Part I

Point process theory

1 Introduction

1.1 What is stochastic geometry?

Stochastic geometry (sometimes used synonymously with the older term geometric probability) deals with random spatial patterns. Random point patterns or *point processes* are the most basic and important such objects, hence *point process theory* is often considered to be the main sub-field of stochastic geometry.

Stochastic geometry provides answers to questions such as the following.

- How can one describe a (random) collection of points in one, two, or higher dimensions?
- How can one derive statistical properties of such a collection of points?
- How can one calculate statistical averages over all the possible realizations of such a random collection?
- How can one condition on having a point at a fixed location?
- Given an empirical set of points, which statistical model is likely to produce this point set?
- How can one describe more general random geometric objects such as a "random line" or a "random triangle"?

Throughout this book, we will use *point processes* to model the distributions of nodes (users, wireless terminals) in a wireless network where node locations are subject to uncertainty. In Part II, we will also encounter random geometric graphs to address the connectivity of wireless networks and random regions in the context of coverage problems.

1.2 Point processes as spatial models for wireless networks

Loosely speaking, a point process is a random collection of points that reside in some space. In this book, we will focus on the one-, two-, and three-dimensional Euclidean spaces $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$, since, in our applications, the points represent the locations of wireless nodes in the real world.

Point process models permit statements about entire classes of wireless networks, instead of just about one specific configuration of the network. In some cases, distributions over the point process can be calculated (for example for the

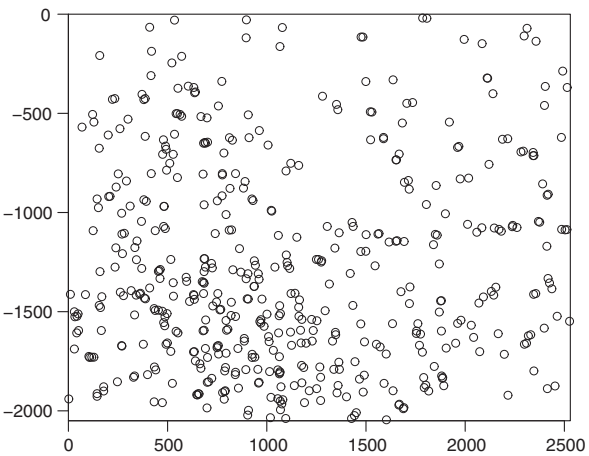

Figure 1.1 Locations of 493 cellular base stations in an area of about 5 km^2 in central London.

interference), in others, *spatial averaging* is performed, which yields expected values of certain performance metrics, such as the likelihood of transmission success.

Figure 1.1 shows the locations of all cellular base stations in an area of 2.5 km$\times$ 2 km in a central part of London. It illustrates that even the base stations (let alone the mobile users) themselves do not form a nice deterministic pattern or lattice. A random approach to modeling these locations is thus sensible.

In some applications, nodes are naturally clustered (attracted to each other), in others, they are separated (repelled from each other), and there exist models for these cases. In many situations, such as when we model interference, the relevant point process is the one of the *transmitters* only, not the point process of all nodes. It is determined by the spatial configuration of nodes together with the channel access (MAC) scheme, which performs a *thinning* procedure on the set of all nodes to produce the set of transmitters.

Emerging classes of large wireless systems such as ad hoc and sensor networks and cellular networks with coverage extensions such as relays or micro-base stations have been the subject of intense investigation over the last decade. Classical methods of communication theory are insufficient to analyze these new types of networks for the following reasons. (i) The performance-limiting metric is the signal-to-interference-plus-noise ratio (SINR) rather than the signal-to-noise ratio (SNR). (ii) The interference depends on the path loss and fading characteristics, which, in turn, are functions of the *network geometry*. (iii) The amount of uncertainty present in large wireless networks far exceeds the amount present in point-to-point systems: it is impossible for each node to know or predict the locations and channels of all but perhaps a few other nodes.

Two main tools have recently proved most helpful in circumventing the above difficulties: stochastic geometry and random geometric graphs. Stochastic

geometry allows one to study the average behavior over many spatial realizations of a network whose nodes are placed according to some probability distribution. Random geometric graphs capture the distance dependence and randomness in the connectivity of the nodes. This book provides an introduction to these mathematical tools and discusses some important applications to problems in wireless networking.

1.3 Asymptotic notation

We will make use of the standard asymptotic notation, summarized in Box 1.1. While the notation $f(x) = O(g(x))$ is common, it is not strictly correct since it implies a symmetry that does not exist. For example, $O(x^7) = O(\mathrm{e}^x)$ as $x \to \infty$, but this does not imply that $O(\mathrm{e}^x) = O(x^7)$, since not every function bounded by the exponential is bounded by the monomial (but the converse is true). A set-theoretic notation is more rigorous, where $O(g(x))$ denotes the class of functions which remain bounded when they are divided by $g(x)$. The above statement then reads $O(x^7) \subset O(\mathrm{e}^x)$, which is obviously not invertible.

Since "remains bounded" is weaker than "goes to zero" $o(\cdot)$ implies and is stronger than $O(\cdot)$, i.e., $f(x) = o(g(x)) \Rightarrow f(x) = O(g(x))$, or $o(g(x)) \subset O(g(x))$.

Box 1.1 Asymptotic notation

Let x tend to a. We write

$f(x) = O(g(x))$	if the ratio $f(x)/g(x)$ remains bounded
$f(x) = o(g(x))$	if the ratio $f(x)/g(x)$ goes to 0
$f(x) = \omega(g(x))$	if $g(x) = o(f(x))$
$f(x) = \Theta(g(x))$	if $f(x) = O(g(x))$ and $g(x) = O(f(x))$
$f(x) \sim g(x)$	if the ratio $f(x)/g(x)$ approaches 1.

More formally:

$$f(x) = O(g(x)) \text{ as } x \to a \quad \Longleftrightarrow \quad \limsup_{x \to a} \left| \frac{f(x)}{g(x)} \right| < \infty.$$

It is important to always indicate the limit point.

Example 1.1 Examples of asymptotic notation:

- $\ln x = O(x)$, $x^4 = O(\mathrm{e}^x)$, $\sin x = O(1)$ as $x \to \infty$;
- $x^2 = O(x)$, $\sin x = O(x)$ as $x \to 0$;
- Taylor expansion (at $x = 0$): $\mathrm{e}^x = 1 + x + O(x^2)$, $\mathrm{e}^x - 1 = x + o(x)$, $\mathrm{e}^x - 1 - x = o(x^{3/2})$, $\mathrm{e}^x = 1 + x + \Theta(x^2)$.

1.4 Sets and measurability

1.4.1 Borel sets and σ-algebra

Here we give a brief overview of the set-theoretic concepts used in the book and define measurability. Starting with the half-open intervals of $\mathbb{R}$, given by

$$[u, v) = \{x \in \mathbb{R} \colon u \le x < v\},$$

we obtain the Borel sets by taking the complement and countable unions and intersections of such intervals. The collection of all Borel sets forms the *Borel σ-algebra* or σ-field, denoted by $\mathcal{B}$. It contains all the singletons $\{x\}$ since

$$\{x\} = \bigcap_{n=1}^{\infty} \left[x, x + \frac{1}{n}\right)$$

and thus also all closed and open intervals since $(u, v) = [u, v) \setminus \{u\}$ and $[u, v] = [u, v) \cup \{v\}$. In fact, $\mathcal{B}$ can also be defined starting with closed or open intervals instead of the half-open ones. The restriction to *countable* unions and intersections is important, otherwise *all* subsets of $\mathbb{R}$ could be constructed simply by taking the union of the corresponding singletons.

The concept of the Borel σ-algebra is easily extended to the d-dimensional space $\mathbb{R}^d$, denoted by $\mathcal{B}^d$, by starting with (hyper)rectangles whose coordinate sets belong to $\mathcal{B}$. It further generalizes to all metric spaces (sets equipped with a distance or metric defined between their elements).

If a set consists of a countable number of singletons, it is a countable (finite or countably infinite) set. Conversely, all open sets are infinite sets, as are all half-open intervals of $\mathbb{R}$.

Letting $b(o, r) \triangleq \{x \in \mathbb{R}^d \colon \|x\| \le r\}$ be the (closed) ball of radius r centered at the origin o, a set $A \subseteq \mathbb{R}^d$ is *bounded* if there is a ball $b(a, r)$ such that $A \subset b(a, r)$. It is *compact* if it is also closed. Since $\mathcal{B}^d$ includes all closed sets it is a superset of the compact sets. Most sets encountered in this book will be Borel or compact.

The Cartesian product of sets is denoted by $\times$.

1.4.2 Measurability

While we will not concern ourselves with the intricacies of measurability, we need to clarify what "measurable" means. Generally, an ordered pair $(A, \mathcal{A})$ forms a *measurable space* if $\mathcal{A}$ is a σ-algebra of subsets of A. An important example is $(\mathbb{R}^d, \mathcal{B}^d)$ as defined before.

A function $f \colon \mathbb{R}^d \mapsto \mathbb{R}$ is *$\mathcal{B}^d$-measurable* if and only if the pre-image of $B \in \mathcal{B}$ is an element of $\mathcal{B}^d$, i.e.,

$$f^{-1}(B) \triangleq \{x \in \mathbb{R}^d \colon f(x) \in B\} \in \mathcal{B}^d.$$

More generally, if $(\mathcal{N}, \mathfrak{N})$ is another measurable space, $f\colon \mathcal{N} \mapsto \mathbb{R}$ is $\mathfrak{N}$-measurable if and only if the pre-image of $B \in \mathcal{B}$ is an element of $\mathfrak{N}$, i.e.,

$$f^{-1}(B) \triangleq \{\varphi \in \mathcal{N}\colon f(\varphi) \in B\} \in \mathfrak{N}.$$

This is the notion of measurability that we will employ in the context of point processes, where $\mathcal{N}$ denotes the space of point patterns.

A *measure* ν is a function $\mathcal{A} \mapsto \mathbb{R} \cup \{\infty\}$, where $\mathcal{A}$ is a σ-algebra, with the property of countable additivity, i.e.,

$$\nu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \nu(A_i)$$

for all pairwise disjoint $A_i \in \mathcal{A}$ and $n \in \mathbb{N} \cup \{\infty\}$ and satisfying $\nu(\emptyset) = 0$.

A fundamental measure is the *Lebesgue measure*, denoted by ν_d in the d-dimensional case or $|\cdot|$ if the number of dimensions is clear from the context. For intervals, it is defined as their length, i.e., $\nu_1((u, v]) = v - u$, and in higher dimensions, it is the area or (hyper)volume. For example, in two dimensions, the area of a disk of radius 2 is $|b(o, 2)| = 4\pi$. The Lebesgue measure is a *diffuse measure* since it gives zero mass to every singleton and thus any countable set. Other measures we will use frequently are probability measures, Dirac measures, and counting measures. The product of measures is denoted by $\otimes$.

Problems

1.1 Which of the following are true?

As $x \downarrow 0$:

(a) $\cosh x = o(1)$

(b) $\sin x = o(1)$

(c) $x^{-1} = O(\log x)$.

As $x \to \infty$:

(d) $O(2^x) = O(2^{ax})$

(e) $\mathrm{e}^{\Theta(1)} = \Theta(\mathrm{e}^x)$

(f) $O(\log x) = O(\log(x^a))$

(g) $f_1(x) = O(g_1(x))$, $f_2(x) = O(g_2(x)) \Rightarrow f_1(x)f_2(x) = O(g_1(x)g_2(x))$.

1.2 Let $X_1, X_2, \ldots, X_k$ be independent and identically distributed (iid) random variables with cumulative distribution function (cdf) $F(x)$. What is the cdf of the minimum $\min_i\{X_i\}$?

1.3 Let X be a Poisson random variable with parameter Y, where Y is itself a Poisson random variable with parameter μ. Show that the generating function of $X + Y$ is

$$G_{X+Y}(x) \triangleq \mathbb{E}(x^{X+Y}) = \exp\{\mu(x\mathrm{e}^{x-1} - 1)\}.$$

1.4 Let $X_1, \ldots, X_N$ be N iid random variables with cdf $F(x)$, where N is Poisson with mean μ. Calculate the cdf $G(x)$ of the *maximum* of the N random variables.

1.5 Let Z be the distance between two points picked independently (uniformly) at random in a disk of radius a. Show that $\mathbb{E}(Z^2) = a^2$.

1.6 Let a, b, and c be iid exponential. Show that the probability that the polynomial $ax^2 + bx + c$ has real roots is $1/3$.

1.7 A goose lays N eggs, where N is Poisson with mean λ. Each egg hatches with probability p, independently of the other eggs. Let K be the number of baby geese. Find $\mathbb{E}(K \mid N)$, $\mathbb{E}K$, and $\mathbb{E}(N \mid K)$.

2 Description of point processes

2.1 Description of one-dimensional point processes

There are several ways to describe a collection of points $(x_1, x_2, \ldots)$ in one dimension. Here are four of them, assuming the points lie on $\mathbb{R}^+$.

1. Direct characterization of the points (or *arrival times* if the axis is a time axis) x_i.
2. Using the increasing step function $A(t) = \sum_{i=1}^{\infty} \mathbf{1}(x_i \in [0, t))$.
3. Using the *interarrival intervals* $S_i = x_{i+1} - x_i$, $i \in \mathbb{N}$. Here it is assumed that the points are ordered, i.e., $x_1 \leq x_2 \leq \ldots$. In the case of *renewal processes*, the increments S_i are independent.
4. *Counting* the number of points falling in a set $B \subset \mathbb{R}$:

$$N(B) = \sum_{i=1}^{\infty} \mathbf{1}(x_i \in B).$$

Method 1 may be convenient if the total number of points n is finite and fixed and if the points are distributed independently and identically, i.e., if the random variables $x_1, x_2, \ldots, x_n$ are iid. Method 2 is a special case of method 4, with B restricted to an interval $[0, t)$. From $A(t)$, $N(B)$ can be calculated (in much the same way as the complete distribution of a random variable can be obtained from just the distribution function). Such cumulative functions are common in networking to count the number of packets that have arrived up to time t. Since method 3 relies on an ordering of the points, it is restricted to the one-dimensional case. Processes with independent interarrival intervals are the subject of study in the field of renewal theory. The most important case is where the S_i are exponentially distributed. This interarrival interval distribution results in a Poisson process.

An important dichotomy in point process theory is whether a point process is permitted to have multiple points at the same location. If not, it can be expressed as a *random set* $\{x_1, x_2, \ldots\}$; if yes, the point process is not a random set, and a random measure formalism as in method 4 is usually employed. We discuss these dual views of point processes in detail in the next section, before exploring point processes in general dimensions.

2.2 Point process duality

We first give a formal definition of a point process.

DEFINITION 2.1 (Point process) A point process is a countable random collection of points that reside in some measure space, usually the Euclidean space $\mathbb{R}^d$. The associated σ-algebra consists of the Borel sets $\mathcal{B}^d$, and the measure is the Lebesgue measure.

Such a point process can be described using two formalisms, a *random set formalism* (Box 2.1) and a *random measure formalism* (Box 2.2).

Box 2.1 Random set formalism

In this formalism, the point process is regarded as a *countable random set* $\Phi = \{x_1, x_2, \ldots\} \subset \mathbb{R}^d$ consisting of random variables $x_i \in \mathbb{R}^d$ as its elements.

Box 2.2 Random measure formalism

In this formalism, a point process is characterized by counting the number of points falling in sets $B \subset \mathbb{R}^d$. The number of points in B is denoted by $N(B)$. Hence $N(B)$ is a random variable that assumes values from the non-negative integers $\mathbb{N}_0$. N is called a (random) *counting measure*.

Similarly to the particle–wave duality in physics, we may speak of a duality in point process theory: On the one hand, point processes can be characterized as random sets, on the other hand, they can be regarded as random counting measures.

If Φ is given, N is obtained by

$$N(B) = \#\{\Phi \cap B\}.$$

Conversely, Φ is retrieved from N by

$$\Phi = \{x \in \mathbb{R}^d \colon N(\{x\}) = 1\}.$$

Hence there is a one-to-one mapping between Φ and N.

By considering the point process as a set $\Phi = \{x_1, x_2, \ldots\}$, we have implicitly assumed that only one point can exist at a given location, since a set can only contain one instance of each element. Such a point process is called *simple*. The random measure formalism is more general; it permits the characterization of point processes that may have multiple co-located points.[1] It is used to formally define simple point processes.

[1] In principle, point processes could be viewed also as *multisets*, which are sets where elements are allowed to appear more than once.