Lecture 16

Mathematical Induction

The **Mathematical Induction** is a very powerful tool for proving infinitely many propositions by using only a few steps. In particular, it can be used often when the propositions are involving the nonnegative integer n.

Theorem I. (**The First Induction**) *Let $\{P_n\}, n \geq m$ be a family of infinitely many propositions (or statements) involving the integer n, where m is a nonnegative integer. If (i) P_m can be proved to be true, and (ii) P_{k+1} must be true provided P_k is true for any $k \geq m$, then P_n is true for all $n \geq m$.*

Theorem II. (**The Second Induction**) *Let $\{P_n\}, n \geq m$ be a family of infinitely many propositions (or statements) involving the integer n, where m is a nonnegative integer. If (i) P_m can be proved to be true, and (ii) P_{k+1} must be true provided $P_m, P_{m+1}, \ldots, P_k$ are all true for any $k \geq m$, then P_n is true for all $n \geq m$.*

Note that it seems to be more difficult for using the second induction, since it require more conditions. However it is actually not so. We can use either one freely if necessary.

Variations of Mathematical Induction

(i) For proving each P_n, $n = 1, 2, \ldots$, we can prove $P_{n_k}, k = 1, 2, \ldots$ first by induction, and then prove P_n for each n.

(ii) For proving $\{P_n, n \in \mathbb{N}\}$ by induction, sometimes it is convenient to prove a stronger proposition P'_n.

(iii) For proving $\{P_n, n \geq n_0\}$ by induction, sometimes it is convenient to prove P_n by induction starting from $n = n_1 > n_0$.

(iv) The induction can be conducted backwards: (i) Prove each proposition in the subsequence $\{P_{n_k}, k \in \mathbb{N}\}$ first, and then (ii) prove that P_k is true provided P_{k+1} is true.

(v) Cross-Active (or Spiral) Induction: For proving propositions $P_n, n = 0, 1, 2, \ldots$ and $Q_n, n = 0, 1, 2, \ldots$, it suffices to show that (i) P_0 and Q_0 are true; (ii) If P_k, Q_k are true, then $P_k \Rightarrow Q_{k+1} \Rightarrow P_{k+1} \Rightarrow P_{k+1}$.

(vi) Multiple Induction: Let $\{P_{n,m}, n, m \in \mathbb{N}\}$ be a sequence of propositions with two independent parameters n and m. If (i) $P_{1,m}$ is true for all $m \in \mathbb{N}$ and $P_{n,1}$ are true for all $n \in \mathbb{N}$; (ii) $P_{n+1,m+1}$ is true provided $P_{n,m+1}$ and $P_{n+1,m}$ are true, then $P_{n,m}$ is true for any $n, m \in \mathbb{N}$.

Examples

Example 1. Prove by induction that $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for each $n \in \mathbb{N}$. Hence prove that $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$ for each $n \in \mathbb{N}$.

Solution Let P_n be the statement

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}, \qquad n = 1, 2, \ldots.$$

For $n = 1$, left hand side $= 1 = \dfrac{1^2(1+1)^2}{4} =$ right hand side, P_1 is proven.

Assume that P_k is true ($k \geq 1$), i.e. $1^3 + 2^3 + \cdots + k^3 = \dfrac{k^2(k+1)^2}{4}$, then for $n = k + 1$,

$$1^3 + 2^3 + \cdots + (k+1)^3 = 1^3 + 2^3 + \cdots + k^3 + (k+1)^3$$
$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2}{4} \cdot [k^2 + 4(k+1)] = \frac{(k+1)^2(k+2)^2}{4}.$$

Thus, P_{k+1} is also true. By induction, P_n is true for each $n \in \mathbb{N}$.

Now for any positive integer n,

$$1^3 + 3^3 + \cdots + (2n-1)^3 = 1^3 + 2^3 + \cdots + (2n)^3 - (2^3 + 4^3 + \cdots + (2n)^3)$$
$$= \left[\frac{(2n)(2n+1)}{2}\right]^2 - 2^3(1^3 + 2^3 + \cdots + n^3) = n^2(2n+1)^2 - 8 \cdot \frac{n^2(n+1)^2}{4}$$
$$= n^2(2n+1)^2 - 2n^2(n+1)^2 = n^2[(2n+1)^2 - 2(n+1)^2] = n^2(2n^2 - 1),$$

as desired.

Example 2. (CWMO/2008) $\{a_n\}$ is a sequence of real numbers, satisfying $a_0 \neq 0, 1$, $a_1 = 1 - a_0$, $a_{n+1} = 1 - a_n(1 - a_n)$ for all $n \in \mathbb{N}$. Prove that for any positive integer n,

$$a_0a_1\cdots a_n\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right)=1.$$

Solution The given condition yields $1-a_{n+1}=a_n(1-a_n), n\in\mathbb{N}_0$, hence

$$1-a_{n+1}=a_na_{n-1}(1-a_{n-1})=\cdots=a_n\cdots a_1(1-a_1)=a_n\cdots a_1a_0,$$

namely

$$a_{n+1}=1-a_0a_1\cdots a_n,\qquad n=1,2,\ldots.$$

We now prove the original conclusion by induction.

For $n=1$, $a_0a_1\left(\frac{1}{a_0}+\frac{1}{a_1}\right)=a_0+a_1=1$, the conclusion is proven.

Assume that the proposition is true for $n=k$, then for $n=k+1$,

$$\begin{aligned}&a_0a_1\cdots a_{k+1}\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_{k+1}}\right)\\&=a_0a_1\cdots a_k\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_k}\right)a_{k+1}+a_0a_1\cdots a_k\\&=a_{k+1}+a_0a_1\cdots a_k=1.\end{aligned}$$

Thus, the proposition is true for $n=k+1$ also. By induction, the conclusion is true for all $n\in\mathbb{N}$.

Example 3. (IMO/Shortlist/2006) The sequence of real numbers $a_0,a_1,a_2,\ldots$ is defined recursively by

$$a_0=-1,\quad \sum_{k=0}^{n}\frac{a_{n-k}}{k+1}=0\ \text{ for } n\geq 1.$$

Show that $a_n>0$ for $n\geq 1$.

Solution The proof goes by induction on n. For $n=1$ the formula yields $a_1=1/2$. Take $n\geq 1$ and assume $a_1,\ldots,a_n>0$. Write the recurrence formula for n and $n+1$ respectively as

$$\sum_{k=0}^{n}\frac{a_k}{n-k+1}=0\qquad\text{and}\qquad\sum_{k=0}^{n+1}\frac{a_k}{n-k+2}=0.$$

Subtraction yields

$$\begin{aligned}0&=(n+2)\sum_{k=0}^{n+1}\frac{a_k}{n-k+2}-(n+1)\sum_{k=0}^{n}\frac{a_k}{n-k+1}\\&=(n+2)a_{n+1}+\sum_{k=0}^{n}\left(\frac{n+2}{n-k+2}-\frac{n+1}{n-k+1}\right)a_k.\end{aligned}$$

The coefficient of a_0 vanishes, so

$$\begin{aligned} a_{n+1} &= -\frac{1}{n+2}\sum_{k=1}^{n}\left(\frac{n+2}{n-k+2}-\frac{n+1}{n-k+1}\right)a_k \\ &= \frac{1}{n+2}\sum_{k=1}^{n}\frac{k}{(n-k+1)(n-k+2)}a_k. \end{aligned}$$

The coefficients of $a_1, \ldots, a_n$ are all positive. Therefore, $a_1, \ldots, a_n > 0$ implies $a_{n+1} > 0$.

Example 4. (CHNMO/2010) (edited) If $0 < x_1 \le x_2 \le \ldots \le x_s, 0 < y_1 \le y_2 \le \ldots \le y_s$ satisfy the condition that

$$x_1^r + x_2^r + \cdots + x_s^r = y_1 + y_2^r + \cdots + y_s^r, \quad \text{for all } r \in \mathbb{N},$$

prove that $x_i = y_i, i = 1, 2, \ldots, s$.

Solution We prove by induction on s. Let $P_s, s = 1, 2, \ldots$ be the statement to be proven.

For $s = 1$, taking $r = 1$ yields $x_1 = y_1$. Assume that P_s is true for $s = k$ $(k \ge 1)$ i.e.,

$$x_1 = y_1, x_2 = y_2, \ldots, x_k = y_k.$$

then for $s = k + 1$, if $x_{k+1} \ne y_{k+1}$, without loss of generality we assume that $x_{k+1} < y_{k+1}$. Then

$$\left(\frac{x_1}{y_{k+1}}\right)^r + \cdots + \left(\frac{x_{k+1}}{y_{k+1}}\right)^r = \left(\frac{y_1}{y_{k+1}}\right)^r + \cdots + \left(\frac{y_k}{y_{k+1}}\right)^r + 1 \ge 1.$$

But on the other hand, since $0 < \dfrac{x_i}{y_{k+1}} < 1, i = 1, 2, \ldots, k+1$, letting $r \to +\infty$ yields $0 \ge 1$, a contradiction. Hence $x_{k+1} = y_{k+1}$, then the inductive assumption implies that $x_i = y_i, i = 1, 2, \ldots, k$. Thus, P_{k+1} is also true. By induction, P_s is true for each $s \in \mathbb{N}$.

Example 5. (**AM-GM Inequality**) For real numbers $a_1, a_2, \ldots, a_n > 0$, prove that

$$a_1 + a_2 + \cdots + a_n \ge n \cdot \sqrt[n]{a_1 a_2 \cdots a_n}, \tag{$*$}$$

and that equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Solution We first prove $(*)$ for $n = 2^t, t \in \mathbb{N}$ by induction on t.

When $t = 1$, $(*) \Leftrightarrow (\sqrt{a_1} - \sqrt{a_2})^2 \ge 0$, so it's true.

Assume that $(*)$ is true for $t = k$, then for $t = k + 1$, by the induction assumption,

$$\begin{aligned}\sum_{j=1}^{2^{k+1}} a_j &= \sum_{j=1}^{2^k} a_j + \sum_{j=1}^{2^k} a_{j+2^k} \geq 2^k \sqrt[2^k]{\prod_{j=1}^{2^k} a_j} + 2^k \cdot \sqrt[2^k]{\prod_{j=1}^{2^k} a_{j+2^k}} \\ &\geq 2^k \cdot 2 \cdot \sqrt[2^{k+1}]{\prod_{j=1}^{2^{k+1}} a_j} = 2^{k+1} \cdot \sqrt[2^{k+1}]{\prod_{j=1}^{2^{k+1}} a_j}.\end{aligned}$$

Thus, the proposition is proven for $t = k + 1$. By induction, $(*)$ is proven for all $n = 2^t, t = 1, 2, \ldots$.

Next we prove $(*)$ for each $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ which is not of the form 2^t, there exist an $s \in \mathbb{N}$ such that $2^{s-1} < n < 2^s$. Since

$$a_1+a_2+\cdots+a_{2^s} \geq 2^s \cdot \sqrt[2^s]{a_1 a_2 \cdots a_{2^s}}, \tag{$**$}$$

By letting $a_{n+1} = a_{n+2} = \cdots = a_{2^s} = \sqrt[n]{a_1 a_2 \cdots a_n}$ in $(**)$, $(**)$ yields

$$\begin{aligned}&a_1 + a_2 + \cdots + a_n + (2^s - n) \sqrt[n]{a_1 a_2 \cdots a_n} \\ &\geq 2^s \cdot \sqrt[2^s]{(a_1 a_2 \cdots a_n)(a_1 a_2 \cdots a_n)^{\frac{2^s - n}{n}}} = 2^s \cdot \sqrt[n]{a_1 a_2 \cdots a_n}, \\ &\therefore a_1 + a_2 + \cdots + a_n \geq n \cdot \sqrt[n]{a_1 a_2 \cdots a_n},\end{aligned}$$

hence $(*)$ is proven for general $n \in \mathbb{N}$.

Example 6. (CMC/2007) For each positive integer $n > 1$, prove that

$$\frac{2n}{3n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{25}{36}.$$

Solution To prove the left inequality, define

$$f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} - \frac{2n}{3n+1}, \quad n \in \mathbb{N}.$$

Then $f(n+1) = \dfrac{1}{n+2} + \dfrac{1}{n+3} + \cdots + \dfrac{1}{2n+2} - \dfrac{2n+2}{3n+4}$, so

$$\begin{aligned}f(n+1) - f(n) &= \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) + \left(\frac{2n}{3n+1} - \frac{2n+2}{3n+4}\right) \\ &= \frac{1}{(2n+1)(2n+2)} - \frac{2}{(3n+1)(3n+4)} \\ &= \frac{n(n+3)}{(2n+1)(2n+2)(3n+1)(3n+4)} > 0,\end{aligned}$$

i.e., $f(n)$ is an increasing function of n. Thus, for $n \geq 2$,

$$f(n) \geq f(2) = \frac{1}{3} + \frac{1}{4} - \frac{4}{7} = \frac{1}{84} > 0,$$

the left inequality is proven.

Below we prove the right inequality by proving a stronger one by induction. Let the proposition $P_n, n \geq 2$ be

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} < \frac{25}{36} - \frac{1}{4n+1}.$$

When $n = 2$, the left hand side of P_2 is $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$, and the right hand side is $\frac{25}{36} - \frac{1}{9} = \frac{7}{12}$, so P_2 is true. Assume P_n is true for $n = k$ ($k \geq 2$), i.e.,

$$\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{k+k} < \frac{25}{36} - \frac{1}{4k+1}.$$

Then for $n = k + 1$, the left hand side of P_{k+1} is

$$\begin{aligned}
\frac{1}{k+2} + \cdots + \frac{1}{2k+2} &= \frac{1}{k+1} + \cdots + \frac{1}{2k+1} - \frac{1}{2k+2} \\
&\leq \frac{25}{36} - \frac{1}{4k+1} + \frac{1}{2k+1} - \frac{1}{2k+2} \\
&= \frac{25}{36} - \frac{1}{4k+5} + \frac{1}{4k+5} - \frac{1}{4k+1} + \frac{1}{2k+1} - \frac{1}{2k+2}.
\end{aligned}$$

Since $\frac{1}{4k+5} - \frac{1}{4k+1} + \frac{1}{2k+1} - \frac{1}{2k+2} = -\frac{3}{(2k+1)(2k+2)(4k+1)(4k+5)} < 0$, therefore $\frac{1}{k+2} + \cdots + \frac{1}{2k+2} < \frac{25}{36} - \frac{1}{4k+5}$.

Thus, P_{k+1} is true also. The inductive proof is completed.

Example 7. Suppose that $a_1, a_2, \cdots, a_n, \cdots$ are real numbers with $0 \leq a_i \leq 1$ for $i = 1, 2, \cdots$. Prove that for every $n \geq 2$,

$$\sum_{i=1}^{n} \frac{1}{1+a_i} \leq \frac{n}{1 + \sqrt[n]{a_1 a_2 \cdots a_n}}. \qquad (*)$$

Solution As the first step, we prove the conclusion for all n with the form 2^m by induction on m. For $m = 1$, the inequality $(*)$ becomes

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} \leq \frac{2}{1+\sqrt{a_1 a_2}}$$

or equivalently:

$$\begin{gathered}2(1+a_1)(1+a_2) \geq (2+a_1+a_2)(\sqrt{a_1a_2}+1),\\ a_1+a_2+2a_1a_2 \geq (2+a_1+a_2)\sqrt{a_1a_2},\\ (a_1+a_2)(1-\sqrt{a_1a_2})-2\sqrt{a_1a_2}(1-\sqrt{a_1a_2}) \geq 0,\\ (\sqrt{a_1}-\sqrt{a_2})^2(1-\sqrt{a_1a_2}) \geq 0.\end{gathered}$$

Therefore the conclusion is true. Assume that for $m = k$ $(k \geq 1)$ the conclusion is true, i.e.

$$\sum_{i=1}^{2^k} \frac{1}{1+a_i} \leq \frac{2^k}{1+\sqrt[2^k]{a_1a_2\cdots a_{2^k}}}.$$

Then, for $m = k+1$

$$\begin{aligned}\sum_{i=1}^{2^{k+1}} \frac{1}{1+a_i} &= \sum_{i=1}^{2^k}\left(\frac{1}{1+a_{2i-1}}+\frac{1}{1+a_{2i}}\right) \leq 2\sum_{i=1}^{2^k}\frac{1}{1+\sqrt{a_{2i-1}a_{2i}}}\\ &\leq 2\cdot\frac{2^k}{1+\sqrt[2^k]{\sqrt{a_1a_2}\sqrt{a_3a_4}\cdots\sqrt{a_{2^{k+1}-1}a_{2^{k+1}}}}}\\ &= \frac{2^{k+1}}{1+\sqrt[2^{k+1}]{a_1a_2\cdots a_{2^{k+1}}}}.\end{aligned}$$

Therefore the conclusion is true for all $n = 2^m,\ m \in \mathbb{N}$.

Next, to show that $(*)$ is true for all $n \in \mathbb{N}$, it suffices to show that $(*)$ is true for $n = k-1$ $(k \geq 2)$ if $(*)$ is true for $n = k$. In fact, by letting $a_k = \sqrt[k-1]{a_1a_2\cdots a_{k-1}}$, we have

$$\begin{aligned}&\sum_{i=1}^{k-1}\frac{1}{1+a_i}+\frac{1}{1+\sqrt[k-1]{a_1a_2\cdots a_{k-1}}}\\ &\leq \frac{k}{1+\sqrt[k]{a_1a_2\cdots a_{k-1}\cdot\sqrt[k-1]{a_1a_2\cdots a_{k-1}}}} = \frac{k}{1+\sqrt[k-1]{a_1a_2\cdots a_{k-1}}},\end{aligned}$$

therefore we have

$$\sum_{i=1}^{k-1}\frac{1}{1+a_i} \leq \frac{k-1}{1+\sqrt[k-1]{a_1a_2\cdots a_{k-1}}}.$$

By backward induction, $(*)$ is true for each $n \in \mathbb{N}$.

Example 8. Let $\{F_n\}$, $n \in \mathbb{N}$ be the Fibonacci sequence. Prove that $F_{n+1}^2+F_n^2 = F_{2n+1}$ for each $n \in \mathbb{N}$.

Solution Let $P_n : F_{n+1}^2 + F_n^2 = F_{2n+1}$, $Q_n : 2F_{n+1}F_n + F_{n+1}^2 = F_{2n+2}$ for $n = 1, 2, \ldots$.

When $n = 1$, then P_1, Q_1 are obvious since $F_1 = F_2 = 1$.

When P_k, Q_k are true for $n = k$ $(k \geq 1)$, i.e., $F_{k+1}^2 + F_k^2 = 2F_{2k+1}$ and $2F_{k+1}F_k + F_{k+1}^2 = F_{2k+2}$, then

$$\begin{aligned} F_{k+2}^2 + F_{k+1}^2 &= (F_{k+1} + F_k)^2 + F_{k+1}^2 \\ &= (F_{k+1}^2 + 2F_{k+1}F_k) + (F_k^2 + F_{k+1}^2) \\ &= F_{2k+2} + F_{2k+1} = F_{2k+3} = F_{2(k+1)+1}, \end{aligned}$$

i.e., P_{k+1} is true. Further, P_{k+1} and Q_k are true yields

$$\begin{aligned} 2F_{k+2}F_{k+1} + F_{k+2}^2 &= 2(F_{k+1} + F_k)F_{k+1} + F_{k+2}^2 \\ &= (F_{k+1}^2 + F_{k+2}^2) + (F_{k+1}^2 + 2F_{k+1}F_k) \\ &= F_{2k+3} + F_{2k+2} = F_{2k+4} = F_{2(k+1)+2}, \end{aligned}$$

namely Q_{k+1} is true. By the cross-active induction, P_n, Q_n are true for each $n \in \mathbb{N}$.

Example 9. m boxes are arranged in a row. Prove that there are $\dfrac{(n+m-1)!}{n!(m-1)!}$ different ways to put n identical balls into these boxes.

Solution By $F(n, m)$ we denote the number of ways to put n identical balls into m boxes. The propositions to be proven are

$$F(n, m) = \frac{(n+m-1)!}{n!(m-1)!}$$

for $n, m \in \mathbb{N}$.

For $n = 1$, we have $F(1, m) = m = \dfrac{m!}{(m-1)!}$, hence the conclusion is true for $n = 1$ and all $m \in \mathbb{N}$. For $m = 1$ and any $n \in \mathbb{N}$, we have $F(n, 1) = 1 = \dfrac{n!}{n!}$, therefore the conclusion is true for $m = 1$ and all $n \in \mathbb{N}$.

Suppose that $F(n+1, m) = \dfrac{(n+m)!}{(n+1)!(m-1)!}$ and $F(n, m+1) = \dfrac{(n+m)!}{n!m!}$. To evaluate $F(n+1, m+1)$, by considering whether the number of balls in the $m+1$st box is 0 or greater than 0, we have

$$\begin{aligned} &F(n+1, m+1) \\ &= F(n+1, m) + F(n, m+1) = \frac{(n+m)!}{(n+1)!(m-1)!} + \frac{(n+m)!}{n!m!} \\ &= \frac{(n+m)!(m+n+1)}{(n+1)!m!} = \frac{(m+n+1)!}{(n+1)!(m!)} = \frac{[(n+1)+(m+1)-1]!}{(n+1)!(m+1-1)!}. \end{aligned}$$

The inductive proof is completed.

Testing Questions (A)

1. (USAMO/2002) Let S be a set with 2002 elements, and let N be an integer with $0 \le N \le 2^{2002}$. Prove that it is possible to color every subset of S either blue or red so that the following conditions hold:

   (a) the union of any two red subsets is red;

   (b) the union of any two blue subsets is blue;

   (c) there are exactly N red subsets.

2. (CHNMO/TST/2005) Let k be a positive integer. Prove that it is always possible to partition the set $\{0, 1, 2, 3, \ldots, 2^{k+1} - 1\}$ into two disjoint subsets $\{x_1, x_2, \ldots, x_{2^k}\}$ and $\{y_1, y_2, \ldots, y_{2^k}\}$ such that $\sum_{i=1}^{2^k} x_i^m = \sum_{i=1}^{2^k} y_i^m$ holds for each $m \in \{1, 2, \ldots, k\}$.

3. (CMC/2006) Let $f(x) = x^2 + a$. Define $f^1(x) = f(x)$, $f^n(x) = f(f^{n-1}(x))$, $n = 2, 3, \ldots$ and $M = \{a \in \mathbb{R} \mid |f^n(0)| \le 2 \text{ for all } n \in \mathbb{N}\}$. Prove that
$$M = \left[-2, \frac{1}{4}\right].$$

4. (BELARUS/2007) For positive real numbers $x_1, x_2, \ldots, x_{n+1}$, prove that
$$\frac{1}{x_1} + \frac{x_1}{x_2} + \frac{x_1 x_2}{x_3} + \cdots + \frac{x_1 x_2 \cdots x_n}{x_{n+1}} \ge 4(1 - x_1 x_2 \cdots x_{n+1}).$$

5. (CMC/2010) Given the sequence $\{a_n\}$, $a_1 = 2$ and $a_{n+1} = a_1 a_2 \cdots a_n + 1$ for $n = 1, 2, \ldots$. Prove that
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \ge \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} \quad n \in \mathbb{N}.$$

6. (AUSTRIA/2009) Prove that $3^{n^2} > (n!)^4$ for all positive integers n.

7. (CHNMO/TST/2008) Let a and b be positive integers with $(a, b) = 1$, and a, b having different parities. Let the set S have the following properties: (1) $a, b \in S$; (ii) If $x, y, z \in S$, then $x + y + z \in S$. Prove that all the integers greater than $2ab$ are in S.

8. (USAMO/2007) Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

Testing Questions (B)

1. (CHNMO/TST/2010) It is given that the real numbers a_i, b_i ($i = 0, 1, \ldots, 2n$) satisfy the following three conditions:

   (i) $a_i + a_{i+1} \geq 0$ for $i = 0, 1, \ldots, 2n-1$;

   (ii) $a_{2j+1} \leq 0$ for $j = 0, 1, \ldots, n-1$;

   (iii) For any integers p, q with $0 \leq p \leq q \leq n$, $\sum_{k=2p}^{2q} b_k > 0$.

   Prove that $\sum_{i=0}^{2n}(-1)^i a_i b_i \geq 0$, and find the condition under which the equality holds.

2. (USAMO/2007) Prove that for every nonnegative integer n, the number $7^{7^n} + 1$ is the product of at least $2n + 3$ (not necessarily distinct) primes.

3. (BULGARIA/2007) Let $a_1 > \frac{1}{12}$, $a_{n+1} = \sqrt{(n+2)a_n + 1}, n = 1, 2, \ldots$. Prove that

   (i) $a_n > n - \frac{2}{n}, n \in \mathbb{N}$; (ii) The sequence $b_n = 2^n\left(\frac{a_n}{n} - 1\right), n \in \mathbb{N}$ is convergent.

4. (USAMO/2010) There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \ldots < h_n$. If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

5. (IMO/2010) In each of six boxes $B_1, B_2, B_3, B_4, B_5, B_6$ there is initially one coin. There are two types of operation allowed:

   Type 1 : Choose a nonempty box B_j with $1 \leq j \leq 5$. Remove one coin from B_j and add two coins to B_{j+1}.

   Type 2 : Choose a nonempty box B_k with $1 \leq k \leq 4$. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2}.

   Determine whether there is a finite sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.)