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Fourier's representation for functions on $\mathbb{R}$, $\mathbb{T}_p$, $\mathbb{Z}$, and $\mathbb{P}_N$

1.1 Synthesis and analysis equations

Introduction

In mathematics we often try to synthesize a rather arbitrary function f using a suitable linear combination of certain elementary basis functions. For example, the power functions $1, x, x^2, \ldots$ serve as such basis functions when we synthesize f using the power series representation

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots . \tag{1}$$

The coefficient a_k that specifies the amount of the basis function x^k needed in the recipe (1) for constructing f is given by the well-known Maclaurin formula

$$a_k = \frac{f^{(k)}(0)}{k!}, \quad k = 0, 1, 2, \ldots$$

from elementary calculus. Since the equations for $a_0, a_1, a_2, \ldots$ can be used only in cases where $f, f', f'', \ldots$ are defined at $x = 0$, we see that not all functions can be synthesized in this way. The class of analytic functions that do have such power series representations is a large and important one, however, and like Newton [who with justifiable pride referred to the representation (1) as "my method"], you have undoubtedly made use of such power series to evaluate functions, to construct antiderivatives, to compute definite integrals, to solve differential equations, to justify discretization procedures of numerical analysis, etc.

Fourier's representation (developed a century and a half after Newton's) uses as basis functions the complex exponentials

$$e^{2\pi isx} := \cos(2\pi sx) + i \cdot \sin(2\pi sx), \tag{2}$$

where s is a real frequency parameter that serves to specify the rate of oscillation, and $i^2 = -1$. When we graph this complex exponential, i.e., when we graph

$$u := \text{Re } e^{2\pi isx} = \cos(2\pi sx)$$
$$v := \text{Im } e^{2\pi isx} = \sin(2\pi sx)$$

as functions of the real variable x in x, u, v-space, we obtain a helix (a Slinky!) that has the spacing $1/|s|$ between the coils. Projections of this helix on the planes $v = 0$, $u = 0$, $x = 0$ give the sinusoids $u = \cos(2\pi sx)$, $v = \sin(2\pi sx)$, and the circle $u^2 + v^2 = 1$, as shown in Fig. 1.1.

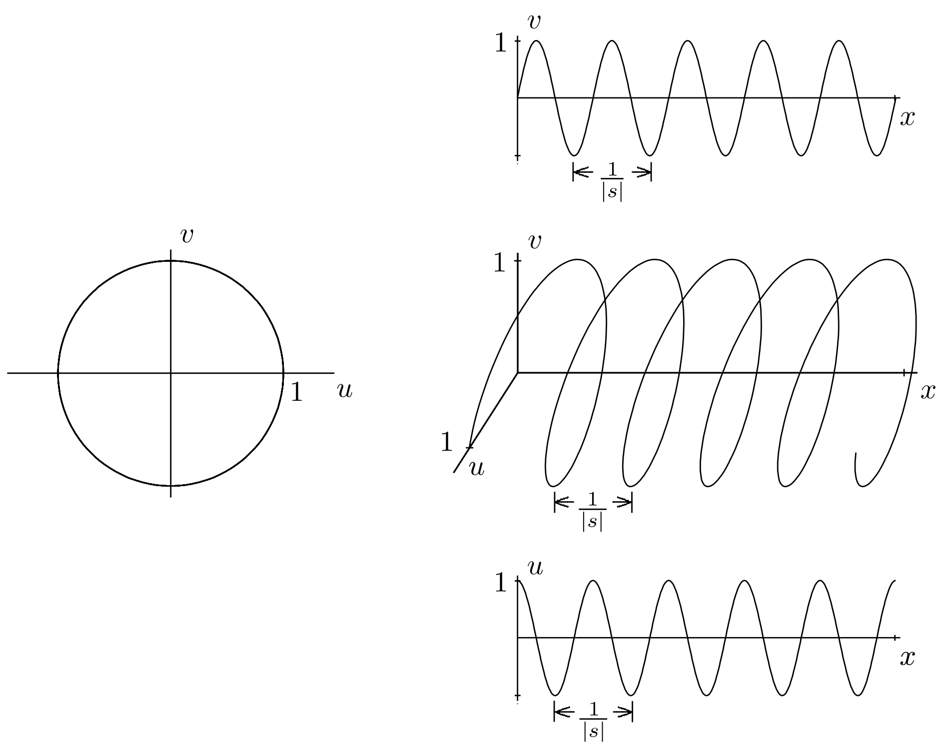

Figure 1.1. The helix $u = \cos(2\pi sx)$, $v = \sin(2\pi sx)$ in x, u, v-space together with projections in the x, u, the x, v, and the u, v planes.

Functions on $\mathbb{R}$

Fourier discovered that any suitably regular complex-valued function f defined on the real line $\mathbb{R}$ can be synthesized by using the integral representation

$$f(x) = \int_{s=-\infty}^{\infty} F(s)e^{2\pi isx}\,ds, \quad -\infty < x < \infty. \tag{3}$$

Here F is also a complex-valued function defined on $\mathbb{R}$, and we think of $F(s)ds$ as being the amount of the exponential $e^{2\pi isx}$ with frequency s that must be used in the recipe (3) for f. At this point we are purposefully vague as to the exact hypotheses that must be imposed on f to guarantee the existence of such a Fourier representation. Roughly speaking, the Fourier representation (3) is possible in all cases where f does not fluctuate too wildly and where the tails of f at $\pm\infty$ are not too large. It is certainly not *obvious* that such functions can be represented in the form (3) [nor is it *obvious* that $\sin x$, $\cos x$, e^x, and many other functions can be represented using the power series (1)]. At this point we are merely announcing that this is, in fact, the case, and we encourage you to become familiar with equation (3) along with analogous equations that will be introduced in the next few paragraphs. Later on we will establish the validity of (3) after giving meaning to the intentionally vague term *suitably regular*.

Fourier found that the auxiliary function F from the representation (3) can be constructed by using the integral

$$F(s) = \int_{x=-\infty}^{\infty} f(x)e^{-2\pi isx}\,dx, \quad -\infty < s < \infty. \tag{4}$$

We refer to (3) as the *synthesis equation* and to (4) as the *analysis equation* for f. The function F is said to be the *Fourier transform* of f. We cannot help but notice the symmetry between (3) and (4), i.e., we can interchange f, F provided that we also interchange $+i$ and $-i$. Other less symmetric analysis-synthesis equations are sometimes used for Fourier's representation, see Ex. 1.4, but we prefer to use (3)–(4) in this text. We will often display the graphs of f, F side by side, as illustrated in Fig. 1.2. Our sketch corresponds to the case where both f and F are real valued. In general, it is necessary to display the four graphs of $\operatorname{Re} f$, $\operatorname{Im} f$, $\operatorname{Re} F$, and $\operatorname{Im} F$. You will find such displays in Chapter 3, where we develop an efficient calculus for evaluating improper integrals having the form (3) or (4).

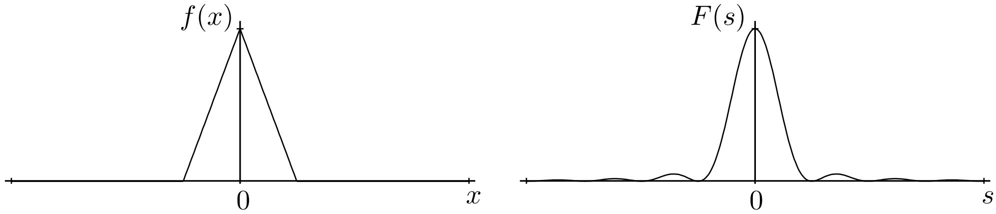

Figure 1.2. The graph of a function f on $\mathbb{R}$ and its Fourier transform F on $\mathbb{R}$.

Functions on $\mathbb{T}_p$

We say that a function f defined on $\mathbb{R}$ is p-*periodic*, $p > 0$, when

$$f(x+p) = f(x), \quad -\infty < x < \infty.$$

Fourier (like Euler, Lagrange, and D. Bernoulli before him) discovered that a suitably regular p-periodic complex-valued function on $\mathbb{R}$ can be synthesized by using the p-periodic complex exponentials from (2). We will routinely identify any p-periodic function on $\mathbb{R}$ with a corresponding function defined on the circle $\mathbb{T}_p$ having the circumference p as illustrated in Fig. 1.3. [To visualize the process, think of wrapping the graph of $f(x)$ versus x around a right circular cylinder just like the paper label is wrapped around a can of soup!] Of course, separate graphs for $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ must be given in cases where f is complex valued.

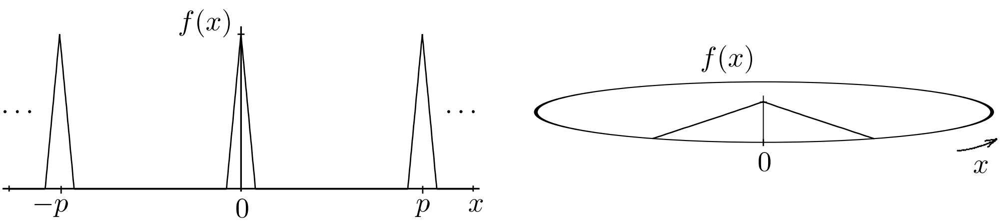

Figure 1.3. Identification of a p-periodic function f on $\mathbb{R}$ with a corresponding function on the circle $\mathbb{T}_p$ having the circumference p.

The complex exponential $e^{2\pi isx}$ will be p-periodic in the argument x, i.e.,

$$e^{2\pi is(x+p)} = e^{2\pi isx}, \quad -\infty < x < \infty,$$

when

$$e^{2\pi isp} = 1,$$

i.e., when

$$s = k/p \quad \text{for some } k = 0, \pm 1, \pm 2, \ldots .$$

In this way we see that the p-periodic exponentials from (2) are given by

$$e^{2\pi ikx/p}, \quad k = 0, \pm 1, \pm 2, \ldots,$$

as shown in Fig. 1.4.

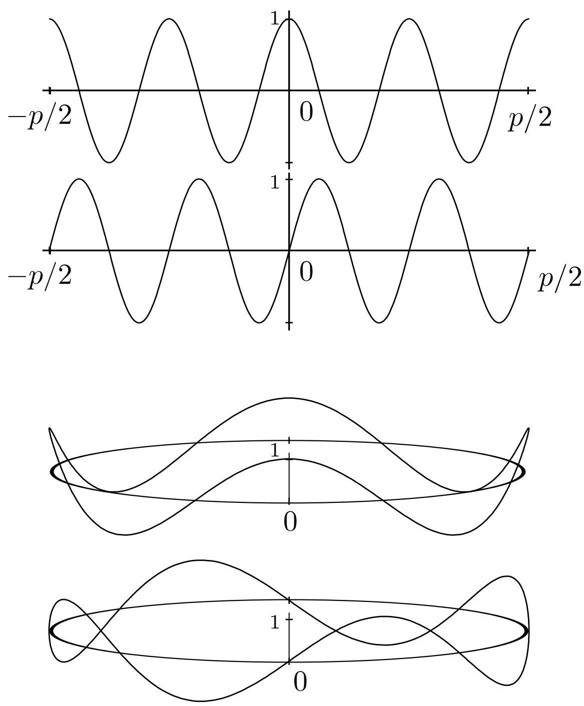

Figure 1.4. Real and imaginary parts of the complex exponential $e^{8\pi ix/p}$ as functions on $\mathbb{R}$ and as functions on $\mathbb{T}_p$.

Fourier's representation

$$f(x) = \sum_{k=-\infty}^{\infty} F[k]e^{2\pi ikx/p}, \quad -\infty < x < \infty, \tag{5}$$

for a p-periodic function f uses all of these complex exponentials. In this case F is a complex-valued function defined on the integers $\mathbb{Z}$ (from the German word *Zahlen*, for *integers*). We use brackets [ ] rather than parentheses ( ) to enclose the independent variable k in order to remind ourselves that this argument is discrete. We think of $F[k]$ as being the amount of the exponential $e^{2\pi ikx/p}$ that we must use in the recipe (5) for f. We refer to (5) as the *Fourier series* for f and we say that $F[k]$ is the kth *Fourier coefficient* for f. You may be familiar with the alternative representation

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\{a_k \cos(2\pi kx/p) + b_k \sin(2\pi kx/p)\}$$

for a Fourier series. You can use Euler's identity (2) to see that this representation is equivalent to (5), see Ex. 1.16. From time to time we will work with such cos, sin series, e.g., this form may be preferable when f is real or when f is known to have even or odd symmetry. For general purposes, however, we will use the compact complex form (5).

Fourier found that the coefficients $F[k]$ for the representation (5) can be constructed for a given function f by using the integrals

$$F[k] = \frac{1}{p}\int_{x=0}^{p} f(x)e^{-2\pi ikx/p}\,dx, \quad k = 0, \pm 1, \pm 2, \dots . \tag{6}$$

[Before discovering the simple formula (6), Fourier made use of clumsy, mathematically suspect arguments based on power series to find these coefficients.] We refer to (5) as the *synthesis equation* and to (6) as the *analysis equation* for the p-periodic function f, and we say that F is the *Fourier transform* of f within this context. We use small circles on line segments, i.e., *lollipops*, when we graph F (a function on $\mathbb{Z}$), and we often display the graphs of f, F side by side as illustrated in Fig. 1.5. Of course, we must provide separate graphs for $\operatorname{Re} f$, $\operatorname{Im} f$, $\operatorname{Re} F$, $\operatorname{Im} F$ in cases where f, F are not real valued. You will find such displays in Chapter 4, where we develop a calculus for evaluating integrals having the form (6).

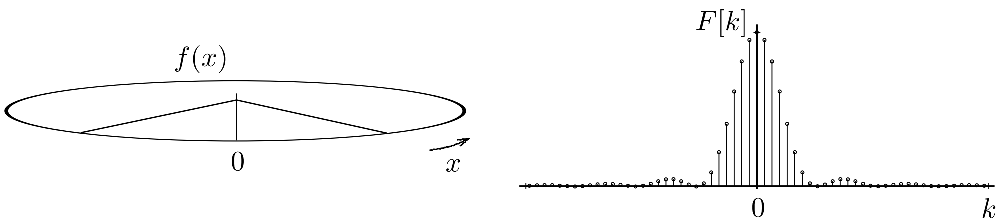

Figure 1.5. The graph of a function f on $\mathbb{T}_p$ and its Fourier transform F on $\mathbb{Z}$.

Functions on $\mathbb{Z}$

There is a Fourier representation for any suitably regular complex-valued function f that is defined on the set of integers, $\mathbb{Z}$. As expected, we synthesize f from the complex exponential functions $e^{2\pi isn}$ on $\mathbb{Z}$, with s being a real parameter. Now for any real s and any integer m we find

$$e^{2\pi i(s+m)n} = e^{2\pi isn}, \quad n = 0, \pm 1, \pm 2, \dots$$

(i.e., the exponentials $e^{2\pi isn}$, $e^{2\pi i(s\pm 1)n}$, $e^{2\pi i(s\pm 2)n}, \dots$ are indistinguishable when n is constrained to take integer values). This being the case, we will synthesize f using

$$e^{2\pi isn}, \quad 0 \le s < 1$$

or equivalently, using

$$e^{2\pi isn/p}, \quad 0 \le s < p,$$

where p is some fixed positive number. Figure 1.6 illustrates what happens when we attempt to use some $s > p$. The high-frequency sinusoid takes on the identity or *alias* of some corresponding low-frequency sinusoid. It is easy to see that $e^{2\pi isn/p}$ oscillates slowly when s is near 0 or when s is near p. The choice $s = p/2$ gives the most rapid oscillation with the complex exponential

$$e^{2\pi i(p/2)n/p} = (-1)^n$$

having the smallest possible period, 2.

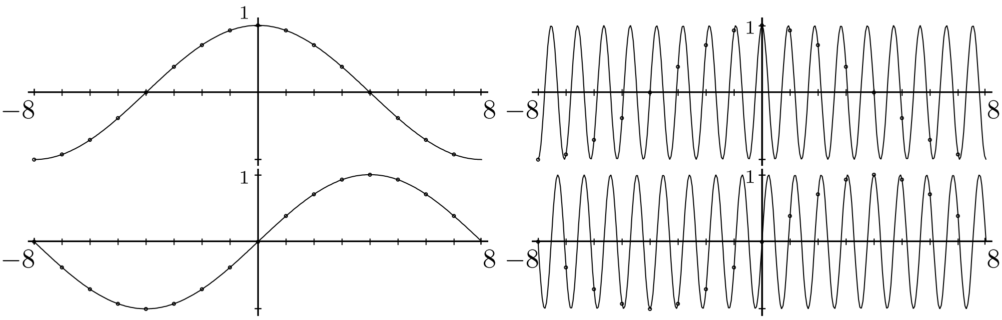

Figure 1.6. The identical samples of $e^{2\pi ix/16}$ and $e^{2\pi i17x/16}$ at $x = 0, \pm1, \pm2, \ldots$.

Fourier's *synthesis equation*,

$$f[n] = \int_{s=0}^{p} F(s)e^{2\pi isn/p}\, ds, \tag{7}$$

for a suitably regular function f on $\mathbb{Z}$, uses all of these complex exponentials on $\mathbb{Z}$, and the corresponding *analysis equation* is given by

$$F(s) = \frac{1}{p} \sum_{n=-\infty}^{\infty} f[n]e^{-2\pi isn/p}. \tag{8}$$

We say that F is the *Fourier transform* of f and observe that this function is p-periodic in s, i.e., that F is a complex-valued function on the circle $\mathbb{T}_p$. Figure 1.7 illustrates such an f, F pair.

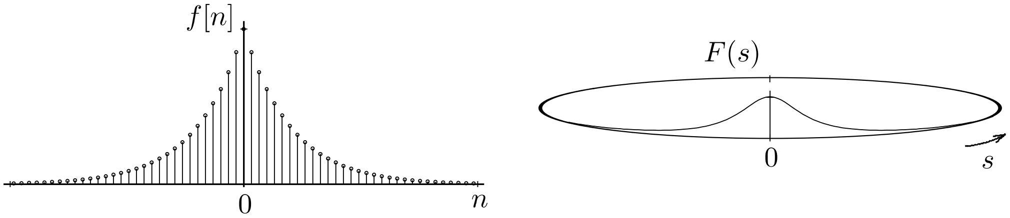

Figure 1.7. The graph of a function f on $\mathbb{Z}$ and its Fourier transform F on $\mathbb{T}_p$.

We have chosen to include the parameter $p > 0$ for the representation (7) (instead of working with the special case $p = 1$) in order to emphasize the *duality* that exists between (5)–(6) and (7)–(8). Indeed, if we replace

$$i, \quad x, \quad k, \quad f, \quad F$$

in (5)–(6) by

$$-i, \quad s, \quad n, \quad pF, \quad f,$$

respectively, we obtain (7)–(8). Thus every Fourier representation of the form (5)–(6) corresponds to a Fourier representation of the form (7)–(8), and vice versa.

Functions on $\mathbb{P}_N$

Let N be a positive integer, and let $\mathbb{P}_N$ consist of N uniformly spaced points on the circle $\mathbb{T}_N$ as illustrated in Fig. 1.8. We will call this discrete circle a *polygon* even in the degenerate cases where $N = 1, 2$.

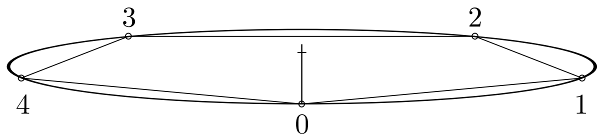

Figure 1.8. The polygon $\mathbb{P}_5$.

The simplest Fourier representation [found by Gauss in the course of his study of interpolation by trigonometric polynomials a few years before Fourier discovered either (3)–(4) or (5)–(6)] occurs when f is a complex-valued N-periodic function defined on $\mathbb{Z}$. We will routinely identify such an N-periodic f with a corresponding function that is defined on $\mathbb{P}_N$ as illustrated in Fig. 1.9. Of course, we must provide separate graphs for $\text{Re}\, f$, $\text{Im}\, f$ when f is complex valued. Since f is completely specified by the N function values $f[n]$, $n = 0, 1, \ldots, N-1$, we will sometimes find that it is convenient to use a complex N-vector

$$f = (f[0], f[1], \ldots, f[N-1])$$

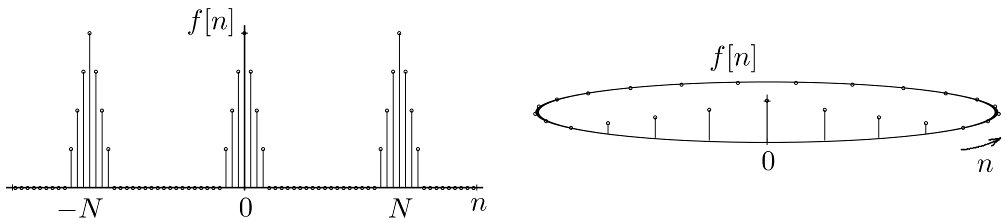

Figure 1.9. Identification of an N-periodic discrete function on $\mathbb{Z}$ with a corresponding function on the polygon $\mathbb{P}_N$.

to represent this function. This is particularly useful when we wish to process f numerically. You will observe that we always use $n = 0, 1, \dots, N-1$ (not $n = 1, 2, \dots, N$) to index the components of f.

The complex exponential $e^{2\pi isn}$ (with s being a fixed real parameter) will be N-periodic in the integer argument n, i.e.,

$$e^{2\pi is(n+N)} = e^{2\pi isn} \quad \text{for all } n = 0, \pm1, \pm2, \dots$$

when

$$e^{2\pi isN} = 1,$$

i.e., when $s = k/N$ for some integer k. On the other hand, when m is an integer we find

$$e^{2\pi ikn/N} = e^{2\pi i(k+mN)n/N} \quad \text{for all } n = 0, \pm1, \pm2, \dots,$$

so the parameters

$$s = \frac{k}{N}, \quad s = \frac{k \pm N}{N}, \quad s = \frac{k \pm 2N}{N}, \dots$$

all give the same function. Thus we are left with precisely N distinct discrete N-periodic complex exponentials

$$e^{2\pi ikn/N}, \quad k = 0, 1, \dots, N-1.$$

The complex exponentials with $k = 1$ or $k = N-1$ make one complete oscillation on $\mathbb{P}_N$, those with $k = 2$ or $k = N-2$ make two complete oscillations, etc., as illustrated in Fig. 1.10. The most rapid oscillation occurs when N is even and $k = N/2$ with the corresponding complex exponential

$$e^{2\pi i(N/2)n/N} = (-1)^n$$

having the smallest possible period, 2.

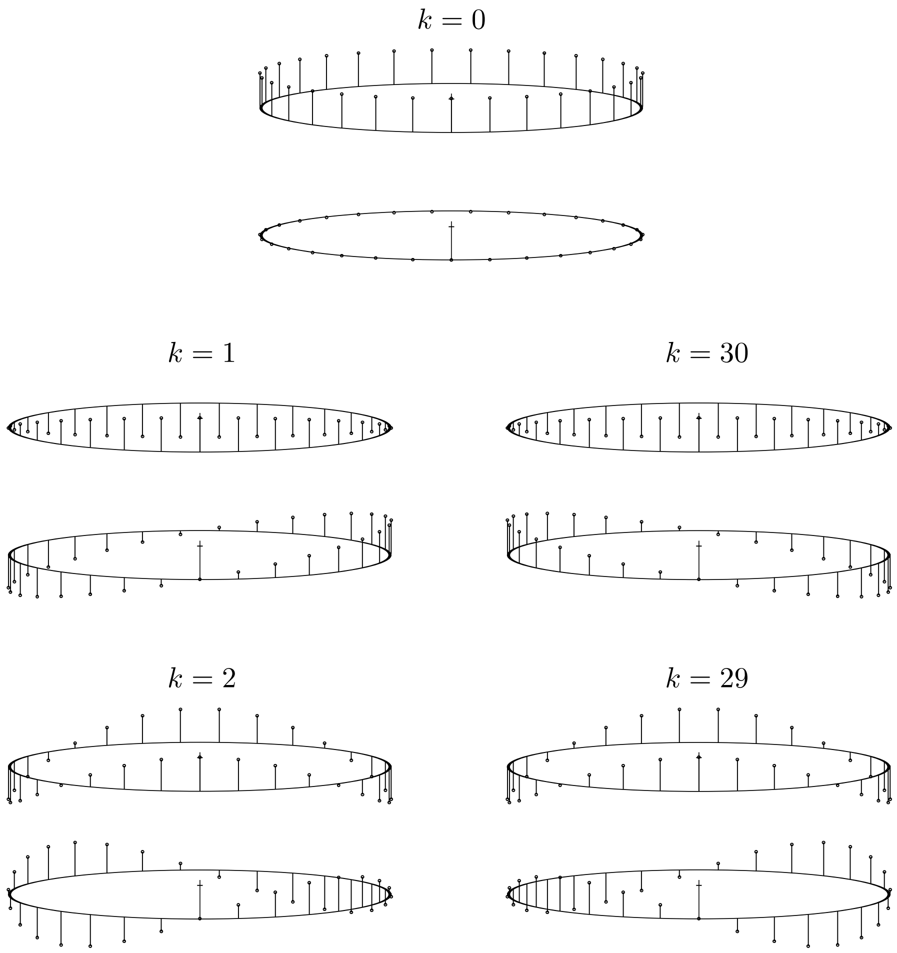

Figure 1.10. Complex exponentials $e^{2\pi ikn/31}$ on $\mathbb{P}_{31}$.

Fourier's *synthesis equation* takes the form

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{2\pi ikn/N}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{9}$$

within this setting. Again we regard $F[k]$ as the amount of the discrete exponential $e^{2\pi ikn/N}$ that must be used in the recipe for f, we refer to (9) as the *discrete Fourier series* for f, and we say that $F[k]$ is the kth *Fourier coefficient* for f. The corresponding *analysis equation*

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-2\pi ikn/N}, \quad k = 0, 1, \ldots, N-1 \tag{10}$$