1 Introduction

1.1 Linear antennas

Wireless communication depends upon the interaction of oscillating electric currents in specially designed, often widely separated configurations of conductors known as antennas. Those considered in this book consist of thin metal wires, rods or tubes arranged in arrays. Electric charges in the conductors of a transmitting array are maintained in systematic accelerated motion by suitable generators that are connected to one or more of the elements by transmission lines. These oscillating charges exert forces on other charges located in the distant conductors of a receiving array of elements of which at least one is connected by a transmission line to a receiver. Fundamental quantities which describe such interactions are the electromagnetic field, the driving-point admittance, and the driving-point impedance. These can be easily determined if the distributions of current on the array elements are known. The determination of the currents on the array elements is the main concern of this book. In this first chapter, the basic electromagnetic equations are formulated and applied to a single antenna in free space. The simplest approach of assuming the current rather than actually determining it is reviewed first. Then, integral equations for the current distributions are derived, and determining the current by numerical methods is discussed. These discussions serve as an introduction to the analytical theory of antennas and arrays based on the solution of integral equations that is presented in subsequent chapters.

Figures 1.1a and 1.1b show two simple practical radiating systems. In Fig. 1.1a, a section at the open end of a two-wire transmission line has been bent outward to form a dipole antenna. In Fig. 1.1b, the inner conductor of a coaxial transmission line is extended above a ground plane. In both cases, the transmission lines are connected to generators which oscillate at a frequency $f = \omega/2\pi$. In a small region (comparable in extent with the distance between the two conductors of the transmission line), the antenna and line are coupled. Owing to the complications involved in this coupling, it is convenient to replace the actual generator/transmission line with an idealized so-called *delta-function* generator, which maintains an impressed electric field $\mathbf{E}^e(z) = \hat{\mathbf{z}}E_z^e(z) = V\delta(z)\hat{\mathbf{z}}$ at the surface of the antenna. This is the linear antenna of Fig. 1.1c. The impressed field is non-zero only at the center $z = 0$ of the cylindrical surface. The delta-function generator is an independent voltage source in the sense of ordinary

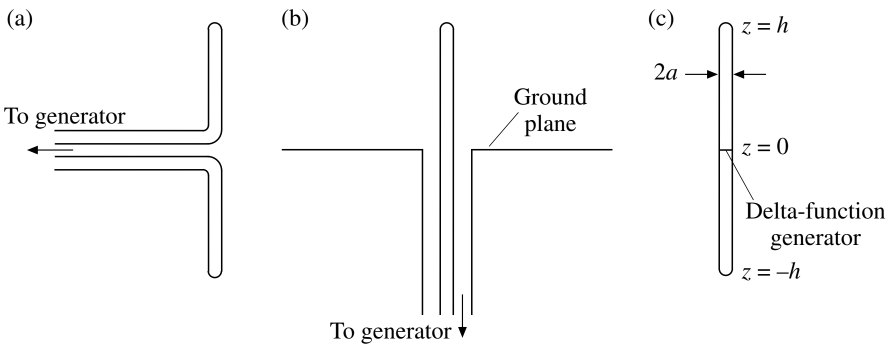

Figure 1.1 (a) Dipole antenna and two-wire transmission line. (b) Monopole antenna over a ground plane. (c) Simplified center-driven linear antenna.

circuit theory. The linear antenna of Fig. 1.1c can also serve as a model for other types of radiating systems. The simplifying assumption of studying the antenna in the absence of the connecting transmission line is particularly useful when the antenna is an array element.

The radius of the linear dipole antenna of Fig. 1.1c is a, and its half-length is h. It is assumed throughout this book that the radius is much smaller than both the wavelength λ and the length $2h$ of the antenna. Under such conditions, one can neglect the small currents on the capped ends of the antenna and assume that only a current $K_z(z) = I(z)/2\pi a$ is maintained on the cylindrical surface of the antenna. Other concepts of circuit theory can be introduced, and are particularly useful to the antenna engineer: the driving-point admittance Y_0 and driving-point impedance Z_0 are defined as

$$Y_0 = G_0 + jB_0 = \frac{I(0)}{V} = \frac{1}{Z_0}, \qquad Z_0 = R_0 + jX_0 = \frac{V}{I(0)} = \frac{1}{Y_0}. \tag{1.1}$$

G_0, B_0, R_0, and X_0 are respectively, the driving-point conductance, susceptance, resistance, and reactance. When h, a, and f are such that the antenna is at resonance, one has $X_0 = 0$ and $B_0 = 0$. As an example of the use of these quantities in a practical situation, consider the problem of designing the antenna so that, at a given frequency f, there is maximum power transfer from a transmission line of given characteristic impedance Z_c. With the assumption that the transmission line and the antenna can be studied separately, the problem is reduced to that of determining h and a so that Z_0 is equal to Z_c^*, the complex conjugate of Z_c.

The delta function $\delta(z)$ is zero except when $z = 0$. Additional, well-known properties of the delta function are

$$\delta(z) = \begin{cases} 0, & \text{if } z \neq 0 \\ \infty, & \text{if } z = 0 \end{cases}, \qquad \int_{-b}^{b} \delta(z)\, dz = 1 \tag{1.2a}$$

$$\delta(kz) = \frac{1}{|k|}\delta(z), \qquad f(z)\delta(z) = f(0)\delta(z) \tag{1.2b}$$

$$\int_{-b}^{b} f(z)\delta(z)\,dz = f(0) \tag{1.2c}$$

$$\frac{d}{dz}H(z) = \delta(z) \quad \text{where} \quad H(z) = \begin{cases} 1, & \text{if } z > 0 \\ 0, & \text{if } z < 0. \end{cases} \tag{1.2d}$$

In (1.2), b is any positive constant, k is any real constant, $f(z)$ is any smooth function of z, and $H(z)$ is the step function.

The next section introduces the fundamental equations of electromagnetic theory that are useful in the antenna problems considered in this book. More details can be found in [1], and in more concise form in [2, Chapter 1].

1.2 Maxwell's equations and the potential functions

The interaction of charges and currents is governed by Maxwell's equations which define the electromagnetic field. With an assumed time dependence $e^{j\omega t}$, they are

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + j\omega\epsilon_0\mathbf{E}), \qquad \nabla \cdot \mathbf{B} = 0 \tag{1.3a}$$

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B}, \qquad \nabla \cdot \mathbf{E} = \rho/\epsilon_0, \tag{1.3b}$$

where the electric vector $\mathbf{E}$ is in volts per meter (V/m), the magnetic vector $\mathbf{B}$ in tesla (T). SI units are used throughout this book. The volume density of current $\mathbf{J}$ in amperes per square meter ($\mathrm{A/m^2}$) is the charge crossing unit area per second. The volume density of charge ρ is in coulombs per cubic meter ($\mathrm{C/m^3}$). $\mathbf{J}$ and ρ satisfy the equation of continuity,

$$\nabla \cdot \mathbf{J} + j\omega\rho = 0. \tag{1.3c}$$

In the interior of perfect conductors, $\mathbf{J} = 0$ and $\rho = 0$. In (1.3), ϵ_0 and μ_0 are the absolute permittivity and permeability of free space. They have the numerical values $\epsilon_0 = 8.854 \times 10^{-12}$ farads per meter (F/m) and $\mu_0 = 4\pi \times 10^{-7}$ henrys per meter (H/m), and are related to the velocity c of light and the characteristic impedance ζ_0 of free space by

$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}, \qquad \zeta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}. \tag{1.4}$$

Transmission lines and antennas are made from highly conducting materials such as brass or copper. In most cases, it is an excellent approximation to assume that the conductors are perfect. The relevant boundary conditions at an interface between a perfect conductor and air are

$$\hat{\mathbf{n}} \times \mathbf{E} = 0, \qquad \hat{\mathbf{n}} \cdot \mathbf{E} = \eta/\epsilon_0 \tag{1.5a}$$

$$\hat{\mathbf{n}} \times \mathbf{B} = \mu_0 \mathbf{K}, \qquad \hat{\mathbf{n}} \cdot \mathbf{B} = 0. \tag{1.5b}$$

In (1.5), $\hat{\mathbf{n}}$ is the unit normal to the conductor–air interface. Its direction is outward from the conductor to the air. $\mathbf{K}$ is the surface density of current in amperes per meter (A/m) and η is the surface density of charge in coulombs per square meter (C/m^2) on the perfect conductor. The left-hand equation in (1.5a) states that the component of the electric field in air tangent to the surface of the perfect conductor must be zero. The left-hand equation in (1.5b) states that the tangential magnetic field in air is proportional to the surface density of current on the conductor.

It is convenient to introduce the scalar and vector potentials ϕ, $\mathbf{A}$. The defining relationships between the potentials and the electromagnetic-field vectors are obtained with the aid of Maxwell's equations. With the vector identity $\nabla \cdot (\nabla \times \mathbf{C}) = 0$ (where $\mathbf{C}$ is any vector) and the equation $\nabla \cdot \mathbf{B} = 0$, the magnetic field may be expressed in the form

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.6}$$

If (1.6) is substituted in (1.3b), it follows that

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0. \tag{1.7}$$

The identity $\nabla \times (\nabla\psi) = 0$, where ψ is a scalar function, then permits the definition of ϕ in the form

$$-\nabla\phi = \mathbf{E} + j\omega\mathbf{A}. \tag{1.8}$$

The substitution of (1.6) and (1.8) into the remaining Maxwell equations leads to coupled partial differential equations for $\mathbf{A}$ and ϕ. They can be decoupled if the following condition relating $\mathbf{A}$ and ϕ is imposed:

$$\nabla \cdot \mathbf{A} = -j\omega\mu_0\epsilon_0\phi \quad \text{or} \quad \nabla \cdot \mathbf{A} = -j\,\frac{\beta_0^2}{\omega}\,\phi, \tag{1.9}$$

where the free-space wave number β_0 (also denoted by k in this book) is given by

$$\beta_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda} \tag{1.10}$$

and λ is the free-space wavelength. Equation (1.9) is known as the Lorentz condition. The resulting equations for $\mathbf{A}$ and ϕ are

$$(\nabla^2 + \beta_0^2)\mathbf{A} = -\mu_0\mathbf{J}, \qquad (\nabla^2 + \beta_0^2)\phi = -\rho/\epsilon_0. \tag{1.11}$$

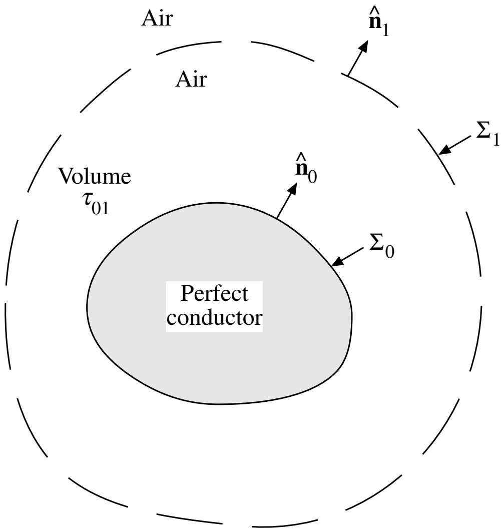

Figure 1.2 Perfect conductor in air.

The solutions to (1.11) can be derived with the use of the retarded Green's function. They are

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{-j\beta_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \, dV' \tag{1.12a}$$

and

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \frac{e^{-j\beta_0|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \, dV', \tag{1.12b}$$

where the volume integrations extend over the entire region occupied by currents or charges. In most cases considered in this book, the conductors are perfect so that only surface current densities $\mathbf{K}$ and surface charge densities η are present. In such cases, the volume integrals in (1.12) reduce to surface integrals. In the limit of infinitely thin wire antennas, the surface integrals in turn reduce to line integrals.

1.3 Power and the Poynting vector

The complex Poynting vector is defined as

$$\mathbf{S} = \frac{1}{2\mu_0} \mathbf{E} \times \mathbf{B}^*, \tag{1.13}$$

where the asterisk denotes the complex conjugate. The integral of the normal component of $\mathrm{Re}\{\mathbf{S}\}$ over a closed surface Σ is the time-average, total power transferred from within Σ. The time average is over a period $T = 2\pi/\omega$. Several useful identities involving the Poynting vector are now derived. The geometry of interest is shown in Fig. 1.2. A perfect conductor surrounded by air is shown. The conductor–air interface is the closed surface Σ_0, and $\hat{\mathbf{n}}_0$ is the unit outward normal. Assume that there is an impressed electric field $\mathbf{E}^e$ tangent to the surface of the conductor. As a result, a surface current density $\mathbf{K}$ exists on the conductor's surface. This, in turn, maintains an electromagnetic field $\mathbf{E}$ and $\mathbf{B}$ in the air. The total electric field on the conductor's surface is $\mathbf{E}+\mathbf{E}^e$, and the boundary conditions on the surface of the perfect conductor are

$$\hat{\mathbf{n}}_0 \times (\mathbf{E}+\mathbf{E}^e) = 0, \qquad \hat{\mathbf{n}}_0 \times \mathbf{B} = \mu_0 \mathbf{K}. \tag{1.14}$$

Suppose that Σ_1 is a closed (mathematical) surface in the air surrounding the perfect conductor, and that $\hat{\mathbf{n}}_1$ is the corresponding unit normal vector. Let τ_{01} be the volume lying between Σ_0 and Σ_1, and consider the quantity

$$\int_{\tau_{01}} \nabla \cdot \mathbf{S}\, dV. \tag{1.15}$$

First, with (1.13), the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}^*) = \mathbf{B}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}^*) \tag{1.16}$$

and the Maxwell equations on the left in (1.3a, b), it is seen that

$$\int_{\tau_{01}} \nabla \cdot \mathbf{S}\, dV = -j\omega \int_{\tau_{01}} \left(\tfrac{1}{2}\mu_0^{-1}|\mathbf{B}|^2 - \tfrac{1}{2}\epsilon_0|\mathbf{E}|^2\right) dV. \tag{1.17}$$

The boundaries of the volume τ_{01} are the surfaces Σ_0 and Σ_1. Application of the divergence theorem to the quantity in (1.15) yields

$$\int_{\tau_{01}} \nabla \cdot \mathbf{S}\, dV = -\int_{\Sigma_0} (\hat{\mathbf{n}}_0 \cdot \mathbf{S})\, d\Sigma + \int_{\Sigma_1} (\hat{\mathbf{n}}_1 \cdot \mathbf{S})\, d\Sigma. \tag{1.18}$$

A comparison of (1.17) and (1.18) yields the identity

$$\int_{\Sigma_1} (\hat{\mathbf{n}}_1 \cdot \mathbf{S})\, d\Sigma = \int_{\Sigma_0} (\hat{\mathbf{n}}_0 \cdot \mathbf{S})\, d\Sigma - j\omega \int_{\tau_{01}} \left(\tfrac{1}{2}\mu_0^{-1}|\mathbf{B}|^2 - \tfrac{1}{2}\epsilon_0|\mathbf{E}|^2\right) dV. \tag{1.19a}$$

If one takes the real part of this equation, no volume integral appears:

$$P \equiv \int_{\Sigma_0} (\hat{\mathbf{n}}_0 \cdot \mathrm{Re}\{\mathbf{S}\})\, d\Sigma = \int_{\Sigma_1} (\hat{\mathbf{n}}_1 \cdot \mathrm{Re}\{\mathbf{S}\})\, d\Sigma. \tag{1.19b}$$

Equation (1.19b) states that P, the total time-average power entering Σ_0, is the same as the total time-average power leaving Σ_1.

The next identity of interest is obtained by expressing $\int_{\Sigma_0} (\hat{\mathbf{n}}_0 \cdot \mathbf{S})\, d\Sigma$ in (1.19a) in terms of $\mathbf{E}^e$ and $\mathbf{K}$. With (1.13), the vector identity $\hat{\mathbf{n}}_0 \cdot (\mathbf{E} \times \mathbf{B}^*) = -\mathbf{E} \cdot (\hat{\mathbf{n}}_0 \times \mathbf{B}^*)$, and the boundary conditions (1.14), it is seen that $\int_{\Sigma_0}(\hat{\mathbf{n}}_0 \cdot \mathbf{S})\, d\Sigma = \int_{\Sigma_0} \frac{1}{2}\mathbf{E}^e \cdot \mathbf{K}^*\, d\Sigma$ so that (1.19a) can be written as

$$\int_{\Sigma_0} \tfrac{1}{2}\mathbf{E}^e \cdot \mathbf{K}^*\, d\Sigma = j\omega \int_{\tau_{01}} \left(\tfrac{1}{2}\mu_0^{-1}|\mathbf{B}|^2 - \tfrac{1}{2}\epsilon_0|\mathbf{E}|^2\right) dV + \int_{\Sigma_1} (\hat{\mathbf{n}}_1 \cdot \mathbf{S})\, d\Sigma. \qquad (1.20a)$$

The real part of this expression is

$$P \equiv \int_{\Sigma_0} \mathrm{Re}\{\tfrac{1}{2}\mathbf{E}^e \cdot \mathbf{K}^*\}\, d\Sigma = \int_{\Sigma_1} (\hat{\mathbf{n}}_1 \cdot \mathrm{Re}\{\mathbf{S}\})\, d\Sigma. \qquad (1.20b)$$

In (1.20), Σ_1 is any surface completely surrounding the air–conductor interface Σ_0. Equations (1.20a, b) can be extended to surfaces Σ_1 that pass through the surface of the perfect conductor, provided that $\mathbf{E}^e = 0$ on any part of Σ_0 excluded by Σ_1. This follows from the boundary condition $\hat{\mathbf{n}}_0 \times \mathbf{E} = 0$ on the part of Σ_0 excluded by Σ_1 and the fact that all fields are zero within the volume occupied by the perfect conductor.

Equation (1.20b) states that the time-average power transferred to the perfect conductor from the "generator" (i.e. the impressed electric field $\mathbf{E}^e$) is all radiated into free space. Equations (1.20a, b) possess analogues for the case of imperfect conductors; these involve a volume integral instead of a surface integral, and include a term due to the ohmic losses in the conductors. It is important to note that in both (1.19) and (1.20), only integrations of $\hat{\mathbf{n}} \cdot \mathbf{S}$ over closed surfaces appear; it is not mathematically justified to attach meaning to an integral of $\hat{\mathbf{n}} \cdot \mathbf{S}$ over only a part of a closed surface.

Consider the limiting case of an infinitely thin, perfectly conducting wire lying on the z-axis between $-h$ and h. The impressed electric field is $E_z^e(z)$, and the current on the wire is $I(z)$. In this limit, (1.20b) reduces to

$$P \equiv \int_{-h}^{h} \mathrm{Re}\{\tfrac{1}{2}E_z^e(z)I^*(z)\}\, dz = \int_{\Sigma_1} (\hat{\mathbf{n}}_1 \cdot \mathrm{Re}\{\mathbf{S}\})\, d\Sigma. \qquad (1.20c)$$

1.4 The field of thin linear antennas: general equations

Now consider the linear antenna of Fig. 1.1c and assume that $a \ll h$ and $\beta_0 a \ll 1$. Both cylindrical coordinates ρ, Φ, z and spherical coordinates r, Θ, Φ are to be used throughout this book. Rotational symmetry obtains, so that all cylindrical or spherical field components are independent of Φ. There is a surface current density $K_z(z)$ on the cylindrical surface $\rho = a$, and also a current on the small capped ends of the antenna. The latter currents can be neglected when calculating the field of the antenna. The total current $I(z)$ and the charge per unit length $q(z)$ are defined to be

$$I(z) = 2\pi a K_z(z), \qquad q(z) = 2\pi a \eta(z). \qquad (1.21)$$

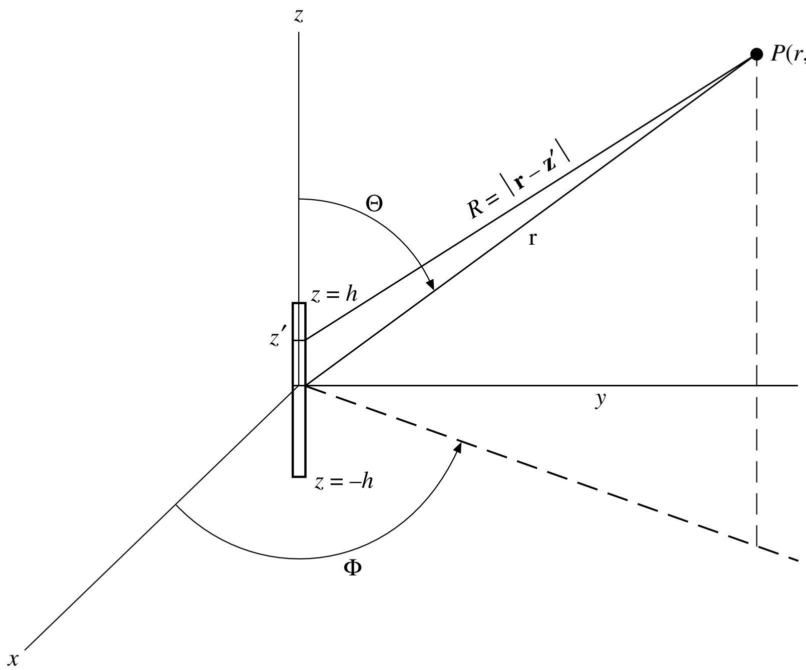

Figure 1.3 Coordinate system for calculations in the far zone.

They are related by the one-dimensional equation of continuity

$$\frac{dI(z)}{dz} = -j\omega q(z). \tag{1.22}$$

$I(z)$ is even with respect to z and $q(z)$ is odd.

When calculating the field of the antenna, one can assume that the current is located at the axis $z = 0$, which is the same as replacing the antenna of radius a by an infinitely thin antenna. With this assumption, but without reference to a particular current distribution $I(z)$, formulas for calculating the field are given in this section and some general characteristics of the field are discussed. The coordinate system is shown in Fig. 1.3.

It is seen from (1.12a) that $\mathbf{A} = \hat{\mathbf{z}}A_z(\rho, z)$. Equations (1.12a, b) reduce to

$$A_z = \frac{\mu_0}{4\pi}\int_{-h}^{h} I(z')\,\frac{e^{-j\beta_0 R}}{R}\,dz' \tag{1.23a}$$

and

$$\phi = \frac{1}{4\pi\epsilon_0}\int_{-h}^{h} q(z')\,\frac{e^{-j\beta_0 R}}{R}\,dz', \tag{1.23b}$$

where $R = |\mathbf{r} - \hat{\mathbf{z}}z'|$ is the distance from a point z' on the infinitely thin antenna to the observation point **r**. The one-dimensional Lorentz condition is

$$\frac{\partial A_z}{\partial z} = -j\,\frac{\beta_0^2}{\omega}\,\phi. \tag{1.23c}$$

The **E** and **B** fields are obtained from (1.6) and (1.8) with (1.23a) and (1.23c). In the cylindrical coordinates ρ, Φ, z, they are $\mathbf{B} = \hat{\mathbf{\Phi}} B_\Phi$ and $\mathbf{E} = \hat{\boldsymbol{\rho}} E_\rho + \hat{\mathbf{z}} E_z$, where

$$B_\Phi = \frac{-\partial A_z}{\partial \rho} \tag{1.24a}$$

$$E_\rho = \frac{-j\omega}{\beta_0^2}\,\frac{\partial^2 A_z}{\partial \rho\,\partial z} \tag{1.24b}$$

$$E_z = \frac{-j\omega}{\beta_0^2}\left(\frac{\partial^2 A_z}{\partial z^2} + \beta_0^2 A_z\right). \tag{1.24c}$$

In the spherical coordinates r, Θ, Φ with origin at the center of the antenna, the electric field is given by

$$E_r = E_z \cos\Theta + E_\rho \sin\Theta \tag{1.25a}$$

$$E_\Theta = -E_z \sin\Theta + E_\rho \cos\Theta. \tag{1.25b}$$

At sufficiently great distances from the antenna ($r^2 \gg h^2$ and $(\beta_0 r)^2 \gg 1$), the field reduces to a simple form known as the radiation or far field. It is given by

$$B_\Phi^r = E_\Theta^r / c, \tag{1.26a}$$

where

$$\mathbf{E}^r \doteq E_\Theta^r \hat{\mathbf{\Theta}}, \quad E_\Theta^r = \frac{j\omega\mu_0}{4\pi}\sin\Theta \int_{-h}^{h} I(z')\,\frac{e^{-j\beta_0 R}}{R}\,dz'. \tag{1.26b}$$

The distance R from an arbitrary point on the antenna to the field point is given in terms of r and z' by the cosine law, namely (Fig. 1.3),

$$R = \sqrt{r^2 + z'^2 - 2rz'\cos\Theta}. \tag{1.27a}$$

In the radiation zone, $r^2 \gg z'^2$. If the binomial expansion is applied to (1.27a) and only the linear term in z' is retained, the following approximate form is obtained for R:

$$R \doteq r - z'\cos\Theta, \quad (\beta_0 r)^2 \gg 1. \tag{1.27b}$$

The phase variation of $\exp(-j\beta_0 R)/R$ is replaced with the linear phase variation given by (1.27b), i.e. by $\exp(-j\beta_0 r + j\beta_0 z'\cos\Theta)$. The amplitude $1/R$ of $\exp(-j\beta_0 R)/R$ is a slowly varying function of z' and is replaced by $1/r$, where r is the distance to the center of the antenna. With these approximations, (1.26b) can be written as

$$E_\Theta^r = \frac{j\zeta_0 I(0)}{2\pi}\,\frac{e^{-j\beta_0 r}}{r}\,F_0(\Theta, \beta_0 h), \tag{1.28a}$$

where $\zeta_0 = \sqrt{\mu_0/\epsilon_0} \doteq 120\pi$ ohms and

$$F_0(\Theta, \beta_0 h) = \frac{\beta_0 \sin\Theta}{2I(0)} \int_{-h}^{h} I(z')e^{j\beta_0 z' \cos\Theta}\, dz'. \tag{1.28b}$$

The term $F_0(\Theta, \beta_0 h)$ contains all the directional properties of a linear radiator of length $2h$. It is called the field characteristic, field factor, or element factor, and will be computed for some commonly used current distributions. The magnetic field $\mathbf{B}^r$ in the far zone is at right angles to $\mathbf{E}^r$ and also perpendicular to the direction of propagation $\mathbf{r}$. It is given by (1.26a). Thus

$$\mathbf{B}^r = \hat{\mathbf{\Phi}} B^r_\Phi, \quad B^r_\Phi = \frac{j\mu_0 I(0)}{2\pi} \frac{e^{-j\beta_0 r}}{r} F_0(\Theta, \beta_0 h). \tag{1.28c}$$

Note that the field in the far zone depends on $F_0(\Theta, \beta_0 h)$ which is a function of the particular distribution of current in the antenna.

It is instructive to consider the instantaneous value of the field in (1.28a), which is obtained by multiplication with $e^{j\omega t}$ and selection of the real part. Except for a phase factor,

$$E^r_\Theta(\mathbf{r}, t) = \mathrm{Re}\, E_\Theta(\mathbf{r})e^{j\omega t} \sim \frac{\sin(\omega t - \beta_0 r)}{r} = \frac{\sin\omega(t - r/c)}{r}. \tag{1.29a}$$

Note that the field at the point r at the instant t is computed from the current at $r = 0$ at the earlier time $(t - r/c)$. This is a consequence of the finite velocity of propagation c.

The equiphase and equipotential surfaces of $\mathbf{E}$ and $\mathbf{B}$ are spherical shells on which r is equal to a constant. There are an infinite number of such shells that have the same phase (differ by an integral multiple of 2π) but only one that has both the same amplitude and the same phase. The velocity of propagation is the outward radial velocity of the surfaces of constant phase where the phase is represented by the argument of the sine term in (1.29a), that is

$$\text{phase} = \Psi = \omega t - \beta_0 r. \tag{1.29b}$$

For a constant phase

$$\frac{d\Psi}{dt} = 0 = \omega - \frac{\beta_0\, dr}{dt}. \tag{1.29c}$$

It follows that

$$\frac{dr}{dt} = \frac{\omega}{\beta_0} = c = 3 \times 10^8 \text{ m/s}. \tag{1.29d}$$

Since the phase repeats itself every 2π radians, a wavelength is the distance between two adjacent equiphase surfaces. For example, if one surface is defined by $r = r_1$ and the other by $r = r_2$, then

$$\omega t - \beta_0 r_1 = 2\pi \quad \text{and} \quad \omega t - \beta_0 r_2 = 4\pi \tag{1.30a}$$