Introduction

This book deals with the controllability problem in distributed parameter systems (DPS). As the following chapters explain, one way to handle this problem is to reduce it to the problem of moments and to study the resulting exponential families $\mathscr{E}_{\text{par}} = \{\eta_n e^{-\lambda_n t}\}_{n=1}^{\infty}$ (the "parabolic" case) or $\mathscr{E}_{\text{hyp}} = \{\eta_n e^{\pm i\sqrt{\lambda_n t}}\}_{n=1}^{\infty}$ (the "hyperbolic" case). Here, $\{\lambda_n\}$ is the spectrum of a system, and vectors η_n belong to an auxiliary Hilbert space $\mathfrak{N}$, dim $\mathfrak{N} \le \infty$, determined by the space of control actions.

Over the past twenty-five years, DPS control theory has been developing rapidly, both because of its important technological applications and its usefulness in resolving a variety of mathematical problems. Indeed, the theory has been described at length in various monographs (Butkovskiĭ 1965, 1975; Lions 1968, 1983, 1988a; Lurie 1975; Curtain and Pritchard 1978; Egorov A. I. 1978; Litvinov 1987; Lagnese and Lions 1989; Krabs 1992).[1]

The controllability question occupies a prominent place in DPS control theory for a number of reasons. First, many practical problems in various fields of engineering, physics, and chemistry are formulated as controllability problems, that is, as questions about how to describe reachability sets. Second, it is essential to have some insight into controllability in order to resolve DPS optimal control problems. Furthermore, controllability plays a vital role in the stabilization and identification of DPS. Recent studies have demonstrated its profound connection to the classical inverse problems of mathematical physics, for example (Belishev 1989;

The first version of this book was published in Russian in 1989 (Avdonin and Ivanov 1989b). For the present edition, the book has been thoroughly revised and new results were added.

[1] Since the number of publications pertinent to the subject of this book is enormous, we are unable to present an exhaustive list of references; an extensive bibliography may be found in Fleming (1988).

Avdonin, Belishev, and Ivanov, 1991a). Note, too, that DPS controllability studies have brought to light a host of interesting and complex questions in several branches of mathematics, such as PDE theory, operator theory, the theory of functions, and the theory of numbers.

The various techniques used to investigate DPS controllability can be divided into three fundamental approaches. The first one is based on theoretical operator methods (see, e.g., Fattorini 1966, 1967; Tsujioka 1970; Fuhrmann 1972; Weiss 1973; Triggiani 1978; Nefedov and Sholokhovich 1985; Sholokhovich 1987). These are rather general methods that make it possible to treat a broad range of systems described by equations in Hilbert and Banach spaces; however, they are not always effective in addressing concrete problems.

The second approach employs a specific technique of the theory of partial differential equations. This technique has been the subject of considerable research and has such broad applications that we can mention only a few of the works that have focused on it.

D. L. Russell (1971a, 1971b, 1972, 1973) and J. E. Lagnese (1983), for example, apply the method of characteristics for hyperbolic equations and the Holmgreen uniqueness theorem (see also Littman 1986).

J.-L. Lions (1986) suggested the Hilbert Uniqueness Method, which is based on the duality between controllability and observability and on a priori estimates of solutions of nonhomogeneous boundary value problems. This method was developed by L. F. Ho (1986), P. Grisvard (1987), E. Zuazua (1987), A. Haraux (1988), and I. Lasiecka and R. Triggiani (1989). A number of searchers (Chen et al. 1987; Leugering and Schmidt 1989; Schmidt 1992; Lagnese, Leugering, and Schmidt 1993) have applied the method to networks of strings and beams.

Bardos, Lebeau, and Rauch (1988a, 1988b, 1992) have developed an approach to the controllability problems for hyperbolic equations using microlocal analysis and propagations of singularities. This approach made it possible to solve the problem of exact controllability in cases where controls act on a part of the boundary or on a subdomain. See also Emanuilov (1990).

The third approach, which reduces the control problem to the problem of moments relative to a family of exponentials, is known as the moment method. It is a powerful tool of control theory in that it provides solutions to many kinds of problems. N. N. Krasovskiĭ (1968) applied the method to the systems described by ordinary differential equations. It has also been used in DPS control theory to investigate optimum time control problems (Egorov Yu. V. 1963a, 1963b; Butkovskiĭ 1965, 1975; Gal′chuk

1968; Korobov and Sklyar 1987) and to solve optimal problems with the quadratic quality criterion (Plotnikov 1968; Egorov A. I. 1978; Vasil′ev, Ishmukhametov, and Potanov 1989). In addition, this method has been used in combination with the Pontryagin maximum principle to address the optimal boundary control problem to parabolic vector equations (Kuzenkov and Plotnikov 1989), and to study some bilinear control problems (Egorov A. I. and Shakirov 1983), as well as observation problems in parabolic-type equations (Mizel and Seidman 1969, 1972; Seidman 1976, 1977).

In investigations of DPS controllability, the moment method has most often been used for systems with one spatial variable and for a scalar control function (see, e.g., Russell 1967, 1978; Fattorini and Russell 1971; Butkovskiĭ 1975; Reid and Russell 1985). Work has also been done on controllability problems associated with several control actions (Fattorini 1968; Sakawa 1974).

The moment method has also been used to analyze controllability in systems permitting separation of spatial variables. Here, the method has made it possible to reduce the controllability problem to a series of scalar problems (see Graham and Russell 1975; Fattorini 1975, 1979; Krabs, Leugering, and Seidman 1985). For the exponential family $\mathscr{E}$ arising in the transition from a control problem for a moment one, the role of auxiliary space $\mathfrak{N}$ is filled by the space to which the values of control actions belong. In the case of a single control action $\dim \mathfrak{N} = 1$, the usual "scalar" families of exponentials appear. If there is a finite number N of scalar control actions, then $\dim \mathfrak{N} = N$, and a family of vector exponentials with the values in a finite dimensional space arises. A string vibration equation with the control actions at both boundary points serves as an example (where $N = 2$). If, for instance, a control acts on the boundary Γ of a multidimensional spatial domain Ω, then it is natural to choose $L^2(\Gamma)$ as $\mathfrak{N}$, whereupon $\dim \mathfrak{N} = \infty$.

The solvability of the resulting problem of moments, and hence of the primary control problem, depends on the properties of the corresponding exponential family. The study of scalar exponential families (nonharmonic Fourier series) dates back to the 1930s (Paley and Wiener 1934) and since then has become a well-known branch of the mathematical analysis. Thus, questions concerning the completeness, minimality, and basis property of such families in space $L^2(0, T)$ have been investigated in some depth (Ingham 1934; Levinson 1940; Duffin and Eachus 1942; Duffin and Schaffer 1952; Levin 1956, 1961; Kadets 1964; Redheffer 1968; Katsnelson 1971; Young 1980). In addition, B. S. Pavlov (1973, 1979) has suggested a geometric (in a Hilbert space sense) approach that has provided a basis property criterion. (For a detailed exposition of this approach and its relation to other problems connected with the theory of functions, see Hrushchev, Nikol′skiĭ, and Pavlov 1981.) We use Pavlov's geometrical approach in this book to develop a theory of exponentials in a space of vector functions. This work has also enabled us to shed new light on scalar families and thus is proving to be useful in DPS control problems.

In fact, DPS control problems were the very reason that we decided to investigate exponential families. Without the results that we have obtained on vector exponential families, it would be very difficult to apply the moment method to problems that cannot be treated in the terms of a scalar exponential family or a series of such families of simple enough structure. In other words, the extension of the moment method to a wider class of DPS is one of the principal objectives of this book.

The book consists of seven chapters, each of which is divided into numbered sections, which in turn contain enumerated assertions (remarks, theorems, corollaries, and so on). When referring to a statement or a formula within a chapter section, we omit the number of that chapter section (e.g., we refer to Proposition 17(a) rather than Proposition I.1.17(a)). When referring to a formula of another section, we add the number of that section. Sections are divided into subsections. Although this arrangement may seem unwieldy, it is difficult to treat this complex subject in any other way.

Chapter I presents the basic information needed to understand projectors in Hilbert spaces, families of elements, and families of subspaces, as well as the problem of moments. Although we cannot claim to be presenting original results (except, perhaps, for some assertions on the problem of moments solvability) or to have made any methodological discoveries, we have brought together for the first time all basic information concerning this subject.

The discussion opens in Section I.1 with the geometry of Hilbert spaces. For two subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of a Hilbert space $\mathfrak{H}$, we introduce the concept of an angle $\varphi(\mathfrak{M}, \mathfrak{N})$ between them,

$$\varphi(\mathfrak{M}, \mathfrak{N}) = \arccos \sup_{m \in \mathfrak{M},\, n \in \mathfrak{N}} \frac{|(m, n)|}{\|m\|\,\|n\|}.$$

In terms of the angles, we elucidate the properties of operators $P_{\mathfrak{M}}|_{\mathfrak{N}}$ (orthoprojectors on $\mathfrak{M}$ restricted to $\mathfrak{N}$). In particular,

$$\|[P_{\mathfrak{M}}|_{\mathfrak{N}}]^{-1}\| = 1/\sin \varphi(\mathfrak{H} \ominus \mathfrak{M}, \mathfrak{N}).$$

Skew projector $\mathscr{P}_{\mathfrak{M}}^{\|\mathfrak{N}}$ from the direct sum $\mathfrak{M} \dotplus \mathfrak{N}$ to $\mathfrak{M}$ parallel to $\mathfrak{N}$ is studied further:

$$\mathscr{P}_{\mathfrak{M}}^{\|\mathfrak{N}}(m+n) = m \qquad \forall m \in \mathfrak{M}, n \in \mathfrak{N}.$$

This projector is bounded if and only if $\varphi(\mathfrak{M}, \mathfrak{N}) > 0$. Such projectors play a central role in the investigation of exponential families in $L^2(0, T)$.

Next, we study families $\Xi = \{\xi_n\}$ of element (and families of subspaces) from the perspective of the "degree" of their linear independence. The linear independence of any finite subfamilies of Ξ is the weakest one. We let it be denoted by $\Xi \in (L)$. The next step is to introduce W-linear independence (notation $\Xi \in (W)$). This property, in somewhat simplified terms, means that the weak convergence of series $\sum c_n \xi_n$ to zero implies that all coefficients c_n are zeros. A stronger property is minimality ($\Xi \in (M)$). It means that for any n, element ξ_n does not lie in the closure of linear span of the remaining elements. To put it another way: there exists a family $\Xi' = \{\xi_n'\}$, called the biorthogonal family, such that

$$(\xi_n, \xi_n') = \delta_n^m.$$

If, along with this, $\|\xi_n'\|$ are jointly bounded, then the family is said to be *-uniformly minimal (notation $\Xi \in (UM)$). For the almost normed families ($\|\xi_n\| \asymp 1$), the (UM) property is equivalent to

$$\varphi\left(\xi_n, \bigvee_{m \neq n} \xi_m\right) \geq \delta > 0.$$

A family for which the latter relation holds is said to be uniformly minimal.

The strongest property, which is the $\mathscr{L}$-basis or Riesz basis property in the closure of the linear span of the family, means that family Ξ is an image of an orthonormal one under the action of some isomorphism; we write $\Xi \in (LB)$ in this case.

Families of subspaces are classified by similar definitions. Thus the hierarchy of the "independence"

$$(LB) \Rightarrow (UM) \Rightarrow (M) \Rightarrow (W) \Rightarrow (L)$$

is established.

In Section I.2, we turn to the problem of moments. For a given family Ξ and some element $c \in \ell^2$, one has to find $f \in \mathfrak{H}$ such that $\{(f, \xi_n)\} = c$. Operator

$$\mathscr{I}_{\Xi}\colon f \mapsto \{(f, \xi_n)\} \tag{1}$$

is called the operator of the problem of moments. We focus our attention on the "quality" of the solvability of the moment problem or, more precisely, on the image R_Ξ of operator $\mathscr{J}_\Xi$. The solvability of the problem of moments is directly associated with "the degree of linear independence" of family Ξ. In particular,

$$\Xi \in (LB) \Rightarrow R_\Xi = \mathfrak{H}, \qquad R_\Xi = \mathfrak{H} \Rightarrow \Xi \in (UM),$$

$$Cl_{\mathfrak{H}} R_\Xi = \mathfrak{H} \Leftrightarrow \Xi \in (W).$$

We also prove that R_Ξ is closed if a Riesz basis can be found in Ξ.

Chapter II examines the properties of family $\mathscr{E}_T$ of vector exponentials in $L^2(0, T; \mathfrak{N})$,

$$\mathscr{E}_T = \{e_n\}_{n\in\mathbb{Z}}, \qquad e_n = e^{-i\bar{\lambda}_n t}\eta_n, \qquad \operatorname{Im} \lambda_n > 0,$$

in detail. Here, $\eta_n \in \mathfrak{N}$, $\dim \mathfrak{N} < \infty$, $T \le \infty$. By the Paley–Wiener theorem, the inverse Fourier transform turns $L^2(0, \infty)$ into Hardy space H^2_+, which consists of analytic functions in the upper half-plane $\mathbb{C}_+$ whose traces are squarely summable over the real axis. Here, the exponentials turn into simple fractions $x_n(k) = (k - \bar{\lambda}_n)^{-1}$ belonging to H^2_+ for $\operatorname{Im} \lambda_n > 0$. This makes it possible to invoke the powerful theory of Hardy spaces in the study of exponentials. Section II.1 explains these spaces and simple fraction families. It should be pointed out that the reader will require some knowledge of the basics of the Hardy space theory in order to understand the theory developed in this book.

To begin with, one needs to be familiar with the concepts associated with inner–outer factorization. Consider, for simplicity, the case of functions bounded in $\mathbb{C}_+$. If such functions have a unit absolute value almost everywhere on the real axis (and are analytic in $\mathbb{C}_+$), then they are said to be inner functions. Among them, Blaschke products (BP), $B(k)$, are recognized,

$$B(k) = \prod_{n\in\mathbb{Z}} \varepsilon_n \frac{k - \lambda_n}{k - \bar{\lambda}_n}, \qquad \operatorname{Im} \lambda_n > 0,$$

where ε_n are the phase factors, $|\varepsilon_n| = 1$, and numbers λ_n – the zeros of the BP – satisfy the Blaschke condition

$$\sum_{n\in\mathbb{Z}} \frac{\operatorname{Im} \lambda_n}{1 + |\lambda_n|^2} < \infty. \tag{B}$$

Functions $\exp(ika)$, $a \ge 0$, are obviously inner functions as well; in contrast to BP's, they have no zeros at all. Such functions are called entire singular inner functions (they have the essentially singular point at infinity). Functions bounded and analytic in $\mathbb{C}_+$ possess factorization

$f = f_i f_e$ in which f_i is an inner function and f_e is an outer function of the form

$$f_e(k) = \exp\left(\frac{i}{\pi}\int_{\mathbb{R}}\frac{h(t)}{k-t}\,dt\right)$$

with $h(t) = \log|f(t)|$. Outer functions have no zeros in $\mathbb{C}_+$. In contrast to entire singular inner functions, they cannot decrease exponentially at Im k + ∞.

Section II.1 introduces the concepts of inner and outer functions for analytic operator functions with the values in finite-dimensional space $\mathfrak{N}$, along with the Blaschke–Potapov (BPP) product, which is the analog of BP, and the entire singular inner operator function (ESF). Matrix exponential $\exp(ikQ)$ with nonnegative matrix (operator) Q is an ESF. Without going into detailed definitions here, suffice it to say that an operator function belongs to the corresponding class if its determinant is a function from a similar scalar class. For analytic operator functions F bounded in $\mathbb{C}_+$ there also exists factorization

$$F = \Pi\Theta F_e^+,$$

where Π is a BPP, Θ is an ESF, and F_e^+ is an outer operator function.

Consider now the exposition of the known results on the properties of families $\mathscr{X}$ of simple fractions $x_n(k)$, $n \in \mathbb{Z}$. It appears that for the minimality of $\mathscr{X}$ (on H_+^2) the validity of Blaschke condition (B) is necessary and sufficient.

The criterion of uniform minimality of $\mathscr{X}$ is the Carleson condition

$$\inf_{m\in\mathbb{Z}}\prod_{n,n\neq m}\left|\frac{\lambda_n-\lambda_m}{\lambda_n-\bar{\lambda}_m}\right| > 0. \tag{C}$$

This condition is well known in the theory of interpolation of bounded analytic functions. Normalized families of simple fractions exhibit a surprising equivalence between the uniform minimality and the $\mathscr{L}$-basis property: that is, the Carleson condition proves to be the $\mathscr{L}$-basis criterion. In the strip $0 < c \leq \text{Im k} \leq C$, condition ($C$) transforms into the separability condition ($\inf_{m\neq n}|\lambda_m - \lambda_n| > 0$). Recall that these properties of family $\mathscr{X}$ are equivalent to similar properties of an exponential family in $L^2(0,\infty)$.

In Section II.2, minimality and $\mathscr{L}$-basis criteria are given for family $\mathscr{E}_\infty$ of vector exponentials. For family $\mathscr{E}_\infty$ to be minimal in $L^2(0,\infty;\mathfrak{N})$, it is necessary and sufficient that λ_n satisfy the Blaschke condition, so the finite dimensional case has no specific character of its own, in comparison with the scalar one.

The situation is more complicated when it comes to the $\mathscr{L}$-basis property. If one takes two Carlesonian sets σ_1 and σ_2, then their unification may be not Carlesonian (in contrast to the Blaschke condition, the Carleson one is not "additive"). Consequently, scalar family $\{e^{i\lambda t}\}_{\lambda\in\sigma_1\cup\sigma_2}$ will not yet be an $\mathscr{L}$-basis. At the same time, vector exponential family

$$\{e^{i\lambda t}\eta_1\}_{\lambda\in\sigma_1}\cup\{e^{i\lambda t}\eta_2\}_{\lambda\in\sigma_2}$$

evidently constitutes an $\mathscr{L}$-basis if η_1 and η_2 are linearly independent. It is known that $\mathscr{L}$-basis family $\mathscr{E}_\infty$ allows a splitting into dim $\mathfrak{N}$ subfamilies, each of which has a Carlesonian spectrum. This single condition is not enough for the $\mathscr{L}$-basis property of $\mathscr{E}_\infty$. To obtain the $\mathscr{L}$-basis criterion it is necessary to demand in addition that every group of "close points" λ has vectors η_λ, which are "linear independent" uniformly in groups. The exact formulation of the criterion is presented in Subsection II.2.2.

A criterion for vector exponential family $\mathscr{E}_T$ to form a basis in space $L^2(0, T; \mathfrak{N})$ is established in Section II.3 in terms of the generating function (GF).

The GF concept was formulated some time ago (Paley and Wiener 1934) and since then has been widely used (see, e.g., Levin 1956, 1961) for the investigation of scalar exponential families $\{e^{i\lambda_n t}\}$ in $L^2(0, T)$. The GF is constructed by its zeros λ_n with the help of the following formula:

$$f(k) = e^{ikT/2} f_0, \qquad f_0 = \text{p.v.} \prod_n (1 - k/\lambda_n), \tag{2}$$

under the assumption that f_0 has the same exponential type $T/2$ both in the upper and the lower half-planes. Later, we assume that the spectrum $\sigma = \{\lambda_n\}$ lies in the strip $0 < c \le \operatorname{Im} \lambda_n \le C$; note that the shift $\lambda_n \mapsto \lambda_n + i\delta$ does not change the minimality and $\mathscr{L}$-basis properties of the family. By employing the GF, B. S. Pavlov (1979) managed to obtain the basis criterion: family $\{e^{i\lambda_n t}\}$ constitutes a Riesz basis in $L^2(0, T)$ if and only if

(i) $\{\lambda_n\}$ is separable and
(ii) $|f(x)|^2$ satisfies the so-called Muckenhoupt condition

$$\sup_{I\in\mathscr{T}} \frac{1}{|I|}\int_I |f(x)|^2\,dx\,\frac{1}{|I|}\int_I |f(x)|^{-2}\,dx < \infty,$$

where $\mathscr{T}$ is the set of intervals of the real axis.

Condition (i) is equivalent to the $\mathscr{L}$-basis property of family $\{e^{i\lambda_n t}\}$ in $L^2(0, \infty)$. Condition (ii) leads to several equivalent statements. One of them, which has just appeared in the Pavlov approach, requires the Hilbert operator

$$(Hu)(x) = \frac{1}{\pi} \text{ p.v. } \int_{\mathbb{R}} \frac{u(t)}{x - t}\, dt \tag{3}$$

to be bounded in the space of functions squarely integrable on the line with the weight $|f(x)|^2$.

Extending this approach, we obtain the necessary and sufficient condition that the vector exponential family forms a basis (see Section II.3). We find that GF f from (ii) may be defined as a function fulfilling factorization conditions

$$f(k) = B(k) f_e^+(k), \qquad k \in \mathbb{C}_+,$$

$$f(k) = e^{ikT} f_e^-(k), \qquad k \in \mathbb{C}_-.$$

Here B is the BP constructed by $\{\lambda_n\}$ while $f_e^{\pm}$ are outer functions in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. These are precisely the relations that provide the grounds for the definition of the GF in the vector case. Entire operator function F with a factorization

$$F = \Pi F_e^+ = e^{ikT} F_e^- \tag{4}$$

is said to be a GF for family $\mathscr{E}_T = \{e^{i\lambda_n t}\eta_n\}$ in $L^2(0, T; \mathfrak{N})$. Here, $F_e^{\pm}$ are outer operator functions in $\mathbb{C}_{\pm}$, and $\Pi(k)$ is the BPP constructed by λ_n and η_n (i.e., the determinant of Π is the BP with zeros λ_n and $\eta_n \in \operatorname{Ker} \Pi^*(\lambda_n)$). Family $\mathscr{E}_T$ is shown to form a Riesz basis in $L^2(0, T; \mathfrak{N})$ if and only if (i) family $\mathscr{E}_\infty$ forms an $\mathscr{L}$-basis on the semiaxis, and (ii) the Hilbert operator (3) is bounded in the space of vector functions squarely summable on the line with the matrix weight $F^*(x)F(x)$.

Properties of scalar exponential families are examined in Section II.4. The known results concerning the minimality and basis property are discussed first, and then some new findings presented. For example, it is now thought that sine-type functions play a significant role in exponential families. Entire function f of the exponential type is called a sine-type function if its zeros lie in the strip $|\operatorname{Im} k| \leq C$ and if both f and $1/f$ are bounded on some line parallel to the real axis. The proximity of numbers λ_n to the zeros of some sine-type function is the known sufficient condition for family $\{e^{i\lambda_n t}\}$ to be a Riesz basis. The converse statement is also proved to be true, and this fact is used to demonstrate the following result.

If family $\mathscr{E}_T = \{e^{i\lambda_n t}\}$ forms a Riesz basis in $L^2(0, T)$, then for any $T' < T$ there exists a subfamily $\mathscr{E}' \subset \mathscr{E}_T$ constituting a Riesz basis in $L^2(0, T')$. We also elaborate an algorithm for the construction of such a subfamily.

In the same section, we prove that statements similar to the one formulated above are valid for families of a more general form:

$$\{t^m e^{i\lambda_n t}\}, \qquad m = 0, 1, \ldots, r_n, \; n \in \mathbb{Z}.$$

In Section II.5, we look at vector exponential families further. We show, for example, that when family $\mathscr{E}_T = \{e^{i\lambda_n t}\eta_n\}$ is minimal in $L^2(0, T; \mathbb{C}^N)$, then scalar family $\{e^{i\lambda_n t}\}$ generally is not minimal in $L^2(0, NT)$, but becomes minimal after N arbitrary elements are removed from it.

Another assertion we make there deals with the stability of the basis property. If $\mathscr{E}_T$ forms a Riesz basis in $L^2(0, T; \mathfrak{N})$, then $\varepsilon > 0$ may be found such that any family $\tilde{\mathscr{E}}$ of the form $\{e^{i\tilde{\lambda}_n t}\tilde{\eta}_n\}$ is also a Riesz basis in $L^2(0, T; \mathfrak{N})$ as soon as

$$|\lambda_n - \tilde{\lambda}_n| + \|\eta_n - \tilde{\eta}_n\| < \varepsilon.$$

Chapter II closes with a discussion of the conditions that provide the weak convergence in $L^2(0, T)$ to zero of series $\sum a_n e^{-\mu_n t}$ (the "parabolic" case) or $\sum a_n e^{i\lambda_n t}$ (the "hyperbolic" case), which implies all the coefficients to be zeros. In the parabolic case, it takes place under very weak limitations on $\{\mu_n\}$. However, to make this implication hold in the hyperbolic case, stringent restrictions on $\{a_n\}$ have to be imposed. As becomes clear later in the book, differences in the behavior of the exponential family lead to a qualitative distinction in the controllability of parabolic and hyperbolic systems.

Evolution equations of the first and second order in time

$$\dot{x}(t) + Ax(t) = f(t), \tag{5}$$

$$\ddot{x}(t) + Ax(t) = f(t) \tag{6}$$

are treated in Chapter III. Here, A is a self-adjoint, semibounded-from-below operator in Hilbert space H; operator $A_\alpha := A + \alpha I$ is positive definite. We assume A to have a set of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with corresponding eigenfunctions φ_n forming an orthonormal basis in H.

We introduce a scale of Hilbert spaces W_s, $s \in \mathbb{R}$. For $s > 0$, W_s is the domain of operator $A_\alpha^{s/2}$; for $s < 0$, $W_s = W'_{-s}$ is the space dual to W_{-s} with respect to inner product in H, $W_0 = H = H'$.