Chapter 2
Nonlinear Schroedinger Equation

The main aim of this thesis is the study of the interaction between nonlinearity and disorder, the pivotal processes underlying the localization of light. Knowing how the manipulation of an optical system by modifying the mutual competition between disorder and nonlinearity can result in localized wave-forms is an intriguing challenge and a significant target of modern optics. In this section, we discuss the Nonlinear Schroedinger equation (NLS), a paradigmatic universal nonlinear model that describes several physical phenomena in the framework of several disciplines, from the nonlinear optics to the quantum condensate. In a simple way, this equation allows to understand how the nonresonant nonlinearity (specifically the Kerr effect) can promote the formation of localized wave-forms through the balance of two opposite effects: wave dispersion and nonlinear response. The NLS represents the starting point to study the complex interplay between nonlinearity and disorder.

2.1 Introduction

We start by the Maxwell's Equations. They describe the propagation of an electromagnetic wave in a medium with refractive index n. In the time domain they read as,

$$\begin{aligned} \nabla \times \mathbf{E} &= -\mu_0 \partial_t \mathbf{H} \\ \nabla \times \mathbf{H} &= \epsilon_0 n^2 \partial_t \mathbf{E}, \end{aligned} \tag{2.1}$$

where x, y, z are the spatial coordinates while t is the temporal variable. The vectors $\mathbf{E}$ and $\mathbf{H}$ are respectively the electric and the magnetic fields. The relative electric permittivity is ϵ_r and the relative magnetic permeability is μ_r. The refractive index is $n = \sqrt{\epsilon_r \mu_r}$, and depends by the relative values of the considered material; ϵ_0 is the electric constant and μ_0 is the magnetic constant. We study the propagation of an electromagnetic field into a dielectric medium in which no current and no charges are present. From Eq. (2.1), we obtain the wave equation for the electric field:

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$$\nabla \times \nabla \times \mathbf{E} = -\mu_0 \epsilon_0 n^2 \partial_t^2 \mathbf{E}. \tag{2.2}$$

We write the monochromatic solution of Eq. (2.2) as:

$$\mathbf{E(x, y, z, t)} = Re[\mathbf{E(x, y, z)} e^{-i\omega t}] \tag{2.3}$$

and by using the paraxial approximation (we are considering a z-collimated laser beam) for which it is possible to approximate $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \approx -\nabla^2 \mathbf{E}$, we write the Helmholtz equation for the electric field:

$$\nabla^2 \mathbf{E} + \frac{\omega^2}{c^2} n^2 \mathbf{E} = 0. \tag{2.4}$$

In an homogeneous medium with $n(r) = n_0$, the simplest solution of Eq. (2.4) in one-dimensional case is the plane wave $\mathbf{E}(z) = E e^{ikz} \hat{\mathbf{x}}$, for which the Helmholtz equation gives the wave vector $k = \frac{\omega}{c} n_0$. We want to treat the interaction of light in a nonlinear medium, in which the permittivity and the permeability depend on the electromagnetic field. In this specific case, for sake of simplicity, we consider the case in which $\mu_r = 1$ while the electric permittivity is a function of the electric field $\epsilon = \epsilon(E)$.[1] If the medium is nonlinear, we can write $n(r) = n_0 + \Delta n(r)$. We let the general solution amplitude to be space-dependent because nonlinearity can spatially modulate the propagating field:

$$\mathbf{E(x, y, z)} = E(x, y, z) e^{ikz} \hat{\mathbf{x}}. \tag{2.5}$$

By inserting Eq. (2.5) in Eq. (2.4) and within the paraxial approximation, for which the longitudinal variation is slow enough and $\partial_z^2 E \approx 0$, we write the amplitude equation:

$$2ik\partial_z E + \nabla_\perp^2 E + 2k^2 \frac{\Delta n}{n_0} E = 0 \tag{2.6}$$

where we have neglected the higher order term in Δn and we have used the relation $k = \frac{\omega}{c} n_0$. The Eq. (2.6) represents the propagation of a laser beam, collimated along $\hat{z}$ (paraxial approximation), in the presence of an index modulation Δn induced by the nonlinear effects. The power flux is related to the z-component of the Poynting vector, $P_z = \frac{nE^2}{2Z_0} = I$, where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of the vacuum. We hence define an optical field A for which the optical intensity is $I = |A|^2$, in this way the complex envelope is related to E by $A = \sqrt{n/2Z_0} E$. The equation for A is simply obtained from which for E:

$$2ik\partial_z A + \nabla_\perp^2 A + 2k^2 \frac{\Delta n}{n_0} A = 0. \tag{2.7}$$

In the following, we put particular attention to the one-dimensional case (that corresponds to take $\nabla_\perp^2 = \partial_x^2$), in order to simplify the successive analysis of the interplay between nonlinearity and disorder.

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[1] Once the disorder is added to the system, the permittivity, and hence the refractive index, will become explicitly dependent on the spatial coordinate, $\epsilon = \epsilon(r, E)$.

2.2 Local Case

In this section, we consider the simplest example of nonlinear system, the Kerr medium. In this kind of material, only the cubic nonlinearity in the paraxial approximation is retained. The refractive index linearly increases with the beam intensity. The refractive index perturbation depends only on the intensity in a given point $\Delta n(r) = n_2 I(r)$, and:

$$n(r) = n_0 + n_2 I(r). \tag{2.8}$$

Equation (2.7) becomes:

$$2ik\partial_z A + \partial_x^2 A + 2k^2 \frac{n_2}{n_0} |A|^2 A = 0. \tag{2.9}$$

This is the well-known nonlinear Schroedinger equation (NLS), a universal nonlinear equation describing several nonlinear phenomena, including the Bose-Einstein condensation when it is expressed in the adimensional form, by rescaling the coordinates $z \to 2kx_0^2 z$, $x \to x_0 x$ and the field amplitude $A \to \sqrt{n_0/(2k^2 x_0^2 |n_2|)}\psi$:

$$i\partial_z \psi + \partial_x^2 \psi \pm |\psi|^2 \psi = 0, \tag{2.10}$$

where the sign plus (minus) is associated to the self-focusing $n_2 > 0$ (self-defocusing $n_2 < 0$) character of the medium.

2.2.1 *Plane Wave Solution*

Equation (2.10) admits a plane wave stationary solution for which the field is not dependent on x. By writing $\psi(z) = \psi_0 \exp(i\beta z)$ and inserting this solution in (2.10), we obtain that $\beta = \psi_0^2$ such that

$$\psi(z) = \psi_0 \exp(iCIz), \tag{2.11}$$

where C is a constant and I is the dimensional intensity. This physically means that, during its propagation in the nonlinear medium, the plane wave field has an intensity dependent phase.

2.2.2 *Modulation Instability*

The plane wave solution is unstable when a small transverse perturbation is applied. Let us analyze the stability properties of (2.11) and write it as:

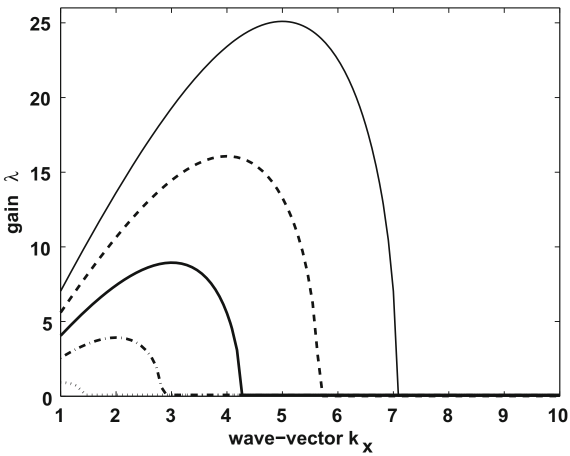

Fig. 2.1 Gain versus wave-vector for various plane-wave amplitudes $\psi_0 = 1$ (*dotted line*), $\psi_0 = 2$ (*dot-dashed line*), $\psi_0 = 3$ (*continuous- bold line*), $\psi_0 = 4$ (*dashed line*), $\psi_0 = 5$ (*continuous-thin line*)

$$\psi(z,x) = [\psi_0 + p(z,x)]e^{i\psi_0^2 z}, \tag{2.12}$$

where $p(z,x) = \alpha_+(z)e^{ik_x x} + \alpha_-^*(z)e^{-ik_x x}$. We put this solution in (2.10) and by linearizing, we obtain an equation system for $\alpha_\pm$:

$$\begin{aligned} i\dot{\alpha}_+ - k_x^2\alpha_+ + \psi_0^2(\alpha_- + \alpha_+) &= 0 \\ -i\dot{\alpha}_- - k_x^2\alpha_- + \psi_0^2(\alpha_- + \alpha_+) &= 0, \end{aligned} \tag{2.13}$$

where k_x is the wave-vector. We assume that $\alpha_\pm = \hat{\alpha}_\pm e^{\lambda z}$ and obtain the equation for the gain λ:

$$\lambda^2 = k_x^2(2\psi_0^2 - k_x^2). \tag{2.14}$$

When (for a specific range of k_x), the second term is positive, we have an unstable growth of the introduced periodic perturbation, despite it can be chosen as small as we want. The perturbation hence grows exponentially during the wave propagation. The gain of the perturbation has a maximum growth rate for a fixed value of the wave vector, $k_x = \psi_0$, as it can be seen in Fig. 2.1. There exists a value of the period of the perturbation that grows more successfully than others. This leads to the formation of a periodical pattern of field distribution for which there are alternating regions where the field is much more intense.

In Fig. 2.2, we show the numerical simulations for a plane-wave solution of the NLS equation when a small perturbation is added to the unperturbed solution. We see that an initial homogeneous field distribution, describing the front wave of the plane wave, breaks into a periodical pattern. It is important to stress that, as it can be seen by Eq. (2.14), the modulation instability at which the light beam is subject, is managed by the self-focusing nonlinearity (through the ψ_0^2 intensity term) and the diffraction, related to the wave-vector k_x.

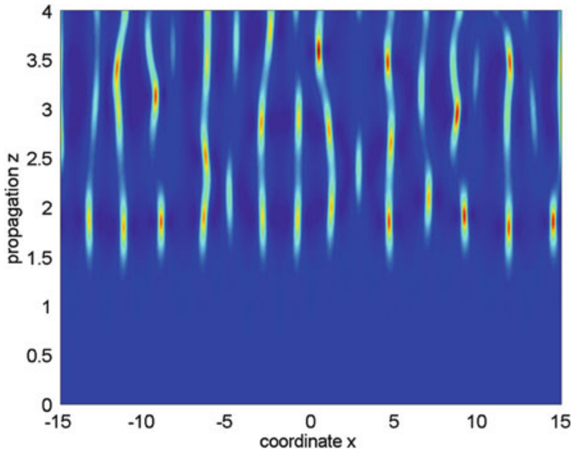

Fig. 2.2 Numerical simulations of the nonlinear Schroedinger equation start from a plane-wave perturbed with 1% perturbation noise. The figure shows that, starting from an homogenous field distribution, periodical pattern emerges, eventually developing into an ensemble of localized waves (solitons)

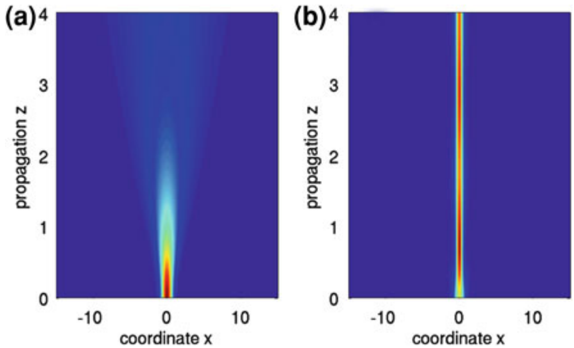

Fig. 2.3 Propagation of a one-dimensional Gaussian beam by integrating the nonlinear Schroedinger equation through the beam propagation method, in the diffraction case (**a**) obtained for a low power input, and in the self-trapped case (**b**) when the input beam is high

2.2.3 The Bound States

As we have seen in the previous section, the plane-wave solution is unstable under transverse spatial perturbation. So, the NLS solution cannot be obtained by applying small perturbations to the plane-wave solution. We must look for new stationary solutions. It can be shown that in nonlinear Kerr media, the nonlinear term in Eq. (2.10) curves the wave phase-front in a way related to the sign of the nonlinear term. We are considering attractive Kerr media for which the positive sign of the nonlinear term involves a convergent effect on the propagating wave. For a beam with a finite transverse extent, we expect that a stable solution can be derived by balancing the diffraction mechanism (a spreading of the propagating beam, a concave curvature of the phase-front) with the nonlinear focusing mechanism. Figure 2.3 shows this mechanism. In panel (a), we consider a low intensity beat. By Eq. (2.10), the nonlinear term can be neglected and the diffraction causes the beam dispersion. In panel (b), the input beam has higher intensity, the nonlinear compensation yields to the formation of a self-trapped beam. The simplest localized solution for Eq. (2.10)

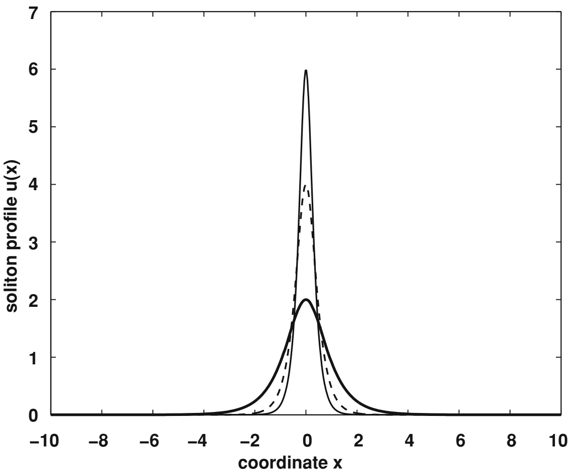

Fig. 2.4 1D kerr soliton shape for three different amplitude values $u_0 = 2$ (*continuous-bold line*), $u_0 = 4$ (*dashed line*), $u_0 = 6$ (*continuous-thin line*)

takes the form $\psi(x, z) = u(x)e^{i\beta z}$, with

$$u(x) = u_0 \operatorname{sech}(x/w_0), \tag{2.15}$$

where the width is $w_0 = \frac{u_0}{\sqrt{2}}$ and $\beta = u_0^2/2$ is directly related to the amplitude of the wave. It is important to observe that the balance of the nonlinearity with the diffraction term results into a flat phase-front and that, in order to exist, the localized solution has to satisfy a fixed relation between the width of the beam (associated to the diffraction) and the beam amplitude u_0, determining the strength of the nonlinearity. As shown in Fig. 2.4, by increasing the amplitude (and hence the power $P = \int dx|u|^2$) of the input beam, the soliton width shrinks such that $u_0 w_0$ = constant. This relationship is known as the "existence curve" of the solitary waves.

2.3 Nonlocal Case

Over the years, the role of a nonlocal nonlinear response, with special emphasis on the optical spatial solitons (OSS) [1], appeared with an increasing degree of importance [2–8]; on one hand because it must be taken into account for the quantitative description of experiments and, on the other hand, because it is a leading mechanism for stabilizing multidimensional solitons [9]. Nonlocality in nonlinear wave propagation is found in those physical systems exhibiting long range correlations, like nematic liquid crystals (LC) [3], photorefractive media (PR) [10], thermal [4, 11, 12] and thermo-diffusive [13] nonlinear susceptibilities, soft-colloidal matter (SM) [14], BEC [15, 16], and plasma-physics [17, 18].

Let us consider a typical nonlocal nonlinear medium in which the change in the refractive index is related to the intensity of the wave in a finite region, it is a nonlocal function of the wave field. This effect can be represented in general form, for the focusing case, as

$$\Delta n(x, I) = \int_{-\infty}^{+\infty} dx' K(x - x') I(x', z), \tag{2.16}$$

where $K(x-x')$ is the response function. Its form depends on the specific nonlinearity under consideration and it is associated to a length-scale σ that measures the degree of nonlocality. Thanks to the paraxial approximation, the nonlocal response along z can be typically neglected. In a local system, σ tends to zero, the response is punctual $R(x-x') = \delta(x-x')$ and the refractive index locally changes with the light intensity (in Kerr media). In general, however, there always exists a degree of nonlocality in the physical systems that support the wave propagation. The local approximation can be done when the spatial correlations in the optical response are much smaller of the wavelength (below 100 nm). By inserting Eq. (2.16) in (2.7) and repeating the rescaling of the physical variables, we obtain the adimensional nonlocal nonlinear Schroedinger equation:

$$i\partial_z \psi + \partial_x^2 \psi + \psi K * |\psi|^2 = 0, \tag{2.17}$$

where $K * |\psi|^2$ is the convolution integral that measures the correlation between the optical field and the punctual response K of the nonlinear medium. In the Fourier domain, the convolution integral becomes $\widetilde{\rho} = S(k_x)|\tilde{\psi}|^2$ where the tilde denotes the Fourier transform and $S(k_x)$ is the "structure factor" (that is the Fourier transform of $K(x)$). It is often useful to write Eq. (2.17) as a system of two differential equations:

$$\begin{aligned} i\partial_z \psi + \nabla_x^2 \psi + \rho\psi &= 0 \\ \mathcal{G}(\rho) &= |\psi|^2, \end{aligned} \tag{2.18}$$

where K is the Green function of the differential operator $\mathcal{G}$.

2.3.1 Plane Wave Solution

Hereafter, we will consider the general nonlocal Kerr nonlinearity with a nonlocal exponential response function:

$$K(x) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right), \tag{2.19}$$

where σ, as we have seen above, represents the degree of the nonlocality. The expression for the structure factor

$$S(k_x) = \frac{1}{1 + \sigma^2 k_x^2} \tag{2.20}$$

is obtained as the Fourier transform of the response function. In this case, we have that the differential operator $\mathcal{G}$ acts on ρ as $-\sigma^2\rho_{xx}+\rho$. This corresponds, for example, to the re-orientational nonlinearity of nematic liquid crystals or to the thermal nonlinearity in lead-glasses [3, 4]. By calculating the corresponding Green function, the solution for ρ takes the form $\rho=\int\frac{e^{-|x-x'|/\sigma}}{2\sigma}|\psi|^2(x')dx'$. As in the local case, the plane wave

$$\begin{aligned}\psi(z)&=\psi_0 e^{i\beta z}\\ \rho&=\rho_0,\end{aligned}\tag{2.21}$$

where $\psi_0=\text{constant}$, is a solution of the Eq. (2.22) if $\beta=\rho_0$ and $\rho_0=\psi_0^2$.

2.3.2 Modulation Instability

The modulation instability theory can be developed through the same procedures used in the local case. Let us start by considering the nonlocal nonlinear Schroedinger equation for an exponential nonlocal response

$$\begin{aligned}i\partial_z\psi+\nabla_x^2\psi+\rho\psi&=0\\ -\sigma^2\rho_{xx}+\rho&=|\psi|^2,\end{aligned}\tag{2.22}$$

and add a small perturbation for the amplitude and the density of the form

$$\begin{aligned}\psi(z,x)&=[\psi_0+p(z,x)]e^{i\psi_0 z}\\ \rho(x)&=\rho_0+r(x),\end{aligned}\tag{2.23}$$

where $p(z,x)=\alpha_+(z)e^{ik_xx}+\alpha_-^*(z)e^{-ik_xx}$ while $r(x)=r_+e^{ik_xx}+r_-^*e^{-ik_xx}$. By inserting the expression for ρ in the second equation of (2.22), one obtains:

$$r_\pm=\psi_0 S(k_x)(\alpha_+ + \alpha_-).\tag{2.24}$$

By linearizing the first equation of (2.22), the dispersion relation give us the following growth rate for the perturbation:

$$\lambda=|k_x|\sqrt{2\psi_0^2 S(k_x)-k_x^2},\tag{2.25}$$

it should be noted that the local limit result (2.14) returns for $\sigma\to 0$, $S(k_x)\to 1$.

In Fig. 2.5, we show the dependence of the growth rate of the transverse perturbation by the wave-vector for the local case ($\sigma=0$) and the nonlocal case ($\sigma=1$). As it can be seen, the nonlocality tends to shrink the bandwidth of the modulation instability phenomenon and to reduce the maximum growth rate of the perturbation [see Fig. 2.6]. This effect will be strongly taken into account in the next chapters where we will study the interplay between nonlinearity and disorder and where the nonlocality will help us to understand and analytically solve the physical involved phenomena.

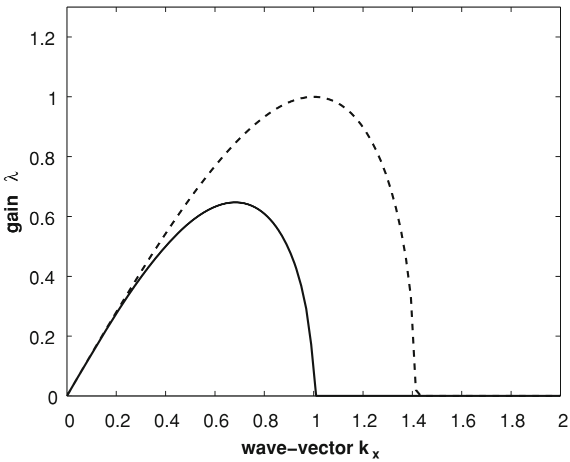

Fig. 2.5 Gain versus wave-vector for $\sigma = 0$, the local case (*dashed line*) and for $\sigma = 1$, the nonlocal case (*continuous line*)

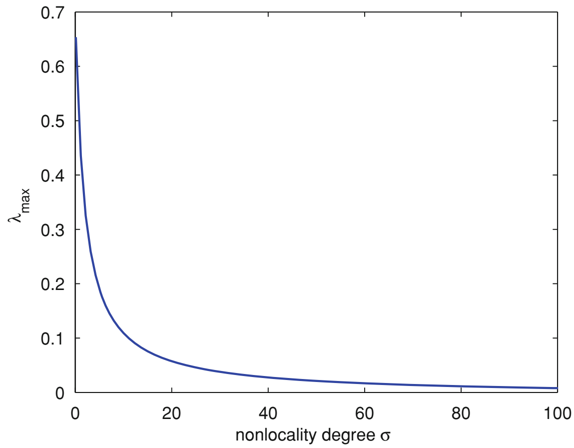

Fig. 2.6 Maximum gain versus the nonlocality degree

2.3.3 The Bound States

Now, we focus on the stable solutions $\psi(x, z) = u(x)e^{i\beta z}$ of the nonlocal nonlinear Schroedinger equation in the case of exponential response:

$$-\beta u + u_{xx} + u\rho = 0, \tag{2.26}$$

where $\rho(x) = \int dx' K(x - x')u^2(x')$ and, as seen above, $K(x) = e^{-|x|/\sigma}/2\sigma$. The exact solutions, at variance with what happens in the local case, cannot be found analytically.

In Fig. 2.7, we show the numerically obtained profiles for $u(x)$ and $\rho(x)$. It can be noticed that, as the nonlocality increases, the response ρ widens with respect to the field profile. This leads to the possibilty of an analytical approach typically denoted

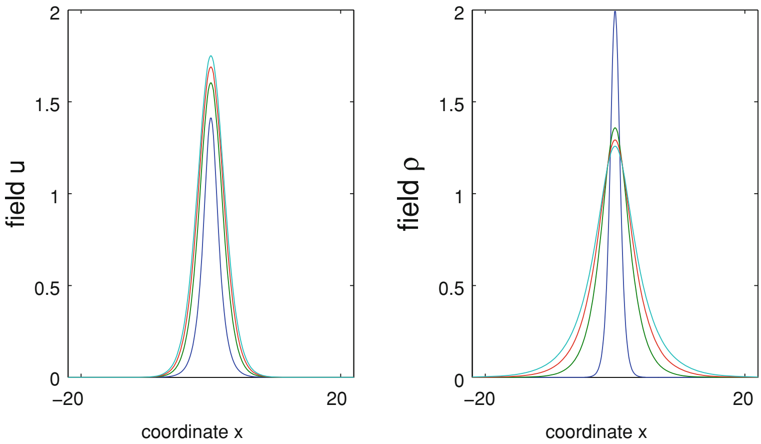

Fig. 2.7 The field profile $u(x)$ and the response function $\rho(x)$ for different degrees of nonlocality, for $\beta = 1$. Shown is the local case (*blue line*), $\sigma = 3$ (*green line*), $\sigma = 7$ (*red line*) and $\sigma = 10$ (*cyan line*)

as the highly nonlocal limit described in the following paragraph, where the response function is approximated by a parabolic profile.

2.3.4 Highly Nonlocal Limit

In this section, we analyze the limit of a strongly nonlocal response. This is the case in which the local response distribution, induced by the optical field, is much broader than the spatial extension of the field itself (see Fig. 2.8. $u(x)$ samples the response function at $x' \approx 0$ in the integral,

$$\rho(x) = \int dx' K(x - x')u^2(x') \simeq K(x)P, \tag{2.27}$$

where $P = \int dx' u^2(x')$ is the power of the soliton. We can then expand the nonlocal response $K(x)$ (which is a bell shaped function) in Taylor series:

$$K(x) \simeq K_0 - \frac{1}{2}K_2 x^2, \tag{2.28}$$

where K_0 and K_2 are positive constants and the minus sign takes into account the focusing solution. In the parabolic approximation, an analytical solution exists. The equation for $u(x)$ takes the following form:

$$-\beta u + u_{xx} + P(K_0 - \frac{K_2}{2}x^2)u = 0. \tag{2.29}$$

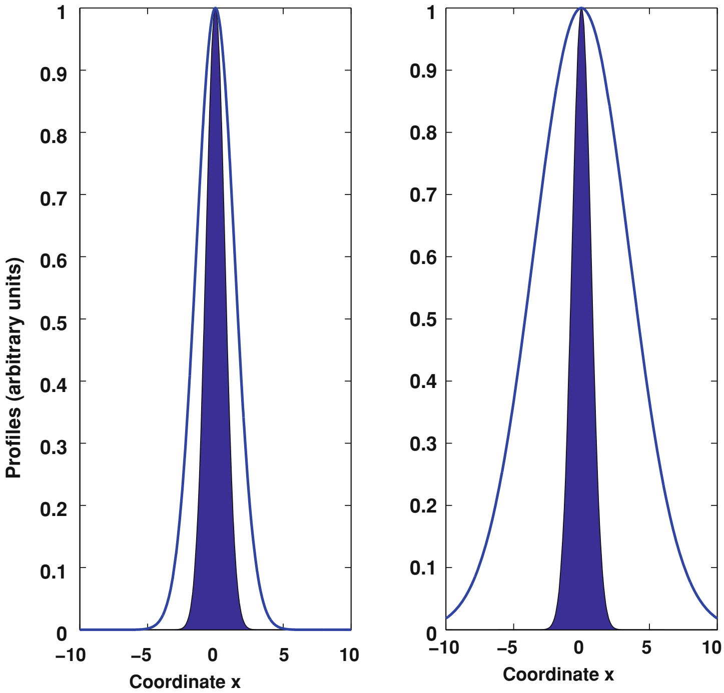

Fig. 2.8 The pulse profile (*filled curve*) and the response function profile (*continuous line*) for the local case (*left* panel) and the highly nonlocal case (*right* panel)

Equation (2.29) admits a Gaussian solution,

$$u(x) = u_0 \exp(-x^2/2x_0^2) \tag{2.30}$$

where $u_0 = \sqrt{\frac{P}{x_0\sqrt{\pi}}}$. The spatial extension of the localized wave is related to the soliton power by the relationship,

$$x_0^4 P = \frac{2}{K_2} = \text{constant}, \tag{2.31}$$

that is also known as the "existence curve" of the nonlocal solitons.

The highly nonlocal limit will allow to solve the disordered version of this kind of systems and the nonlocality, as we will see, play a crucial role in the management of the localized light phenomena, acting as a filter between nonlinearity and disorder, and averaging out several effects related to randomness.

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