, so $d_2 = 0$. As the joint axes intersect,

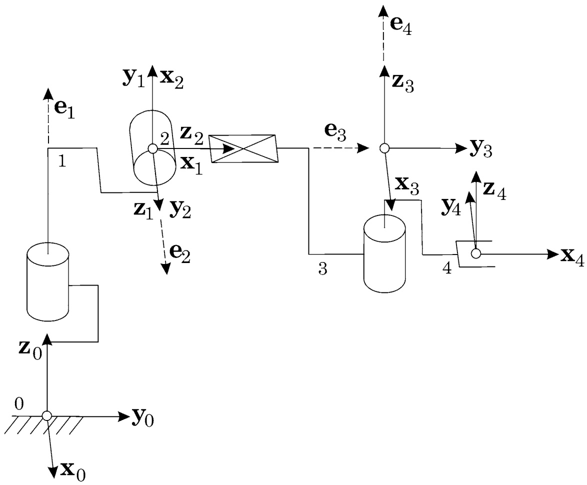

Fig. 2.19 Placement of the coordinate frames for the mechanism with four degrees of freedom

Table 2.2 Denavit-Hartenberg parameters of the mechanism from Fig. 2.19

i	b_i	α_i	d_i	θ_i
1	l_1	$\pi/2$	h_0+h_1	$q_1+\pi/2$
2	0	$\pi/2$	0	$q_2+\pi/2$
3	0	$\pi/2$	$q_3+l_2+l_3$	$\pi/2$
4	l_4	0	$-h_3$	$q_4+\pi/2$

we have $\mathbf{x}_2 = \mathbf{z}_1 \times \mathbf{z}_2$ (this vector could have been taken in the opposite direction). The angle of inclination α_2 is measured from $\mathbf{z}_1$ to $\mathbf{z}_2$, about axis $\mathbf{x}_2$, and $\alpha_2 = \pi/2$. The variable of the second joint is $\theta_2 = q_2 + \pi/2$. It is measured from $\mathbf{x}_1$ to $\mathbf{x}_2$ about $\mathbf{z}_1$.

Axis $\mathbf{z}_3$ lies on joint axis 4. Its origin is at the intersection of the axes of joints 3 and 4. The length of the common normal is $b_3 = 0$. The offset distance between the frames in the $\mathbf{z}_2$ direction is the variable of the third joint, and is given by $d_3 = q_3 + l_2 + l_3$. As the joint axes intersect, we have $\mathbf{x}_3 = \mathbf{z}_2 \times \mathbf{z}_3$ (this vector could have been selected in the opposite direction). The inclination angle α_3 is measured from $\mathbf{z}_2$ and $\mathbf{z}_3$ about $\mathbf{x}_3$, and $\alpha_3 = \pi/2$. Observe that the angle between $\mathbf{x}_2$ and $\mathbf{x}_3$ about the $\mathbf{z}_2$ axis does not change and is a constant $\theta_3 = \pi/2$.

Attachment of axis $\mathbf{z}_4$ on the gripper is arbitrary, since there is no joint 5. Take $\mathbf{z}_4$ parallel to axis 4. The origin of this frame is placed on a reference point on the gripper, which might represent a tool tip, or the midpoint of the gripper. Axis $\mathbf{x}_4$ is in the direction of the common normal from $\mathbf{z}_3$ to $\mathbf{z}_4$. The length of the normal is $b_4 = l_4$ and the offset distance between the frames in the direction of $\mathbf{z}_3$ is $d_4 = -h_3$. The inclination angle between $\mathbf{z}_3$ and $\mathbf{z}_4$, measured about $\mathbf{x}_4$ is $\alpha_4 = 0$. The variable of the fourth joint is $\theta_4 = q_4 + \pi/2$. It is measured from $\mathbf{x}_3$ to $\mathbf{x}_4$ about $\mathbf{z}_3$. The Denavit-Hartenberg parameters, belonging to the mechanism from Fig. 2.19, are given in Table 2.2.

Even in this case of a simple robot mechanism, we observe that placement of the coordinate frames according to Denavit-Hartenberg method is rather complex. An advantage of the Denavit and Hartenberg method is that the transformation matrices between the frames always have the same general form. We only have to enter the values of the parameters from Table 2.2 into (2.9). In our case we have

$$\mathbf{H}_{0,1} = \begin{bmatrix} -\sin q_1 & 0 & \cos q_1 & -l_1 \sin q_1 \\ \cos q_1 & 0 & \sin q_1 & l_1 \cos q_1 \\ 0 & 1 & 0 & h_0 + h_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{H}_{1,2} = \begin{bmatrix} -\sin q_2 & 0 & \cos q_2 & 0 \\ \cos q_2 & 0 & \sin q_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{H}_{2,3} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_3 + l_2 + l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{H}_{3,4} = \begin{bmatrix} -\sin q_4 & -\cos q_4 & 0 & -l_4 \sin q_4 \\ \cos q_4 & -\sin q_4 & 0 & l_4 \cos q_4 \\ 0 & 0 & 1 & -h_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have substituted $\sin(q_i + \pi/2) = \cos q_i$ and $\cos(q_i + \pi/2) = -\sin q_i$.

2.3.2 Vector Parameters of a Mechanism

Consider applying the Method of Vector Parameters to the same example. The method of Vector Parameters allows arbitrary placement of the coordinate frames. The method is applied in the following five steps [46], with the coordinate frames attached to the bodies as follows:

Step 1 The mechanism is placed into the desired reference position (initial pose). One can consider the reference position as the zero position, where the values of all joint variables are taken as zero, $\theta_i = 0$, $d_i = 0$, $i = 1, 2, \ldots, n$. The fixed coordinate frame $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$ which is attached to the base, is arbitrarily located in the space. Usually it is attached to a reference point on the 0 link, which may be the point where the mechanism is attached to the ground;

Step 2 The centers of the joints $i = 1, 2, \ldots, n$ are selected. The center of the i-th joint can be taken anywhere along the axis of joint i. A local coordinate frame $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ is placed on the i-th joint center in such a way that its axes are parallel to the axes of the fixed coordinate frame $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$. The local frame $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ is attached to link i and displaces with it;

Step 3 A joint vector $\mathbf{e}_i$ is placed onto each axis of the mechanism. The translational variable d_i is measured in the direction of the joint vector, while the rotational variable θ_i is measured about it;

Step 4 Link vectors $\mathbf{b}_{i-1,i}$ are directed from the origin of coordinate frame $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$ to the origin of frame $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$ $i = 1, 2, \ldots, n$, locating the origin of frame i relative to frame $i-1$. Link vector $\mathbf{b}_{n,n+1}$ lies between the origin of the coordinate frame $\mathbf{x}_n$, $\mathbf{y}_n$, $\mathbf{z}_n$ and the robot end-point;

Step 5 Joint vectors $\mathbf{e}_i$ and the link vectors $\mathbf{b}_{i-1,i}$ are expressed in the coordinate frame $\mathbf{x}_{i-1}$, $\mathbf{y}_{i-1}$, $\mathbf{z}_{i-1}$ (the vector $\mathbf{b}_{n,n+1}$ in the frame $\mathbf{x}_n$, $\mathbf{y}_n$, $\mathbf{z}_n$), as the vectors $\mathbf{e}_i^{(i-1)}$, $\mathbf{b}_{i-1,i}^{(i-1)}$, and $\mathbf{b}_{n,n+1}^{(n)}$ do not depend upon the variables θ_i and d_i, $i = 1, 2, \ldots, n$. Taking into account that in the reference (initial) pose of the mechanism all local frames $\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$, $i = 1, 2, \ldots, n$, are parallel to the reference frame $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$, the joint and link vectors can be determined in the following way

$$\mathbf{e}_i^{(i-1)} = \mathbf{e}_i^{(0)}, \tag{2.16}$$

$$\mathbf{b}_{i-1,i}^{(i-1)} = \mathbf{b}_{i-1,i}^{(0)}, \qquad \mathbf{b}_{n,n+1}^{(n)} = \mathbf{b}_{n,n+1}^{(0)}. \tag{2.17}$$

Sometimes another frame, $\mathbf{x}_{n+1}$, $\mathbf{y}_{n+1}$, $\mathbf{z}_{n+1}$, is introduced at the reference point on the robot gripper. There is no movement between the frames $\mathbf{x}_n$, $\mathbf{y}_n$, $\mathbf{z}_n$ and $\mathbf{x}_{n+1}$, $\mathbf{y}_{n+1}$, $\mathbf{z}_{n+1}$ since both are attached to the same link. Thus the transformation between them is constant. From the point of view of the kinematics the frame $\mathbf{x}_{n+1}$, $\mathbf{y}_{n+1}$, $\mathbf{z}_{n+1}$ is not necessary, but it may assist in defining the grasp of a tool in the gripper of the mechanism.

In order to simplify the development and form of the kinematic equations, it is desirable that the maximum number of the components of vectors $\mathbf{e}_i^{(0)}$, $\mathbf{b}_{i-1,i}^{(0)}$, and $\mathbf{b}_{n,n+1}^{(0)}$ are zero in the reference position. When a mechanism has consecutively perpendicular or parallel joint axes, which is common in practical robot structures, the mechanism can be placed into a reference position in such a way that particular joint axes are parallel to the axes of the fixed reference frame. As well, with appropriate choice of the joint centers, the link vectors can be chosen so that they are directed parallel to one of the axes of the fixed reference frame.

The method of Vector Parameters will be demonstrated for the same example of a four degree of freedom mechanism shown in Fig. 2.18. The result is presented in Fig. 2.20. The reference position of the mechanism, which corresponds to when the joint variables are zero, $q_1 = q_2 = q_3 = q_4 = 0$, is shown in Fig. 2.20. The vector parameters and the joint variables corresponding to the reference position of the mechanism and the selected positions of the joint centers as shown in Fig. 2.20, are presented in Table 2.3.

The rotational variables θ_1, θ_2, and θ_4 are measured in planes which are perpendicular to axes $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_4$ respectively, while translational variable d_3 is measured in the direction of axis $\mathbf{e}_3$. All joint variables are zero when the mechanism is in its reference position (initial pose). In Fig. 2.21 the mechanism is shown in a pose, where all four variables are nonzero and positive. The variable θ_1 is the angle between the initial (reference position) direction and the current direction of the

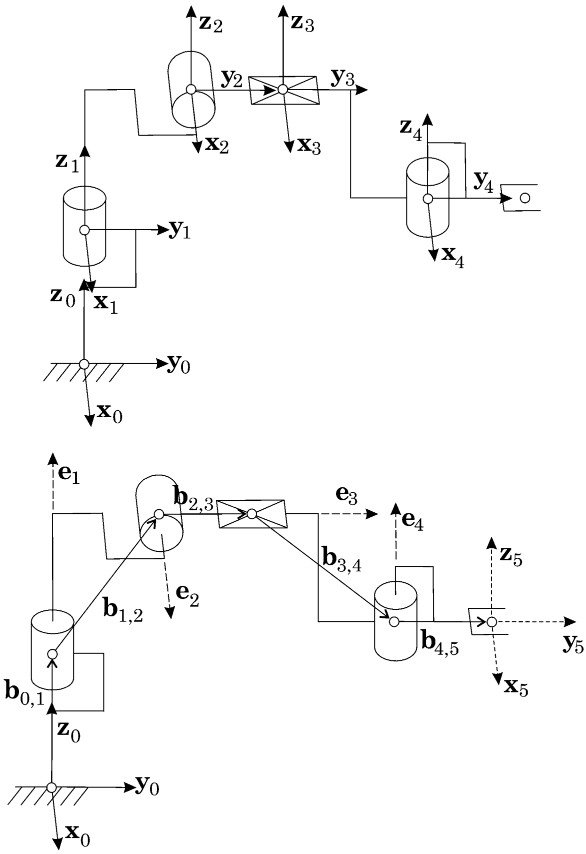

Fig. 2.20 Placement of the coordinate frames for the mechanism with four degrees of freedom

Table 2.3 Vector parameters and variables of the mechanism from Fig. 2.20

i	1	2	3	4
θ_i	q_1	q_2	0	q_4
d_i	0	0	q_3	0

i	1	2	3	4
$\mathbf{e}_i^{(i-1)}$	0	1	0	0
	0	0	1	0
	1	0	0	1

i	1	2	3	4	5
$\mathbf{b}_{i-1,i}^{(i-1)}$	0	0	0	0	0
	0	l_1	l_2	l_3	l_4
	h_0	h_1	0	$-h_3$	0

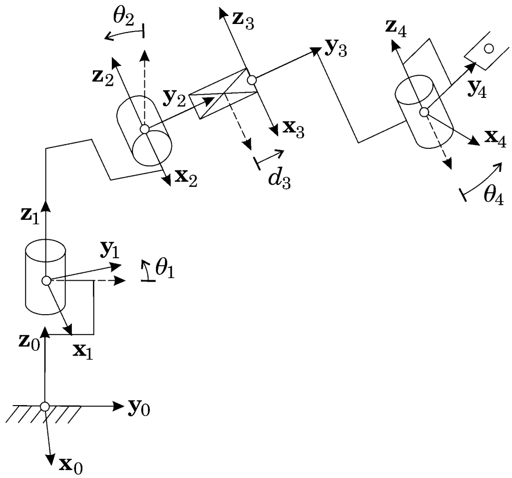

Fig. 2.21 Rotational and translational variables for the mechanism with four degrees of freedom

axis $\mathbf{y}_1$, the variable θ_2 is the angle from the initial (reference position) direction to the current direction of the axis $\mathbf{z}_2$, the variable d_3 is given by the distance from the initial (reference position) position to the current position of the axis $\mathbf{x}_3$, and θ_4 is the angle from the initial (reference position) direction to the current direction of the axis $\mathbf{x}_4$. These variables correspond to the joint displacements q_1, q_2, q_3, and q_4, which are defined in Fig. 2.18.

The selected parameters are inserted in (2.12)

$$\mathbf{H}_{0,1} = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & 0 \\ \sin q_1 & \cos q_1 & 0 & 0 \\ 0 & 0 & 1 & h_0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{H}_{1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos q_2 & -\sin q_2 & l_1 \\ 0 & \sin q_2 & \cos q_2 & h_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{H}_{2,3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q_3 + l_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{H}_{3,4} = \begin{bmatrix} \cos q_4 & -\sin q_4 & 0 & 0 \\ \sin q_4 & \cos q_4 & 0 & l_3 \\ 0 & 0 & 1 & -h_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here, the simplified expressions for the rotation matrices were used (1.51), (1.52), (1.53), describing the rotations about the axes of the coordinate frame. This is made possible by our choice of reference frame and reference position.

Placing a coordinate frame $\mathbf{x}_5, \mathbf{y}_5, \mathbf{z}_5$ onto the reference point of the robot gripper, we obtain the following additional homogeneous transformation matrix

$$\mathbf{H}_{4,5} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which is independent of the joint variables, as the frames $\mathbf{x}_4, \mathbf{y}_4, \mathbf{z}_4$ and $\mathbf{x}_5, \mathbf{y}_5, \mathbf{z}_5$ are both attached to body 4 and are parallel and only displaced for by a constant distance l_4. In a purely kinematic sense the additional frame $\mathbf{x}_5$, $\mathbf{y}_5$, $\mathbf{z}_5$ need not be attached to the mechanism, as the position and orientation of the gripper can be described by the $\mathbf{x}_4$, $\mathbf{y}_4$, $\mathbf{z}_4$ frame alone. This additional frame however may be used to define the grasp of a tool, body 5, by the gripper, body 4, since in many applications control of the tool's position and orientation is the ultimate goal.

This chapter has introduced the Denavit and Hartenberg method and the method of Vector Parameters. Both methods were applied to a kinematic pair and then to an example of a four degree of freedom mechanism. In the Denavit-Hartenberg method, the attachment of local coordinate frames to the links is precisely specified and relative to these frames a minimum number of translational and rotational parameters which describe the relative pose of two neighboring links are defined. In this method the transformation matrices between the coordinate frames have a generic form, requiring fewer algebraic operations in developing the kinematic equations of motion The disadvantage of the approach is in the forced placements of the coordinate frames on the links, placements which may seem irregular or unnatural. The origins of these frames are often at points which are distant from the corresponding joint centers or links. In many cases the configuration of a robot mechanism corresponding to when the joint variables are zero is unnatural and because of physical constraints may be inaccessible.

The method of Vector Parameters uses link and joint vectors to describe the geometry of a link and the variables at the joints. It is important in this method to select an appropriate reference position of the mechanism where all the coordinate frames are parallel to the reference coordinate frame and the translational and rotational joint variables are zero. As the reference position of a mechanism is a free choice, we can select the most appropriate reference position from the point of view of clarity of the approach, requirements of the robot task, or the number of mathematical operations included in the transformation matrices. In general the matrix describing the rotation about an arbitrary axis must be used in the transformation matrices. However, when selecting the reference position in such a way that particular joint axes are parallel to one of the axes of the reference frame, the rotational matrices are simplified and the number of required arithmetic operations is not higher than with the Denavit-Hartenberg method.